Propositional Logic
Computation and propositional tableau

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Outline

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Let us reduce all of them to (un)satisfiability!

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A set Γ of wffs is unsatisfiable iff $\Gamma \models \bot$

Validity

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Algorithm: SAT by truth table

Input : a set Γ of wffs
Output: true if Γ is SAT; false otherwise

begin
  foreach *interpretation I* **do**
    if *evaluate*(Γ,*I*) **then**
      return *true*;
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- Usually quite inefficient
- Works on the semantic level (by trying interpretations)

Satisfiability: Complexity

Theorem (Cook 1971)

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   (from 2^n possibilities, for n variables)
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Hardness

- Simulate nondeterministic Turing machine with polynomial time bound

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They require input in some normal form

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For all unsatisfiable sets Γ of wffs:

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3. Transform Γ' to $\bot$ (if possible, i.e., if Γ is unsatisfiable)

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Formula substitution

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Consider $\gamma = \neg A \vee (A \wedge B)$

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For wffs ϕ, ψ and variable A, if $\phi \equiv \psi$ then for each wff γ we have $\gamma[\phi/A] \equiv \gamma[\psi/A]$

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Theorem

For wffs ϕ, ψ and variable A, if $\phi \equiv \psi$ then for each wff γ we have $\gamma[\phi/A] \equiv \gamma[\psi/A]$

Proof.

By induction on the formula structure
(shown on the blackboard)

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  - Can we find a formula structure, such that any wff is equivalent to a formula of this structure?

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- $\phi \leftrightarrow \psi \equiv (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi) \equiv (\neg\phi \vee \psi) \wedge (\neg\psi \vee \phi)$

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Every formula is equivalent to one
containing only connectives $\neg$, $\wedge$, $\vee$

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Order of 2-3 does not matter!

Example of NNF transformation (1)

$$\neg((A \to B) \land (B \leftrightarrow C))$$

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3. $\neg((\neg A \vee B) \wedge ((\neg B \vee C) \wedge (\neg C \vee B))) \equiv$
   $\neg(\neg A \vee B) \vee \neg((\neg B \vee C) \wedge (\neg C \vee B))$

Example of NNF transformation (1)

$\neg((A \rightarrow B) \wedge (B \leftrightarrow C))$

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$\neg(\neg A \vee B) \vee \neg((\neg B \vee C) \wedge (\neg C \vee B))$

3 $\neg(\neg A \vee B) \vee \neg((\neg B \vee C) \wedge (\neg C \vee B)) \equiv$
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2. $(\neg\neg A \wedge \neg B) \vee \neg((\neg B \vee C) \wedge (\neg C \vee B)) \equiv$
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Example of NNF transformation (1)

$$\neg((A \to B) \land (B \leftrightarrow C))$$

1. $\neg((A \to B) \land (B \leftrightarrow C)) \equiv \neg((\neg A \lor B) \land (B \leftrightarrow C))$
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3. $\neg((\neg A \lor B) \land ((\neg B \lor C) \land (\neg C \lor B))) \equiv$
   $\neg(\neg A \lor B) \lor \neg((\neg B \lor C) \land (\neg C \lor B))$
3. $\neg(\neg A \lor B) \lor \neg((\neg B \lor C) \land (\neg C \lor B)) \equiv$
   $(\neg\neg A \land \neg B) \lor \neg((\neg B \lor C) \land (\neg C \lor B))$
2. $(\neg\neg A \land \neg B) \lor \neg((\neg B \lor C) \land (\neg C \lor B)) \equiv$
   $(A \land \neg B) \lor \neg((\neg B \lor C) \land (\neg C \lor B))$
3. $(A \land \neg B) \lor \neg((\neg B \lor C) \land (\neg C \lor B)) \equiv$
   $(A \land \neg B) \lor (\neg(\neg B \lor C) \lor \neg(\neg C \lor B))$

Example of NNF transformation (2)

3. $(A \land \neg B) \lor (\neg(\neg B \lor C) \lor \neg(\neg C \lor B)) \equiv$
   $(A \land \neg B) \lor ((\neg\neg B \land \neg C) \lor \neg(\neg C \lor B))$

Example of NNF transformation (2)

3. $(A \wedge \neg B) \vee (\neg(\neg B \vee C) \vee \neg(\neg C \vee B)) \equiv$
   $(A \wedge \neg B) \vee ((\neg\neg B \wedge \neg C) \vee \neg(\neg C \vee B))$
2. $(A \wedge \neg B) \vee ((\neg\neg B \wedge \neg C) \vee \neg(\neg C \vee B)) \equiv$
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Example of NNF transformation (2)

3 $(A \wedge \neg B) \vee (\neg(\neg B \vee C) \vee \neg(\neg C \vee B)) \equiv$
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2 $(A \wedge \neg B) \vee ((\neg\neg B \wedge \neg C) \vee \neg(\neg C \vee B)) \equiv$
$(A \wedge \neg B) \vee ((B \wedge \neg C) \vee \neg(\neg C \vee B))$

3 $(A \wedge \neg B) \vee ((B \wedge \neg C) \vee \neg(\neg C \vee B)) \equiv$
$(A \wedge \neg B) \vee ((B \wedge \neg C) \vee (\neg\neg C \wedge \neg B))$

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Observation

With the exception of 1, we are reducing the size of the formula

Outline

Propositional Tableau (1)

- A decision procedure for unsatisfiability

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- The idea is to break down logical connectives into their constituent parts
- We consider the tableau for formulas in Negation Normal Form (NNF)

Negation Normal Form

1. Eliminate $\top$, $\bot$, $\rightarrow$, $\leftrightarrow$
2. Apply double negation equivalences:
   - $\neg\neg\phi \equiv \phi$
3. Apply De Morgan equivalences:
   - $\neg(\phi \vee \psi) \equiv \neg\psi \wedge \neg\phi$
   - $\neg(\phi \wedge \psi) \equiv \neg\psi \vee \neg\phi$

Propositional Tableau (2)

Let Γ be a set of wffs in NNF.

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Notation

$$\frac{\phi \wedge \psi}{\begin{array}{c}\phi \\ \psi\end{array}}\,(\wedge) \qquad\qquad \frac{\phi \vee \psi}{\phi \mid \psi}\,(\vee)$$

Propositional Tableau (3)

Example: $\Gamma := \{(A \vee \neg B) \wedge B, \neg A\}$

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$$
\begin{array}{c}
(A \vee \neg B) \wedge B \\
| \\
\neg A
\end{array}
$$

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$$
\begin{array}{c}
(A \vee \neg B) \wedge B \\
| \\
\neg A \\
| \\
A \vee \neg B \\
| \\
B
\end{array}
$$

Propositional Tableau (3)

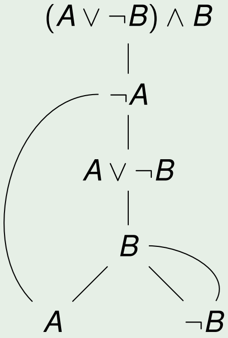
Example: $\Gamma := \{(A \vee \neg B) \wedge B, \neg A\}$

$$
\begin{array}{ccc}
 & (A \vee \neg B) \wedge B & \\
 & | & \\
 & \neg A & \\
 & | & \\
 & A \vee \neg B & \\
 & | & \\
 & B & \\
 / & & \backslash \\
A & & \neg B
\end{array}
$$

Example: $\Gamma := \{(A \lor \neg B) \land B, \neg A\}$

$$
\begin{tikzcd}[arrows=dash]
 & (A \lor \neg B) \land B \arrow[d] & \\
 & \neg A \arrow[d] & \\
 & A \lor \neg B \arrow[d] & \\
 & B \arrow[dl] \arrow[dr] & \\
A & & \neg B
\end{tikzcd}
$$

Example: $\Gamma := \{(A \vee \neg B) \wedge B, \neg A\}$



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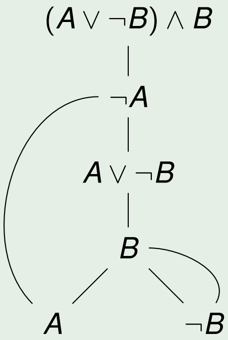


Γ is unsatisfiable!

We can simplify the notation used for a closure by putting a marker on the leaf of the closed path!

Example: $\Gamma := \{(A \vee \neg B) \wedge B, \neg A\}$

$$(A \vee \neg B) \wedge B$$

$$\neg A$$

$$A \vee \neg B$$

$$B$$

$$A \qquad\qquad \neg B$$

Γ is unsatisfiable!

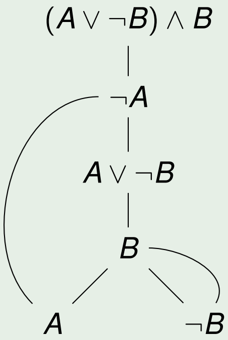
We can simplify the notation used for a closure by putting a marker on the leaf of the closed path!

On the blackboard

$A \wedge C, \neg A \vee B$

Propositional Tableau (3)

Example: $\Gamma := \{(A \vee \neg B) \wedge B, \neg A\}$



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$A \wedge C, \neg A \vee B$

Theorem

Propositional tableau is sound and complete, i.e.,
if the tableau for Γ is closed then $\Gamma \models \bot$, and
if $\Gamma \models \bot$ then the tableau for Γ is closed.

Outline

Exercises (1)

(From *Logic for Computer Science: Foundations of Automatic Theorem Proving*)

Give proof trees (by means of tableau) for the following tautologies:

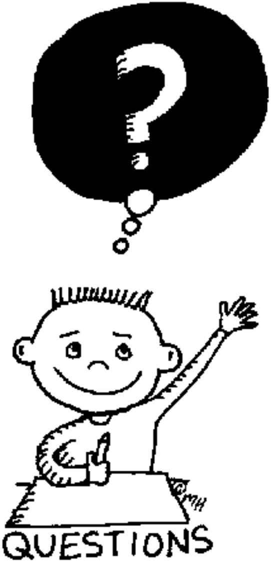
1. $A \rightarrow (B \rightarrow A)$
2. $(A \rightarrow B) \rightarrow ((A \rightarrow (B \rightarrow C)) \rightarrow (A \rightarrow C))$
3. $A \rightarrow (B \rightarrow A \wedge B)$
4. $A \rightarrow A \vee B$
5. $B \rightarrow A \vee B$
6. $(A \rightarrow B) \rightarrow ((A \rightarrow \neg B) \rightarrow \neg A)$
7. $A \wedge B \rightarrow A$
8. $A \wedge B \rightarrow B$
9. $(A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow (A \vee B \rightarrow C))$
10. $\neg\neg A \rightarrow A$

Exercises (2)

(From *Logic for Computer Science: Foundations of Automatic Theorem Proving*)

Give proof trees (by means of tableau) for the following equivalences:

1. $(A \vee B) \vee C \equiv A \vee (B \vee C)$ (associativity)
2. $(A \wedge B) \wedge C \equiv A \wedge (B \wedge C)$ (associativity)
3. $A \vee B \equiv B \vee A$ (commutativity)
4. $A \wedge B \equiv B \wedge A$ (commutativity)
5. $A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$ (distributivity)
6. $A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$ (distributivity)
7. $\neg(A \vee B) \equiv \neg A \wedge \neg B$ (De Morgan)
8. $\neg(A \wedge B) \equiv \neg A \vee \neg B$ (De Morgan)
9. $A \vee A \equiv A$ (idempotency)
10. $A \wedge A \equiv A$ (idempotency)
11. $\neg\neg A \equiv A$ (double negation)
12. $(A \vee B) \wedge (\neg A \vee C) \equiv (A \vee B) \wedge (\neg A \vee C) \wedge (B \vee C)$ (resolution)



END OF THE LECTURE