First-order logic
Semantic notions and sequent calculus

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Outline

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Model

Let ϕ be a wff.

1. ϕ is satisfied in a structure $\mathcal{A}$ wrt a variable assignment $\xi^{\mathcal{A}}$, denoted $(\mathcal{A}, \xi^{\mathcal{A}}) \models \phi$, if $\nu^{(\mathcal{A}, \xi^{\mathcal{A}})}(\phi) = 1$

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5. ϕ is valid, denoted $\models \phi$, if $\mathcal{A} \models \phi$ for every structure $\mathcal{A}$

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Let Γ be a set of wffs.

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Let Γ be a set of wffs and ϕ a wff.

- ϕ is a semantic (or logic) consequence of Γ, denoted $\Gamma \models \phi$, if $(\mathcal{A}, \xi^{\mathcal{A}}) \models \phi$ whenever Γ is satisfied in $\mathcal{A}$ wrt $\xi^{\mathcal{A}}$, i.e., whenever $(\mathcal{A}, \xi^{\mathcal{A}}) \models \psi$ for each $\psi \in \Gamma$

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2. $\forall x\ (Human(x) \rightarrow Mortal(x))$
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- Yes, now we can!

$$1, 2 \models 3$$

Closures

Definition

Let ϕ be a wff such that $free(\phi) = \{x_1, \ldots, x_n\}$.

- $Cl(\phi) = \forall x_1 \cdots \forall x_n\ \phi$ is the universal closure of ϕ
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Theorems

- $\mathcal{A} \models \phi$ if and only if $\mathcal{A} \models Cl(\phi)$
- $\models \phi$ if and only if $\models Cl(\phi)$
- ϕ is satisfiable if and only if $Ex(\phi)$ is satisfiable

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Equivalence (1)

Definition

$\phi \equiv \psi$ if for every $(\mathcal{A}, \xi^{\mathcal{A}})$ we have

$$\nu^{(\mathcal{A}, \xi^{\mathcal{A}})}(\phi) = \nu^{(\mathcal{A}, \xi^{\mathcal{A}})}(\psi)$$

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  - Substitution Theorem

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  - Substitution Theorem
- Can we do more?

Equivalence (2)

Substitution Lemma

Let $(\mathcal{A}, \xi^{\mathcal{A}})$ be an interpretation, x, y variables, and d an object in $D_{\mathcal{A}}$.

1. If y does not occur in a term t then

$$\nu^{(\mathcal{A},\xi^{\mathcal{A}}[d/x])}(t) = \nu^{(\mathcal{A},\xi^{\mathcal{A}}[d/y])}(t[y/x])$$

2. If y does not occur in a formula ϕ then

$$\nu^{(\mathcal{A},\xi^{\mathcal{A}}[d/x])}(\phi) = \nu^{(\mathcal{A},\xi^{\mathcal{A}}[d/y])}(\phi[y/x])$$

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where y is not free in the bottom sequents of $(\forall r)$ and $(\exists l)$

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1. Given the formula

$$\exists x \forall y \ P(g(x, y), f(x), g(y, z))$$

   define one structure which is a model and
   one which is not a model

2. Given the formulas
   1. $\forall x \ R(x, x)$
   2. $\forall x \forall y \ (R(x, y) \rightarrow R(y, x))$
   3. $\forall x \forall y \forall z \ ((R(x, y) \wedge R(y, z)) \rightarrow R(x, z))$

   show that no formula is entailed by the other two formulas.

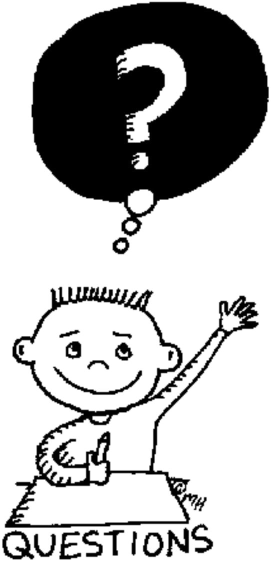
   An easy way of showing this is by defining three structures:
   The first structure should satisfy formulas 1 and 2, but not
   3; the second should satisfy 1 and 3, but not 2; the third
   should satisfy 2 ad 3, but not 1.

1. Find a model for the following set of wffs:
   - $\forall x\ R(x, g(x))$
   - $\forall y\ \neg R(y, y)$
   - $\forall x \forall y \forall z\ ((R(x, y) \wedge R(y, z)) \rightarrow R(x, z))$

   Then, show that no model for this set of wffs has a finite domain.
2. Show the followings using the sequent calculus:
   1. $\models \exists x\ R(x, x) \rightarrow \exists y\ R(y, y)$
   2. $\models \forall x\ R(x, x) \rightarrow \neg \exists y\ \neg R(y, y)$
   3. $\models \neg \exists x\ \neg R(x, x) \rightarrow \forall y\ R(y, y)$
   4. $\models \neg \exists y \forall x\ (\neg P(x, x) \leftrightarrow P(y, x))$
   5. $\neg \forall x \phi \equiv \exists x \neg \phi$

Prove the followings or provide interpretations proving their falsity:

1. $\forall x\, A(x) \vdash \exists x\, A(x)$
2. $\forall x\, (A(x) \wedge B(x)) \vdash \forall x\, A(x) \wedge \forall x\, B(x)$
3. $\neg\exists x\, A(x) \vdash \exists x\, \neg A(x)$
4. $\forall x \forall y\, C(x,y) \vdash \forall y \forall x\, C(x,y)$
5. $\exists x \forall y\, C(x,y) \vdash \forall y \exists x\, C(x,y)$
6. $\forall x \forall y\, (C(x,y) \rightarrow \neg C(y,x)) \vdash \forall x\, \neg C(x,x)$
7. $\forall x\, (\exists y\, C(x,y) \rightarrow A(x)) \vdash \forall x \exists y\, (C(x,y) \rightarrow A(x))$
8. $\forall x\, (\forall y\, C(x,y) \rightarrow A(x)) \vdash \forall x \exists y\, (C(x,y) \rightarrow A(x))$
9. $\forall x\, (\exists y\, C(x,y) \rightarrow A(x)) \vdash \forall x \forall y\, (C(x,y) \rightarrow A(x))$
10. $\forall x\, (\forall y\, C(x,y) \rightarrow A(x)) \vdash \forall x \forall y\, (C(x,y) \rightarrow A(x))$



END OF THE LECTURE