Propositional Logic
Resolution and DPLL

Mario Alviano

University of Calabria, Italy

A.Y. 2018/2019

Outline

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CNF transformation

1. Eliminate $\top$, $\bot$, $\rightarrow$, $\leftrightarrow$

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   - $\phi \vee (\psi \wedge \gamma) \equiv (\phi \vee \psi) \wedge (\phi \vee \gamma)$

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2. Apply double negation equivalences:
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3. Apply De Morgan equivalences:
   - $\neg(\phi \lor \psi) \equiv \neg\psi \land \neg\phi$
   - $\neg(\phi \land \psi) \equiv \neg\psi \lor \neg\phi$
4. Apply distributivity equivalence:
   - $\phi \lor (\psi \land \gamma) \equiv (\phi \lor \psi) \land (\phi \lor \gamma)$

Order of 2-4 does not matter!

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$$\neg((A \rightarrow B) \wedge (B \leftrightarrow C))$$

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Example of CNF transformation (1)

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Example of CNF transformation (1)

$\neg((A \rightarrow B) \wedge (B \leftrightarrow C))$

1 $\neg((A \rightarrow B) \wedge (B \leftrightarrow C)) \equiv \neg((\neg A \vee B) \wedge (B \leftrightarrow C))$

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3 $\neg((\neg A \vee B) \wedge ((\neg B \vee C) \wedge (\neg C \vee B))) \equiv$
$\neg(\neg A \vee B) \vee \neg((\neg B \vee C) \wedge (\neg C \vee B))$

Example of CNF transformation (1)

$$\neg((A \to B) \wedge (B \leftrightarrow C))$$

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3 $\neg((\neg A \vee B) \wedge ((\neg B \vee C) \wedge (\neg C \vee B))) \equiv$
$\neg(\neg A \vee B) \vee \neg((\neg B \vee C) \wedge (\neg C \vee B))$

3 $\neg(\neg A \vee B) \vee \neg((\neg B \vee C) \wedge (\neg C \vee B)) \equiv$
$(\neg\neg A \wedge \neg B) \vee \neg((\neg B \vee C) \wedge (\neg C \vee B))$

Example of CNF transformation (1)

$$\neg((A \rightarrow B) \land (B \leftrightarrow C))$$

1 $\neg((A \rightarrow B) \land (B \leftrightarrow C)) \equiv \neg((\neg A \lor B) \land (B \leftrightarrow C))$

1 $\neg((\neg A \lor B) \land (B \leftrightarrow C)) \equiv \neg((\neg A \lor B) \land ((\neg B \lor C) \land (\neg C \lor B)))$

3 $\neg((\neg A \lor B) \land ((\neg B \lor C) \land (\neg C \lor B))) \equiv$
$\neg(\neg A \lor B) \lor \neg((\neg B \lor C) \land (\neg C \lor B))$

3 $\neg(\neg A \lor B) \lor \neg((\neg B \lor C) \land (\neg C \lor B)) \equiv$
$(\neg\neg A \land \neg B) \lor \neg((\neg B \lor C) \land (\neg C \lor B))$

2 $(\neg\neg A \land \neg B) \lor \neg((\neg B \lor C) \land (\neg C \lor B)) \equiv$
$(A \land \neg B) \lor \neg((\neg B \lor C) \land (\neg C \lor B))$

$$\neg((A \rightarrow B) \wedge (B \leftrightarrow C))$$

1 $\neg((A \rightarrow B) \wedge (B \leftrightarrow C)) \equiv \neg((\neg A \vee B) \wedge (B \leftrightarrow C))$

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3 $\neg(\neg A \vee B) \vee \neg((\neg B \vee C) \wedge (\neg C \vee B)) \equiv$
$(\neg\neg A \wedge \neg B) \vee \neg((\neg B \vee C) \wedge (\neg C \vee B))$

2 $(\neg\neg A \wedge \neg B) \vee \neg((\neg B \vee C) \wedge (\neg C \vee B)) \equiv$
$(A \wedge \neg B) \vee \neg((\neg B \vee C) \wedge (\neg C \vee B))$

3 $(A \wedge \neg B) \vee \neg((\neg B \vee C) \wedge (\neg C \vee B)) \equiv$
$(A \wedge \neg B) \vee (\neg(\neg B \vee C) \vee \neg(\neg C \vee B))$

Example of CNF transformation (2)

3. $(A \wedge \neg B) \vee (\neg(\neg B \vee C) \vee \neg(\neg C \vee B)) \equiv$
   $(A \wedge \neg B) \vee ((\neg\neg B \wedge \neg C) \vee \neg(\neg C \vee B))$

Example of CNF transformation (2)

3. $(A \wedge \neg B) \vee (\neg(\neg B \vee C) \vee \neg(\neg C \vee B)) \equiv$
   $(A \wedge \neg B) \vee ((\neg\neg B \wedge \neg C) \vee \neg(\neg C \vee B))$
2. $(A \wedge \neg B) \vee ((\neg\neg B \wedge \neg C) \vee \neg(\neg C \vee B)) \equiv$
   $(A \wedge \neg B) \vee ((B \wedge \neg C) \vee \neg(\neg C \vee B))$

Example of CNF transformation (2)

3. $(A \land \neg B) \lor (\neg(\neg B \lor C) \lor \neg(\neg C \lor B)) \equiv$
   $(A \land \neg B) \lor ((\neg\neg B \land \neg C) \lor \neg(\neg C \lor B))$
2. $(A \land \neg B) \lor ((\neg\neg B \land \neg C) \lor \neg(\neg C \lor B)) \equiv$
   $(A \land \neg B) \lor ((B \land \neg C) \lor \neg(\neg C \lor B))$
3. $(A \land \neg B) \lor ((B \land \neg C) \lor \neg(\neg C \lor B)) \equiv$
   $(A \land \neg B) \lor ((B \land \neg C) \lor (\neg\neg C \land \neg B))$

Example of CNF transformation (2)

3 $(A \wedge \neg B) \vee (\neg(\neg B \vee C) \vee \neg(\neg C \vee B)) \equiv$
$(A \wedge \neg B) \vee ((\neg\neg B \wedge \neg C) \vee \neg(\neg C \vee B))$

2 $(A \wedge \neg B) \vee ((\neg\neg B \wedge \neg C) \vee \neg(\neg C \vee B)) \equiv$
$(A \wedge \neg B) \vee ((B \wedge \neg C) \vee \neg(\neg C \vee B))$

3 $(A \wedge \neg B) \vee ((B \wedge \neg C) \vee \neg(\neg C \vee B)) \equiv$
$(A \wedge \neg B) \vee ((B \wedge \neg C) \vee (\neg\neg C \wedge \neg B))$

2 $(A \wedge \neg B) \vee ((B \wedge \neg C) \vee (\neg\neg C \wedge \neg B)) \equiv$
$(A \wedge \neg B) \vee ((B \wedge \neg C) \vee (C \wedge \neg B))$

Example of CNF transformation (2)

3 $(A \wedge \neg B) \vee (\neg(\neg B \vee C) \vee \neg(\neg C \vee B)) \equiv$
$(A \wedge \neg B) \vee ((\neg\neg B \wedge \neg C) \vee \neg(\neg C \vee B))$

2 $(A \wedge \neg B) \vee ((\neg\neg B \wedge \neg C) \vee \neg(\neg C \vee B)) \equiv$
$(A \wedge \neg B) \vee ((B \wedge \neg C) \vee \neg(\neg C \vee B))$

3 $(A \wedge \neg B) \vee ((B \wedge \neg C) \vee \neg(\neg C \vee B)) \equiv$
$(A \wedge \neg B) \vee ((B \wedge \neg C) \vee (\neg\neg C \wedge \neg B))$

2 $(A \wedge \neg B) \vee ((B \wedge \neg C) \vee (\neg\neg C \wedge \neg B)) \equiv$
$(A \wedge \neg B) \vee ((B \wedge \neg C) \vee (C \wedge \neg B))$

So far so good!

With the exception of 1, we are reducing the size of the formula

Example of CNF transformation (2)

- 3 $(A \wedge \neg B) \vee (\neg(\neg B \vee C) \vee \neg(\neg C \vee B)) \equiv$
  $(A \wedge \neg B) \vee ((\neg\neg B \wedge \neg C) \vee \neg(\neg C \vee B))$
- 2 $(A \wedge \neg B) \vee ((\neg\neg B \wedge \neg C) \vee \neg(\neg C \vee B)) \equiv$
  $(A \wedge \neg B) \vee ((B \wedge \neg C) \vee \neg(\neg C \vee B))$
- 3 $(A \wedge \neg B) \vee ((B \wedge \neg C) \vee \neg(\neg C \vee B)) \equiv$
  $(A \wedge \neg B) \vee ((B \wedge \neg C) \vee (\neg\neg C \wedge \neg B))$
- 2 $(A \wedge \neg B) \vee ((B \wedge \neg C) \vee (\neg\neg C \wedge \neg B)) \equiv$
  $(A \wedge \neg B) \vee ((B \wedge \neg C) \vee (C \wedge \neg B))$

So far so good!

With the exception of 1, we are reducing the size of the formula

Example of CNF transformation (3)

4. $(A \land \neg B) \lor ((B \land \neg C) \lor (C \land \neg B)) \equiv$
   $(A \lor ((B \land \neg C) \lor (C \land \neg B))) \land (\neg B \lor ((B \land \neg C) \lor (C \land \neg B)))$

Example of CNF transformation (3)

- $(A \wedge \neg B) \vee ((B \wedge \neg C) \vee (C \wedge \neg B)) \equiv (A \vee ((B \wedge \neg C) \vee (C \wedge \neg B))) \wedge (\neg B \vee ((B \wedge \neg C) \vee (C \wedge \neg B)))$
- $(A \vee ((B \wedge \neg C) \vee (C \wedge \neg B))) \wedge (\neg B \vee ((B \wedge \neg C) \vee (C \wedge \neg B))) \equiv (A \vee (((B \wedge \neg C) \vee C) \wedge ((B \wedge \neg C) \vee \neg B))) \wedge (\neg B \vee ((B \wedge \neg C) \vee (C \wedge \neg B)))$

Example of CNF transformation (3)

- $(A \wedge \neg B) \vee ((B \wedge \neg C) \vee (C \wedge \neg B)) \equiv (A \vee ((B \wedge \neg C) \vee (C \wedge \neg B))) \wedge (\neg B \vee ((B \wedge \neg C) \vee (C \wedge \neg B)))$
- $(A \vee ((B \wedge \neg C) \vee (C \wedge \neg B))) \wedge (\neg B \vee ((B \wedge \neg C) \vee (C \wedge \neg B))) \equiv (A \vee (((B \wedge \neg C) \vee C) \wedge ((B \wedge \neg C) \vee \neg B))) \wedge (\neg B \vee ((B \wedge \neg C) \vee (C \wedge \neg B)))$
- $(A \vee (((B \wedge \neg C) \vee C) \wedge ((B \wedge \neg C) \vee \neg B))) \wedge (\neg B \vee ((B \wedge \neg C) \vee (C \wedge \neg B))) \equiv (A \vee (((B \vee C) \wedge (\neg C \vee C)) \wedge ((B \wedge \neg C) \vee \neg B))) \wedge (\neg B \vee ((B \wedge \neg C) \vee (C \wedge \neg B)))$

Example of CNF transformation (3)

- $(A \land \neg B) \lor ((B \land \neg C) \lor (C \land \neg B)) \equiv (A \lor ((B \land \neg C) \lor (C \land \neg B))) \land (\neg B \lor ((B \land \neg C) \lor (C \land \neg B)))$
- $(A \lor ((B \land \neg C) \lor (C \land \neg B))) \land (\neg B \lor ((B \land \neg C) \lor (C \land \neg B))) \equiv (A \lor (((B \land \neg C) \lor C) \land ((B \land \neg C) \lor \neg B))) \land (\neg B \lor ((B \land \neg C) \lor (C \land \neg B)))$
- $(A \lor (((B \land \neg C) \lor C) \land ((B \land \neg C) \lor \neg B))) \land (\neg B \lor ((B \land \neg C) \lor (C \land \neg B))) \equiv (A \lor (((B \lor C) \land (\neg C \lor C)) \land ((B \land \neg C) \lor \neg B))) \land (\neg B \lor ((B \land \neg C) \lor (C \land \neg B)))$
- $(A \lor (((B \lor C) \land (\neg C \lor C)) \land ((B \land \neg C) \lor \neg B))) \land (\neg B \lor ((B \land \neg C) \lor (C \land \neg B))) \equiv (A \lor (((B \lor C) \land (\neg C \lor C)) \land ((B \lor \neg B) \land (\neg C \lor \neg B)))) \land (\neg B \lor ((B \land \neg C) \lor (C \land \neg B)))$

- $(A \land \neg B) \lor ((B \land \neg C) \lor (C \land \neg B)) \equiv (A \lor ((B \land \neg C) \lor (C \land \neg B))) \land (\neg B \lor ((B \land \neg C) \lor (C \land \neg B)))$
- $(A \lor ((B \land \neg C) \lor (C \land \neg B))) \land (\neg B \lor ((B \land \neg C) \lor (C \land \neg B))) \equiv (A \lor (((B \land \neg C) \lor C) \land ((B \land \neg C) \lor \neg B))) \land (\neg B \lor ((B \land \neg C) \lor (C \land \neg B)))$
- $(A \lor (((B \land \neg C) \lor C) \land ((B \land \neg C) \lor \neg B))) \land (\neg B \lor ((B \land \neg C) \lor (C \land \neg B))) \equiv (A \lor (((B \lor C) \land (\neg C \lor C)) \land ((B \land \neg C) \lor \neg B))) \land (\neg B \lor ((B \land \neg C) \lor (C \land \neg B)))$
- $(A \lor (((B \lor C) \land (\neg C \lor C)) \land ((B \land \neg C) \lor \neg B))) \land (\neg B \lor ((B \land \neg C) \lor (C \land \neg B))) \equiv (A \lor (((B \lor C) \land (\neg C \lor C)) \land ((B \lor \neg B) \land (\neg C \lor \neg B)))) \land (\neg B \lor ((B \land \neg C) \lor (C \land \neg B)))$

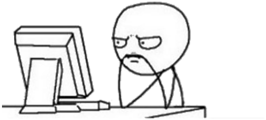
4. $(A \lor (((B \lor C) \land (\neg C \lor C)) \land ((B \lor \neg B) \land (\neg C \lor \neg B)))) \land (\neg B \lor ((B \land \neg C) \lor (C \land \neg B))) \equiv ((A \lor ((B \lor C) \land (\neg C \lor C))) \land (A \lor ((B \lor \neg B) \land (\neg C \lor \neg B)))) \land (\neg B \lor ((B \land \neg C) \lor (C \land \neg B)))$

- $(A \vee (((B \vee C) \wedge (\neg C \vee C)) \wedge ((B \vee \neg B) \wedge (\neg C \vee \neg B)))) \wedge (\neg B \vee ((B \wedge \neg C) \vee (C \wedge \neg B))) \equiv ((A \vee ((B \vee C) \wedge (\neg C \vee C))) \wedge (A \vee ((B \vee \neg B) \wedge (\neg C \vee \neg B)))) \wedge (\neg B \vee ((B \wedge \neg C) \vee (C \wedge \neg B)))$
- $((A \vee ((B \vee C) \wedge (\neg C \vee C))) \wedge (A \vee ((B \vee \neg B) \wedge (\neg C \vee \neg B)))) \wedge (\neg B \vee ((B \wedge \neg C) \vee (C \wedge \neg B))) \equiv (((A \vee (B \vee C)) \wedge (A \vee (\neg C \vee C))) \wedge (A \vee ((B \vee \neg B) \wedge (\neg C \vee \neg B)))) \wedge (\neg B \vee ((B \wedge \neg C) \vee (C \wedge \neg B)))$

- $(A \vee (((B \vee C) \wedge (\neg C \vee C)) \wedge ((B \vee \neg B) \wedge (\neg C \vee \neg B)))) \wedge (\neg B \vee ((B \wedge \neg C) \vee (C \wedge \neg B))) \equiv ((A \vee ((B \vee C) \wedge (\neg C \vee C))) \wedge (A \vee ((B \vee \neg B) \wedge (\neg C \vee \neg B)))) \wedge (\neg B \vee ((B \wedge \neg C) \vee (C \wedge \neg B)))$
- $((A \vee ((B \vee C) \wedge (\neg C \vee C))) \wedge (A \vee ((B \vee \neg B) \wedge (\neg C \vee \neg B)))) \wedge (\neg B \vee ((B \wedge \neg C) \vee (C \wedge \neg B))) \equiv (((A \vee (B \vee C)) \wedge (A \vee (\neg C \vee C))) \wedge (A \vee ((B \vee \neg B) \wedge (\neg C \vee \neg B)))) \wedge (\neg B \vee ((B \wedge \neg C) \vee (C \wedge \neg B)))$
- $(((A \vee (B \vee C)) \wedge (A \vee (\neg C \vee C))) \wedge (A \vee ((B \vee \neg B) \wedge (\neg C \vee \neg B)))) \wedge (\neg B \vee ((B \wedge \neg C) \vee (C \wedge \neg B))) \equiv (((A \vee (B \vee C)) \wedge (A \vee (\neg C \vee C))) \wedge ((A \vee (B \vee \neg B)) \wedge (A \vee (\neg C \vee \neg B)))) \wedge (\neg B \vee ((B \wedge \neg C) \vee (C \wedge \neg B)))$

Example of CNF transformation (4)

- $(A \lor (((B \lor C) \land (\neg C \lor C)) \land ((B \lor \neg B) \land (\neg C \lor \neg B)))) \land (\neg B \lor ((B \land \neg C) \lor (C \land \neg B))) \equiv ((A \lor ((B \lor C) \land (\neg C \lor C))) \land (A \lor ((B \lor \neg B) \land (\neg C \lor \neg B)))) \land (\neg B \lor ((B \land \neg C) \lor (C \land \neg B)))$
- $((A \lor ((B \lor C) \land (\neg C \lor C))) \land (A \lor ((B \lor \neg B) \land (\neg C \lor \neg B)))) \land (\neg B \lor ((B \land \neg C) \lor (C \land \neg B))) \equiv (((A \lor (B \lor C)) \land (A \lor (\neg C \lor C))) \land (A \lor ((B \lor \neg B) \land (\neg C \lor \neg B)))) \land (\neg B \lor ((B \land \neg C) \lor (C \land \neg B)))$
- $(((A \lor (B \lor C)) \land (A \lor (\neg C \lor C))) \land (A \lor ((B \lor \neg B) \land (\neg C \lor \neg B)))) \land (\neg B \lor ((B \land \neg C) \lor (C \land \neg B))) \equiv (((A \lor (B \lor C)) \land (A \lor (\neg C \lor C))) \land ((A \lor (B \lor \neg B)) \land (A \lor (\neg C \lor \neg B)))) \land (\neg B \lor ((B \land \neg C) \lor (C \land \neg B)))$



Example of CNF transformation (5)

4. $(((A \lor (B \lor C)) \land (A \lor (\neg C \lor C))) \land ((A \lor (B \lor \neg B)) \land (A \lor (\neg C \lor \neg B)))) \land (\neg B \lor ((B \land \neg C) \lor (C \land \neg B))) \equiv$
$(((A \lor (B \lor C)) \land (A \lor (\neg C \lor C))) \land ((A \lor (B \lor \neg B)) \land (A \lor (\neg C \lor \neg B)))) \land (\neg B \lor (((B \land \neg C) \lor C) \land ((B \land \neg C) \lor \neg B)))$

Example of CNF transformation (5)

- 4 $(((A \vee (B \vee C)) \wedge (A \vee (\neg C \vee C))) \wedge ((A \vee (B \vee \neg B)) \wedge (A \vee (\neg C \vee \neg B)))) \wedge (\neg B \vee ((B \wedge \neg C) \vee (C \wedge \neg B))) \equiv (((A \vee (B \vee C)) \wedge (A \vee (\neg C \vee C))) \wedge ((A \vee (B \vee \neg B)) \wedge (A \vee (\neg C \vee \neg B)))) \wedge (\neg B \vee (((B \wedge \neg C) \vee C) \wedge ((B \wedge \neg C) \vee \neg B)))$
- 4 $(((A \vee (B \vee C)) \wedge (A \vee (\neg C \vee C))) \wedge ((A \vee (B \vee \neg B)) \wedge (A \vee (\neg C \vee \neg B)))) \wedge (\neg B \vee (((B \wedge \neg C) \vee C) \wedge ((B \wedge \neg C) \vee \neg B))) \equiv (((A \vee (B \vee C)) \wedge (A \vee (\neg C \vee C))) \wedge ((A \vee (B \vee \neg B)) \wedge (A \vee (\neg C \vee \neg B)))) \wedge (\neg B \vee (((B \vee C) \wedge (\neg C \vee C)) \wedge ((B \wedge \neg C) \vee \neg B)))$

Example of CNF transformation (5)

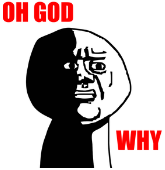
4. $(((A \vee (B \vee C)) \wedge (A \vee (\neg C \vee C))) \wedge ((A \vee (B \vee \neg B)) \wedge (A \vee (\neg C \vee \neg B)))) \wedge (\neg B \vee ((B \wedge \neg C) \vee (C \wedge \neg B))) \equiv$
$(((A \vee (B \vee C)) \wedge (A \vee (\neg C \vee C))) \wedge ((A \vee (B \vee \neg B)) \wedge (A \vee (\neg C \vee \neg B)))) \wedge (\neg B \vee (((B \wedge \neg C) \vee C) \wedge ((B \wedge \neg C) \vee \neg B)))$
4. $(((A \vee (B \vee C)) \wedge (A \vee (\neg C \vee C))) \wedge ((A \vee (B \vee \neg B)) \wedge (A \vee (\neg C \vee \neg B)))) \wedge (\neg B \vee (((B \wedge \neg C) \vee C) \wedge ((B \wedge \neg C) \vee \neg B))) \equiv$
$(((A \vee (B \vee C)) \wedge (A \vee (\neg C \vee C))) \wedge ((A \vee (B \vee \neg B)) \wedge (A \vee (\neg C \vee \neg B)))) \wedge (\neg B \vee (((B \vee C) \wedge (\neg C \vee C)) \wedge ((B \wedge \neg C) \vee \neg B)))$
4. $(((A \vee (B \vee C)) \wedge (A \vee (\neg C \vee C))) \wedge ((A \vee (B \vee \neg B)) \wedge (A \vee (\neg C \vee \neg B)))) \wedge (\neg B \vee (((B \vee C) \wedge (\neg C \vee C)) \wedge ((B \wedge \neg C) \vee \neg B))) \equiv$
$(((A \vee (B \vee C)) \wedge (A \vee (\neg C \vee C))) \wedge ((A \vee (B \vee \neg B)) \wedge (A \vee (\neg C \vee \neg B)))) \wedge (\neg B \vee (((B \vee C) \wedge (\neg C \vee C)) \wedge ((B \vee \neg B) \wedge (\neg C \vee \neg B))))$

Example of CNF transformation (5)

4 $(((A \vee (B \vee C)) \wedge (A \vee (\neg C \vee C))) \wedge ((A \vee (B \vee \neg B)) \wedge (A \vee (\neg C \vee \neg B)))) \wedge (\neg B \vee ((B \wedge \neg C) \vee (C \wedge \neg B))) \equiv$
$(((A \vee (B \vee C)) \wedge (A \vee (\neg C \vee C))) \wedge ((A \vee (B \vee \neg B)) \wedge (A \vee (\neg C \vee \neg B)))) \wedge (\neg B \vee (((B \wedge \neg C) \vee C) \wedge ((B \wedge \neg C) \vee \neg B)))$

4 $(((A \vee (B \vee C)) \wedge (A \vee (\neg C \vee C))) \wedge ((A \vee (B \vee \neg B)) \wedge (A \vee (\neg C \vee \neg B)))) \wedge (\neg B \vee (((B \wedge \neg C) \vee C) \wedge ((B \wedge \neg C) \vee \neg B))) \equiv$
$(((A \vee (B \vee C)) \wedge (A \vee (\neg C \vee C))) \wedge ((A \vee (B \vee \neg B)) \wedge (A \vee (\neg C \vee \neg B)))) \wedge (\neg B \vee (((B \vee C) \wedge (\neg C \vee C)) \wedge ((B \wedge \neg C) \vee \neg B)))$

4 $(((A \vee (B \vee C)) \wedge (A \vee (\neg C \vee C))) \wedge ((A \vee (B \vee \neg B)) \wedge (A \vee (\neg C \vee \neg B)))) \wedge (\neg B \vee (((B \vee C) \wedge (\neg C \vee C)) \wedge ((B \wedge \neg C) \vee \neg B))) \equiv$
$(((A \vee (B \vee C)) \wedge (A \vee (\neg C \vee C))) \wedge ((A \vee (B \vee \neg B)) \wedge (A \vee (\neg C \vee \neg B)))) \wedge (\neg B \vee (((B \vee C) \wedge (\neg C \vee C)) \wedge ((B \vee \neg B) \wedge (\neg C \vee \neg B))))$



4. $(((A \lor (B \lor C)) \land (A \lor (\neg C \lor C))) \land ((A \lor (B \lor \neg B)) \land (A \lor (\neg C \lor \neg B)))) \land (\neg B \lor (((B \lor C) \land (\neg C \lor C)) \land ((B \lor \neg B) \land (\neg C \lor \neg B)))) \equiv (((A \lor (B \lor C)) \land (A \lor (\neg C \lor C))) \land ((A \lor (B \lor \neg B)) \land (A \lor (\neg C \lor \neg B)))) \land ((\neg B \lor ((B \lor C) \land (\neg C \lor C))) \land (\neg B \lor ((B \lor \neg B) \land (\neg C \lor \neg B))))$

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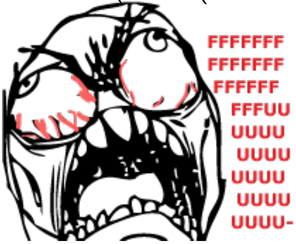
- $(((A \vee (B \vee C)) \wedge (A \vee (\neg C \vee C))) \wedge ((A \vee (B \vee \neg B)) \wedge (A \vee (\neg C \vee \neg B)))) \wedge (\neg B \vee (((B \vee C) \wedge (\neg C \vee C)) \wedge ((B \vee \neg B) \wedge (\neg C \vee \neg B)))) \equiv (((A \vee (B \vee C)) \wedge (A \vee (\neg C \vee C))) \wedge ((A \vee (B \vee \neg B)) \wedge (A \vee (\neg C \vee \neg B)))) \wedge ((\neg B \vee ((B \vee C) \wedge (\neg C \vee C))) \wedge (\neg B \vee ((B \vee \neg B) \wedge (\neg C \vee \neg B))))$
- $(((A \vee (B \vee C)) \wedge (A \vee (\neg C \vee C))) \wedge ((A \vee (B \vee \neg B)) \wedge (A \vee (\neg C \vee \neg B)))) \wedge ((\neg B \vee ((B \vee C) \wedge (\neg C \vee C))) \wedge (\neg B \vee ((B \vee \neg B) \wedge (\neg C \vee \neg B)))) \equiv (((A \vee (B \vee C)) \wedge (A \vee (\neg C \vee C))) \wedge ((A \vee (B \vee \neg B)) \wedge (A \vee (\neg C \vee \neg B)))) \wedge (((\neg B \vee (B \vee C)) \wedge (\neg B \vee (\neg C \vee C))) \wedge (\neg B \vee ((B \vee \neg B) \wedge (\neg C \vee \neg B))))$



Example of CNF transformation (7)

4. $(((A \vee (B \vee C)) \wedge (A \vee (\neg C \vee C))) \wedge ((A \vee (B \vee \neg B)) \wedge (A \vee (\neg C \vee \neg B)))) \wedge (((\neg B \vee (B \vee C)) \wedge (\neg B \vee (\neg C \vee C))) \wedge (\neg B \vee ((B \vee \neg B) \wedge (\neg C \vee \neg B)))) \equiv (((A \vee (B \vee C)) \wedge (A \vee (\neg C \vee C))) \wedge ((A \vee (B \vee \neg B)) \wedge (A \vee (\neg C \vee \neg B)))) \wedge (((\neg B \vee (B \vee C)) \wedge (\neg B \vee (\neg C \vee C))) \wedge ((\neg B \vee (B \vee \neg B)) \wedge (\neg B \vee (\neg C \vee \neg B))))$

Example of CNF transformation (7)

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Flattening

$(A \vee B \vee C) \wedge (A \vee \neg C \vee C) \wedge (A \vee B \vee \neg B) \wedge (A \vee \neg C \vee \neg B) \wedge (\neg B \vee B \vee C) \wedge (\neg B \vee \neg C \vee C) \wedge (\neg B \vee B \vee \neg B) \wedge (\neg B \vee \neg C \vee \neg B)$

4. $(((A \vee (B \vee C)) \wedge (A \vee (\neg C \vee C))) \wedge ((A \vee (B \vee \neg B)) \wedge (A \vee (\neg C \vee \neg B)))) \wedge (((\neg B \vee (B \vee C)) \wedge (\neg B \vee (\neg C \vee C))) \wedge (\neg B \vee ((B \vee \neg B) \wedge (\neg C \vee \neg B)))) \equiv (((A \vee (B \vee C)) \wedge (A \vee (\neg C \vee C))) \wedge ((A \vee (B \vee \neg B)) \wedge (A \vee (\neg C \vee \neg B)))) \wedge (((\neg B \vee (B \vee C)) \wedge (\neg B \vee (\neg C \vee C))) \wedge ((\neg B \vee (B \vee \neg B)) \wedge (\neg B \vee (\neg C \vee \neg B))))$

Flattening

$(A \vee B \vee C) \wedge (A \vee \neg C \vee C) \wedge (A \vee B \vee \neg B) \wedge (A \vee \neg C \vee \neg B) \wedge (\neg B \vee B \vee C) \wedge (\neg B \vee \neg C \vee C) \wedge (\neg B \vee B \vee \neg B) \wedge (\neg B \vee \neg C \vee \neg B)$

Eliminate clauses containing A and $\neg A$ (those are equivalent to $\top$ and are hence neutral for $\wedge$)

$(A \vee B \vee C) \wedge (A \vee \neg C \vee \neg B) \wedge (\neg B \vee \neg C)$

Example of CNF transformation (7)

4. $(((A \lor (B \lor C)) \land (A \lor (\neg C \lor C))) \land ((A \lor (B \lor \neg B)) \land (A \lor (\neg C \lor \neg B)))) \land (((\neg B \lor (B \lor C)) \land (\neg B \lor (\neg C \lor C))) \land (\neg B \lor ((B \lor \neg B) \land (\neg C \lor \neg B)))) \equiv (((A \lor (B \lor C)) \land (A \lor (\neg C \lor C))) \land ((A \lor (B \lor \neg B)) \land (A \lor (\neg C \lor \neg B)))) \land (((\neg B \lor (B \lor C)) \land (\neg B \lor (\neg C \lor C))) \land ((\neg B \lor (B \lor \neg B)) \land (\neg B \lor (\neg C \lor \neg B))))$

Flattening

$(A \lor B \lor C) \land (A \lor \neg C \lor C) \land (A \lor B \lor \neg B) \land (A \lor \neg C \lor \neg B) \land (\neg B \lor B \lor C) \land (\neg B \lor \neg C \lor C) \land (\neg B \lor B \lor \neg B) \land (\neg B \lor \neg C \lor \neg B)$

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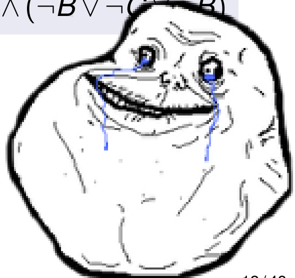

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Flattening

$(A \lor B \lor C) \land (A \lor \neg C \lor C) \land (A \lor B \lor \neg B) \land (A \lor \neg C \lor \neg B) \land (\neg B \lor B \lor C) \land (\neg B \lor \neg C \lor C) \land (\neg B \lor B \lor \neg B) \land (\neg B \lor \neg C \lor \neg B)$

Eliminate clauses containing A and $\neg A$ (those are equivalent to $\top$ and are hence neutral for $\land$)

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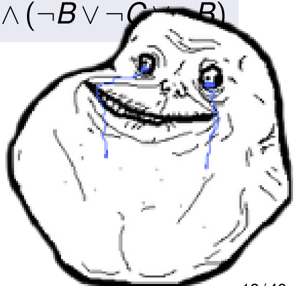
Be careful! This algorithm outputs formulas of exponential size in general



CNF transformation in practice (1)

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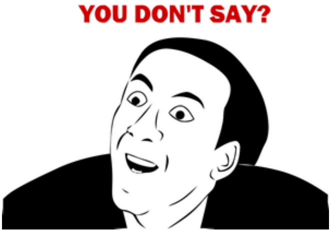

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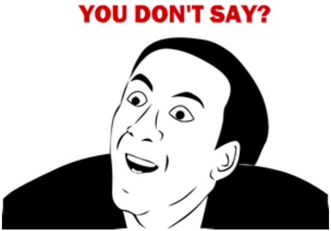
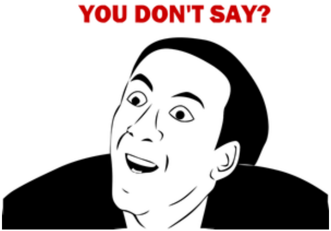


Yes, I say!



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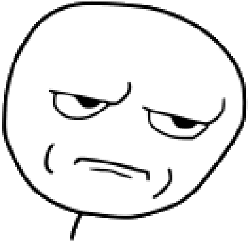

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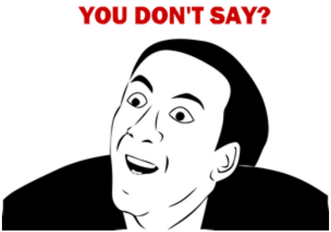

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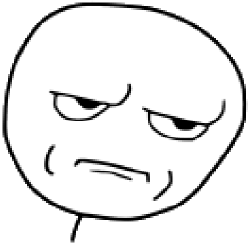

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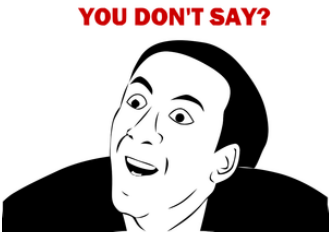

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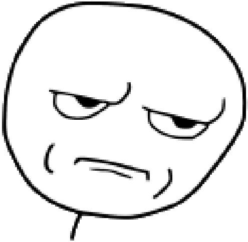

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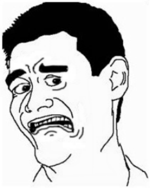

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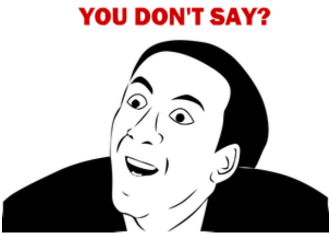

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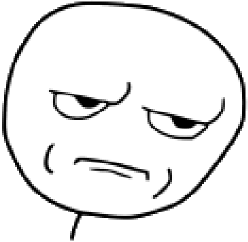

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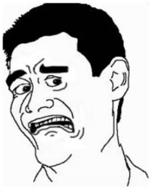

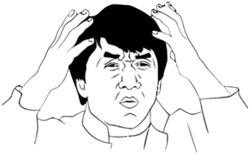

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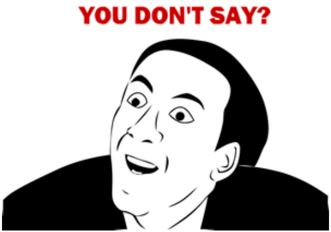

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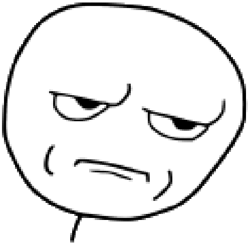

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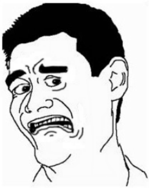

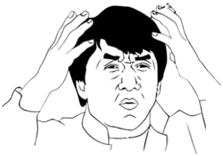

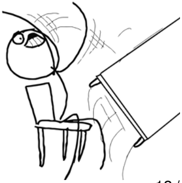

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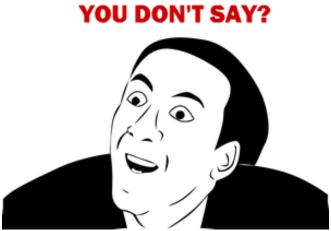

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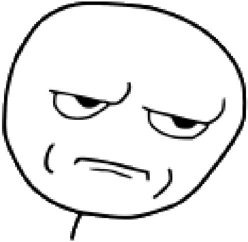

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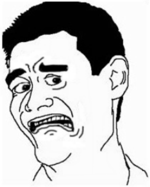
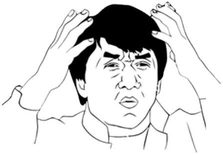
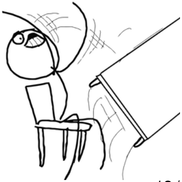
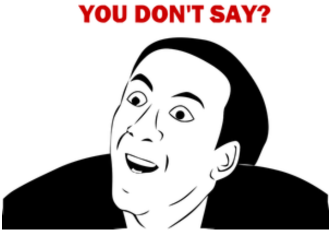




Trust me...





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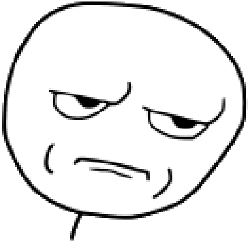

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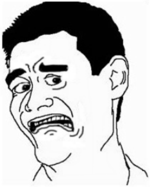

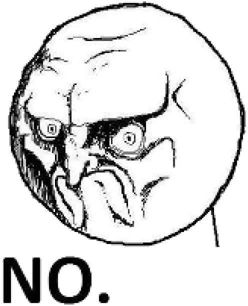

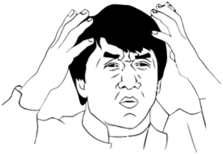

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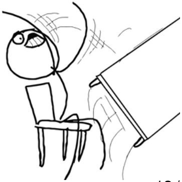

Outline

CNF transformation in practice (2)

Work bottom-up on the structure of the formula and build a set of clauses preserving only (un)satisfiability (not equivalence)

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1. Replace each $L_1 \odot L_2$ (L_1, L_2 literals, $\odot \in \{\vee, \wedge, \rightarrow, \leftrightarrow\}$) by an auxiliary (fresh) variable X and introduce formula $X \leftrightarrow L_1 \odot L_2$, i.e.,

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($\odot = \vee$) $\neg X \vee L_1 \vee L_2$ $[X \rightarrow L_1 \vee L_2]$

$X \vee \neg L_1, X \vee \neg L_2$ $[L_1 \vee L_2 \rightarrow X]$

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$(\odot = \vee)$	$\neg X \vee L_1 \vee L_2$	$[X \rightarrow L_1 \vee L_2]$
	$X \vee \neg L_1, X \vee \neg L_2$	$[L_1 \vee L_2 \rightarrow X]$
$(\odot = \wedge)$	$\neg X \vee L_1, \neg X \vee L_2$	$[X \rightarrow L_1 \wedge L_2]$
	$X \vee \neg L_1 \vee \neg L_2$	$[L_1 \wedge L_2 \rightarrow X]$

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$(\odot = \vee)$	$\neg X \vee L_1 \vee L_2$	$[X \rightarrow L_1 \vee L_2]$
	$X \vee \neg L_1, X \vee \neg L_2$	$[L_1 \vee L_2 \rightarrow X]$
$(\odot = \wedge)$	$\neg X \vee L_1, \neg X \vee L_2$	$[X \rightarrow L_1 \wedge L_2]$
	$X \vee \neg L_1 \vee \neg L_2$	$[L_1 \wedge L_2 \rightarrow X]$
$(\odot = \rightarrow)$	$\neg X \vee \neg L_1 \vee L_2$	$[X \rightarrow (L_1 \rightarrow L_2)]$
	$X \vee L_1, X \vee \neg L_2$	$[(L_1 \rightarrow L_2) \rightarrow X]$

Work bottom-up on the structure of the formula and build a set of clauses preserving only (un)satisfiability (not equivalence)

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$(\odot = \vee)$	$\neg X \vee L_1 \vee L_2$	$[X \rightarrow L_1 \vee L_2]$
	$X \vee \neg L_1,\ X \vee \neg L_2$	$[L_1 \vee L_2 \rightarrow X]$
$(\odot = \wedge)$	$\neg X \vee L_1,\ \neg X \vee L_2$	$[X \rightarrow L_1 \wedge L_2]$
	$X \vee \neg L_1 \vee \neg L_2$	$[L_1 \wedge L_2 \rightarrow X]$
$(\odot = \rightarrow)$	$\neg X \vee \neg L_1 \vee L_2$	$[X \rightarrow (L_1 \rightarrow L_2)]$
	$X \vee L_1,\ X \vee \neg L_2$	$[(L_1 \rightarrow L_2) \rightarrow X]$
$(\odot = \leftrightarrow)$	$\neg X \vee \neg L_1 \vee L_2,\ \neg X \vee \neg L_2 \vee L_1$	$[X \rightarrow (L_1 \leftrightarrow L_2)]$
	$X \vee \neg L_1 \vee \neg L_2,\ X \vee L_1 \vee L_2$	$[(L_1 \leftrightarrow L_2) \rightarrow X]$

CNF transformation in practice (2)

Work bottom-up on the structure of the formula and build a set of clauses preserving only (un)satisfiability (not equivalence)

1. Replace each $L_1 \odot L_2$ (L_1, L_2 literals, $\odot \in \{\vee, \wedge, \rightarrow, \leftrightarrow\}$) by an auxiliary (fresh) variable X and introduce formula $X \leftrightarrow L_1 \odot L_2$, i.e.,

$(\odot = \vee)$	$\neg X \vee L_1 \vee L_2$	$[X \rightarrow L_1 \vee L_2]$
	$X \vee \neg L_1, X \vee \neg L_2$	$[L_1 \vee L_2 \rightarrow X]$
$(\odot = \wedge)$	$\neg X \vee L_1, \neg X \vee L_2$	$[X \rightarrow L_1 \wedge L_2]$
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And do not apply distributivity equivalence!

Example of CNF transformation (reloaded)

- We started with $\neg((A \rightarrow B) \land (B \leftrightarrow C))$

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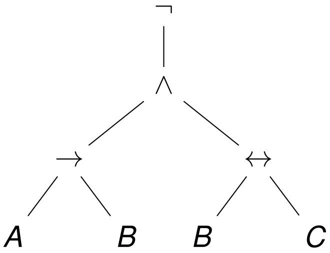

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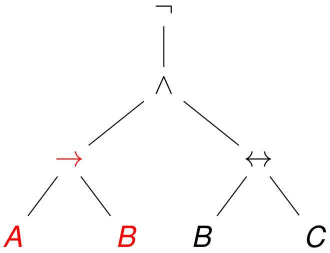

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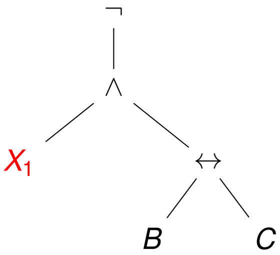

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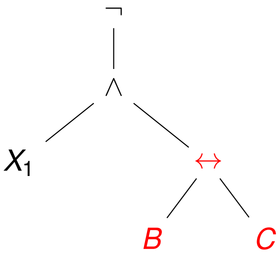

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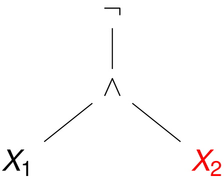

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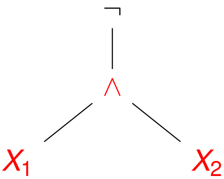

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$\neg$

X_3

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DNF transformation

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Order of 2-4 does not matter!

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More general

$$C_1 \wedge \cdots \wedge C_k \wedge (a \vee L_1^1 \vee \ldots \vee L_n^1) \wedge (\neg a \vee L_1^2 \vee \ldots \vee L_m^2)$$
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This is known as **resolution**!

Propositional resolution (2)

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Factorization comes "for free"!

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Resolvent

Given two clauses C_1 and C_2 such that $a \in C_1$ and $\neg a \in C_2$, $(C_1 \setminus \{a\}) \cup (C_2 \setminus \{\neg a\})$ is the resolvent of C_1 and C_2.

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Given a set Γ of clauses, a derivation by resolution of a clause C from Γ, denoted $\Gamma \vdash_R C$, is a sequence $C_1, \ldots, C_n$ such that $C_n = C$ and for each C_i $(1 \leq i \leq n)$ we have

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Lemma

If $\Gamma \vdash_R C$ then $\Gamma \models C$.

Proof. By induction on the sequence $C_1, \ldots, C_n$.

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- $\Gamma = \{\mathit{rain} \rightarrow \mathit{streetwet},\ \mathit{rain}\}$

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$\{streetwet\}$

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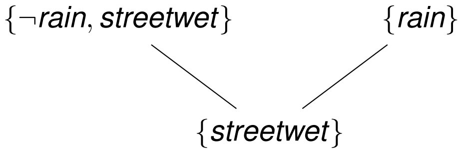

$$rain \rightarrow streetwet,\ rain \vdash_R streetwet$$

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A derivation by resolution of $\Box$ from Γ is called a refutation of Γ.

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A derivation by resolution of $\Box$ from Γ is called a refutation of Γ.

Resolution Theorem

$\Gamma \vdash_R \Box$ if and only if Γ is unsatisfiable.

Proof. Soundness: $\Gamma \vdash_R \Box$ implies $\Gamma \models \Box$ (by the previous Lemma).
Completeness: by induction over the number of variables in Γ.

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Goal: $\{rain \rightarrow streetwet,\ rain\} \models streetwet$

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$\{rain \rightarrow streetwet,\ rain,\ \neg streetwet\}$:

Goal: $\{rain \rightarrow streetwet,\ rain\} \models streetwet$
IFF: $\{rain \rightarrow streetwet,\ rain\} \cup \{\neg streetwet\} \models \bot$

The following is a refutation of
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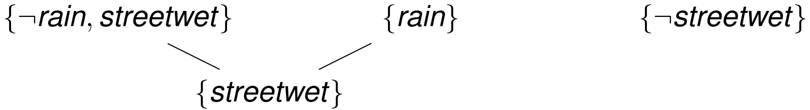

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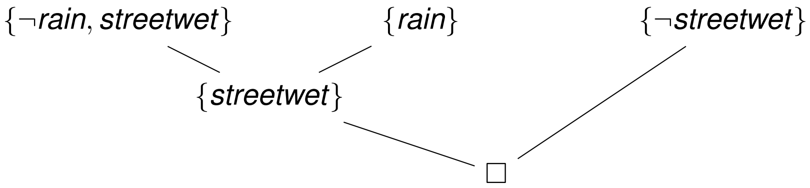

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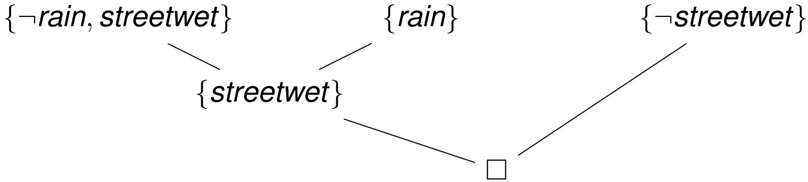

$rain \to streetwet,\ rain,\ \neg streetwet \vdash_R \Box$

Validity, equivalence, entailment, ...

Validity

$\models \phi$ iff $\neg\phi \models \bot$

- Test whether $\neg\phi \vdash_R \Box$

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$\Gamma \models \phi$ iff $\Gamma, \neg\phi \models \bot$

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Satisfiability

ϕ is satisfiable iff $\neg\phi$ is not valid

- Test whether $\phi \nvdash_R \Box$

Algorithm: SAT by resolution

Input : a set Γ of wffs
Output: true if Γ is SAT; false otherwise

```
begin
    Γ^CNF := trasformToCNF(Γ);
    repeat
        if □ ∈ Γ^CNF then
            return false;
        Γ_old := Γ^CNF;
        Γ^CNF := Γ^CNF ∪ resolveAll(Γ^CNF);
    until Γ_old = Γ^CNF;
    return true;
```

Function resolveAll(Γ^{CNF}: set of clauses)

```
begin
    Γ_res := ∅;
    foreach C_1 ∈ Γ^CNF and foreach atom a ∈ C_1 do
        foreach C_2 ∈ Γ^CNF such that ¬a ∈ C_2 do
            Γ_res := Γ_res ∪ {C1 \ {a} ∪ C2 \ {¬a}};
    return Γ_res
```

Implementing resolution (2)

Complexity

Deciding $\Gamma \vdash_R \Box$ requires up to an exponential number of steps (with respect to the size of the formula)

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Since unsatisfiability of a formula is coNP-complete,
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Example

Is the following formula satisfiable?

$$(A \vee B) \wedge (A \leftrightarrow B) \wedge (\neg A \vee \neg B)$$

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A clause C is a tautology if there is $a \in V$ such that $a \in C$ and $\neg a \in C$

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Example

1. $\{\{A, B\}, \{A, \neg B\}, \{\neg A, B\}, \{\neg A, \neg B\}\}$
2. $\{\{A\}, \{B\}, \{A, \neg B\}, \{\neg A, B\}, \{\neg A, \neg B\}\}$

Examples

1. Is $(A_1 \lor A_2) \land (\neg A_2 \lor \neg A_3) \land (A_3 \lor A_4) \land (\neg A_4 \lor \neg A_1)$ satisfiable?
2. Does A follow from $(A \lor B \lor C) \land (\neg C \lor B) \land (A \lor \neg B)$?
3. Does $\neg A$ follow from $(A \lor B \lor C) \land (\neg C \lor B) \land (A \lor \neg B)$?
4. Does $A \land B$ follow from $(\neg A \to B) \land (A \to B) \land (\neg A \to \neg B)$?

Outline

DPLL algorithm

- By $\neg\ell$ we denote the opposite literal of ℓ
  - if $\ell = \neg a$ then $\neg\ell = a$
- Simplify is often called **Unit Propagation**
  - Call-by-name parameters (references)

Algorithm: DPLL

Input : a set Γ of clauses
Output: true if Γ is SAT; false otherwise

begin
  Simplify(Γ);
  if $\Gamma = \emptyset$ **then return** *true* ;
  if $\Box \in \Gamma$ **then return** *false* ;
  $\ell :=$ ChooseLiteral(Γ);
  return $DPLL(\Gamma \cup \{\{\ell\}\})$ **or** $DPLL(\Gamma \cup \{\{\neg\ell\}\})$;

Procedure Simplify(Γ)

begin
  while $\{\ell\} \in \Gamma$ **do**
    foreach $c \in \Gamma$ **do**
      if $\ell \in c$ **then** $\Gamma := \Gamma \setminus \{c\}$;
      else if $\neg\ell \in c$ **then** $\Gamma := (\Gamma \setminus \{c\}) \cup \{(c \setminus \{\neg\ell\})\}$;

Properties

- DPLL(Γ) returns true if Γ is satisfiable, and false otherwise

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- DPLL(Γ) works in polynomial-space

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Backtracking

When a contradiction (empty clause) arises, the search resumes from some previous assumption ℓ by assuming $\neg\ell$ instead

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$\Gamma = \{\{x_1, x_2, \neg x_3\}, \{\neg x_2\}, \{x_4, \neg x_3\}\}$

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Branching order (ChooseLiteral) can make big differences!

Examples (2)

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- Can we reuse something from a previous computation?
  - Progressive SAT

Outline

(From *Logic for Computer Science: Foundations of Automatic Theorem Proving*)

1. Show that the following set of clauses are unsatisfiable using the resolution method:
    1. $\{\{A, B, \neg C\}, \{A, B, C\}, \{A, \neg B\}, \{\neg A\}\}$
    2. $\{\{A, \neg B, C\}, \{B, C\}, \{\neg A, C\}, \{B, \neg C\}, \{\neg B\}\}$
    3. $\{\{A, \neg B\}, \{A, C\}, \{\neg B, C\}, \{\neg A, B\}, \{B, \neg C\}, \{\neg A, \neg C\}\}$
    4. $\{\{A, B\}, \{\neg A, B\}, \{A, \neg B\}, \{\neg A, \neg B\}, \}$
2. Find all resolvents of the following pairs of clauses:
    1. $\{A, B\}$, $\{\neg A, \neg B\}$
    2. $\{A, \neg B\}$, $\{B, C, D\}$
    3. $\{\neg A, B, \neg C\}$, $\{B, C\}$
    4. $\{A, \neg A\}$, $\{A, \neg A\}$
3. Find all resolvents of the following sets of clauses:
    1. $\{\{A, \neg B\}, \{A, B\}, \{\neg A\}\}$
    2. $\{\{A, B, C\}, \{\neg B, \neg C\}, \{\neg A, \neg C\}\}$
    3. $\{\{\neg A, \neg B\}, \{B, C\}, \{\neg C, A\}\}$
    4. $\{\{A, B, C\}, \{A\}, \{B\}\}$

Exercises (2)

1. Show using resolution whether the following statements hold:
   1. $x \vee y \vee \neg z \models (x \vee z) \leftrightarrow (\neg y \rightarrow x)$
   2. $((\neg X \vee \neg Y) \rightarrow \neg(\neg Y \vee X))$ is satisfiable

(From *Logic for Computer Science: Foundations of Automatic Theorem Proving*)

1. Show that the following set of clauses are unsatisfiable using the DPLL algorithm:
    1. $\{\{A, B, \neg C\}, \{A, B, C\}, \{A, \neg B\}, \{\neg A\}\}$
    2. $\{\{A, \neg B, C\}, \{B, C\}, \{\neg A, C\}, \{B, \neg C\}, \{\neg B\}\}$
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    4. $\{\{A, B\}, \{\neg A, B\}, \{A, \neg B\}, \{\neg A, \neg B\}, \}$
2. Show using DPLL whether the following statements hold:
    1. $x \vee y \vee \neg z \models (x \vee z) \leftrightarrow (\neg y \rightarrow x)$
    2. $((\neg X \vee \neg Y) \rightarrow \neg(\neg Y \vee X))$ is satisfiable

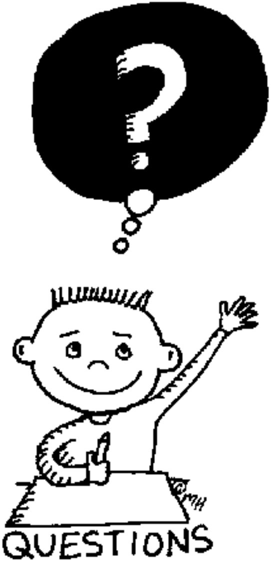
1. Find formulas in CNF and DNF having the following truth table:

A	B	C	D	ϕ
0	0	0	0	0
0	0	0	1	1
0	0	1	0	1
0	0	1	1	0
0	1	0	0	1
0	1	0	1	1
0	1	1	0	0
0	1	1	1	1

A	B	C	D	ϕ
1	0	0	0	0
1	0	0	1	0
1	0	1	0	1
1	0	1	1	0
1	1	0	0	1
1	1	0	1	0
1	1	1	0	1
1	1	1	1	1

2. Decide whether the following formula is satisfiable:

$$A_1 \wedge (\neg A_1 \vee \neg A_2) \wedge (A_2 \vee A_3) \wedge (\neg A_3 \vee \neg A_4) \wedge (A_4 \vee A_5)$$



END OF THE LECTURE