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Quantum Methods for Interacting Particle Systems. II. Glauber Dynamics for Ising Spin Systems

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Abstract. Using the formalism and the results described in [3] and [1], we discuss the approach to termodynamic equilibrium for discrete spin systems in a framework that generalizes the one originally proposed by R. Glauber. We prove a lower bound estimate for their exponetial rate of convergence to equilibrium in the high temperature regime which is better than those previously known (the case of $d = 1$ is amenable to a more detailed analysis, see [10]). We also give application to some (not necessarily ferromagnetic) Ising-spin models. These results provide an upper bound for the critical temperature of the d-dimensional Ising model.

KEYWORDS: Glauber dynamics, spectral gap, interacting particle systems

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1. Introduction

In this paper we give a first example of the application of the algebraic techniques borrowed from equilibrium quantum statistical mechanics to interacting particle systems. We refer the reader to [3] for a general discussion and reference to other results obtained by a similar approach, and to [1] for some application to (equilibrium) classical statistical mechanics.

The formal equivalence of stochastic Ising models and quantum spin systems has earlier been employed by T. Matsui [5–7] who worked directly with infinite systems. Here we are interested in getting estimates for finite volume systems which are (eventually) uniform in its size.

We consider Ising spins on a $\mathbf{Z}^d$ lattice undergoing a Glauber dynamics. The aim is to derive in a straightforward way a bound for the spectral gap of the generator in the high temperature regime. In particular short range interactions satisfy the hypotesis of our theorem and the technique turns out to be particularly useful in these cases, where it provides sharper bounds than those already known. Another study of the spectral properties of the Glauber generator is [9], where the construction of the invariant subspaces for the dynamics is also given (one can also get good bounds for the gap from the representation formulas in [9]). As a byproduct we get estimates for the critical temperature of the models considered.

We cannot expect our bounds to be good in dimension one, actually it is off by a factor of two for small β, as can be seen from the comparison with the explicit expression given in [10].

This is due to the fact that in our approach dynamics is written in form of a Markov chain whose state space is the set of the parts of Λ, $\mathfrak{P}_\Lambda$. Therefore, the dependence of the generator of the process on the lattice dimension is hidden in the cardinality of the state space. This is also why the foregoing analysis works better in higher dimensions.

As an introduction to the formalism, we start with the "single site dynamics" (as in [2]). Following the prescription given in [3] we consider the Hilbert space of complex square summable functions on the single site configuration space $\mathbf{Z}_2^{(x)}$ with respect to the symmetric Bernoulli measure. Namely for all $x \in \mathbf{Z}^d$

$$\begin{aligned} \mathcal{H}_x &:= \operatorname{span}\{|\emptyset\rangle_x, |\mathrm{x}\rangle_x\} \cong \mathbf{C}^2, \\ |\emptyset\rangle_x &\equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}_x, \qquad |\mathrm{x}\rangle_x \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}_x, \end{aligned}$$

$\mathcal{U}_x = M(2, \mathbf{C})$ is the algebra of bounded operators on $\mathcal{H}_x$.[1] Let us define the spin operator $\sigma_x \in \mathcal{U}_x$:

$$\begin{aligned} \sigma_x \ |\emptyset\rangle_x &= |\mathrm{x}\rangle_x \\ \sigma_x \ |\mathrm{x}\rangle_x &= |\emptyset\rangle_x \end{aligned}$$

equivalent to the Pauli matrix $\sigma^{(1)}$

$$\sigma^{(1)} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and the spin flip operator $\mathbf{f}_x \in \mathcal{U}_x$:

$$\begin{aligned} \mathbf{f}_x \, |\emptyset\rangle_x &= |\emptyset\rangle_x \\ \mathbf{f}_x \, |\mathrm{x}\rangle_x &= - |\mathrm{x}\rangle_x \end{aligned}$$

equivalent to the Pauli matrix $\sigma^{(3)}$

$$\sigma^{(3)} \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

[1] Here, see [3] and [8] for more details, we think of $\mathcal{H}_x$ as spanned by two (orthonormal) vectors labelled by the "empty site" and the "full site" configurations. Consequently any operator acting on the configuration space is lifted to a linear operator acting on $\mathcal{H}_x$ and a probability density on the configuration space becomes a convex combination of the projectors on the subspaces spanned by the basis vectors of $\mathcal{H}_x$.

Consider also the automorphism $\mathbf{U}_x$ on $\mathcal{U}_x$ such that

$$\mathbf{U}_x \in \mathcal{U}_x : \mathbf{U}_x \sigma_x \mathbf{U}_x = \mathbf{f}_x, \qquad \mathbf{U}_x^2 = \mathbf{I}_x.$$

Setting $\mathcal{C}_x^{(s)}, \mathcal{C}_x^{(f)} \subset \mathcal{U}_x$ to be the commutative algebras generated by the identity operator $\mathbf{I}_x$ and, respectively, by σ_x, $\mathbf{f}_x$, one can see easily from the above definition that $\mathbf{U}_x$ maps one into another: $\mathbf{U}_x \mathcal{C}_x^{(s)} \mathbf{U}_x = \mathcal{C}_x^{(f)}$. For this reason, in the following, we will denote them both by $\mathcal{C}_x$ (the algebra of observables, that is the algebra of diagonal operators on $\mathcal{H}_x$).

This is the kinematical structure $(\mathcal{H}_x, \mathcal{U}_x, \mathcal{C}_x)$ for a single site system. Let Λ be any finite subset of the $\mathbf{Z}^d$ lattice. We define its analog on Λ by tensor product $(\mathcal{H}_\Lambda, \mathcal{U}_\Lambda, \mathcal{C}_\Lambda)$ ($\mathbf{Z}_2^\Lambda = \bigotimes_{x\in\Lambda} \mathbf{Z}_2^{(x)}$). Then we have (see [3])

$$\begin{aligned}
|\alpha\rangle_\Lambda &= \bigotimes_{x\in\alpha} |\mathrm{x}\rangle_x \bigotimes_{x\in\Lambda\setminus\alpha} |\emptyset\rangle_x \,, \\
\mathcal{H}_\Lambda &= \mathrm{span}\left\{ |\alpha\rangle_\Lambda : \alpha \subseteq \Lambda \right\}.
\end{aligned}$$

Moreover, $\mathcal{U}_\Lambda = M\left(2^{|\Lambda|}, \mathbf{C}\right)$ and $\mathcal{C}_\Lambda$ is the algebra of polynomials in σ_α ($\mathbf{f}_\alpha$) for all $\alpha \subset \Lambda$. Then

$$\begin{aligned}
\sigma_\alpha &= \bigotimes_{x\in\alpha} \sigma_x \bigotimes_{x\in\Lambda\setminus\alpha} \mathbf{I}_x, \\
\mathbf{f}_\alpha &= \mathbf{U}_\Lambda \sigma_\alpha \mathbf{U}_\Lambda = \sum_{\emptyset\subseteq\gamma\subseteq\Lambda} (-1)^{|\alpha\cap\gamma|} \, |\gamma\rangle \langle\gamma|_\Lambda \,, \\
\sigma_\emptyset &= \mathbf{f}_\emptyset = \mathbf{I}_\Lambda,
\end{aligned}$$

where $|\gamma\rangle\langle\gamma|_\Lambda$ denotes the projector on the subspace spanned by $|\gamma\rangle_\Lambda$. The preceding construction can be extended to the configuration space of the whole particle system [3] if we set: $\mathcal{H} = \mathrm{span}\left\{|\alpha\rangle : \alpha \subset \mathbf{Z}^d\right\}$, $\mathcal{U} = \mathcal{B}(\mathcal{H})$ the algebra of bounded operators on $\mathcal{H}$, and $\mathcal{C}$ the algebra of observables for the system.

The following analysis involves a finite size system whose kinematical environment is $\left(\mathcal{H}_\Lambda, \mathcal{U}_\Lambda, \mathcal{C}_\Lambda, \omega_\Lambda^0\right)$, where

$$\omega_\Lambda^0(\mathbf{A}_\Lambda) := \mathrm{tr}_{\mathcal{H}_\Lambda}(\mathbf{A}_\Lambda) = \sum_{\emptyset\subseteq\gamma\subseteq\Lambda} \langle\gamma| \mathbf{A}_\Lambda |\gamma\rangle_\Lambda \;, \quad \mathbf{A}_\Lambda \in \mathcal{U}_\Lambda,$$

or equivalently

$$\begin{aligned}
|0\rangle_\Lambda &= \mathbf{U}_\Lambda |\emptyset\rangle_\Lambda = 2^{-\frac{|\Lambda|}{2}} \sum_{\emptyset\subseteq\gamma\subseteq\Lambda} |\gamma\rangle_\Lambda \,, \\
\bar{\mathbf{A}}_\Lambda &= \mathbf{U}_\Lambda \mathbf{A}_\Lambda \mathbf{U}_\Lambda, \\
\omega_\Lambda^0(\mathbf{A}_\Lambda) &= 2^{|\Lambda|} \langle 0|\mathbf{A}_\Lambda|0\rangle_\Lambda = 2^{|\Lambda|} \langle\emptyset| \bar{\mathbf{A}}_\Lambda |\emptyset\rangle_\Lambda \,.
\end{aligned}$$

2. Free diffusion

The operator ℓ_x (discrete derivative) acting on $\mathcal{H}_x$ is

$$\ell_x = \frac{\mathbf{I}_x - \mathbf{f}_x}{2},$$

$$\ell_x \begin{cases} |\emptyset\rangle_x = 0 \\ |\mathrm{x}\rangle_x = |\mathrm{x}\rangle_x . \end{cases}$$

Note that ℓ_x is selfadjoint and $\ell_x \ell_x = \ell_x$, so ℓ_x acts as the projection operator along the $|\mathrm{x}\rangle_x$ direction in $\mathcal{H}_x$. In the same fashion we define ℓ_x^Λ acting on $\mathcal{H}_\Lambda$, namely

$$\ell_x^\Lambda = \frac{\mathbf{I}_\Lambda - \mathbf{f}_x}{2}$$

such that

$$\ell_x^\Lambda |\alpha\rangle_\Lambda = \delta(x \in \alpha) |\alpha\rangle_\Lambda , \quad \alpha \subseteq \Lambda,$$

thus

$$\ell_x^\Lambda = \sum_{\emptyset \subseteq \gamma \subseteq \Lambda} \delta(x \in \gamma) |\gamma\rangle \langle\gamma|_\Lambda \ .$$

Definition 2.1. For all $\Lambda \subset \mathbf{Z}^d : |\Lambda| < \infty$ the generator of the single spin flip dynamics, i.e. the dynamics of a non interacting spin system ($\beta = 0$) is the operator

$$\mathcal{L}_\Lambda := \sum_{x \in \Lambda} \ell_x^\Lambda \ , \quad \mathcal{L}_\Lambda \in \mathcal{C}_\Lambda .$$

Obviously $\mathcal{L}_\Lambda$ is diagonal on $\mathcal{H}_\Lambda$

$$\begin{aligned} \mathcal{L}_\Lambda &= \sum_{x \in \Lambda} \sum_{\emptyset \subseteq \gamma \subseteq \Lambda} \delta(x \in \gamma) |\gamma\rangle \langle\gamma| \\ &= \sum_{\emptyset \subseteq \gamma \subseteq \Lambda} |\gamma| \, |\gamma\rangle \langle\gamma|_\Lambda . \end{aligned}$$

Since each eigenvalue has multiplicity $\binom{|\Lambda|}{|\gamma|}$, if we set P_k to be the projector along the $|\gamma\rangle$ direction in $\mathcal{H}_\Lambda$ such that $|\gamma| = k \in \mathbf{N}$, we obtain

$$\mathcal{L}_\Lambda = \sum_{n=0}^{|\Lambda|} \sum_{k=0}^{n} \binom{n}{k} k P_k .$$

Similar arguments are valid for the semigroup generated by $\mathcal{L}_\Lambda$

$$\begin{aligned} \mathbf{S}_\Lambda(t) &= \sum_{\emptyset \subseteq \gamma \subseteq \Lambda} e^{-t|\gamma|} |\gamma\rangle \langle\gamma|_\Lambda \\ &= \sum_{n=0}^{|\Lambda|} \sum_{k=0}^{n} \binom{n}{k} e^{-kt} P_k \end{aligned}$$

whose associated unique Perron – Frobenius eigenvector is $|\emptyset\rangle_\Lambda$.

Since the ℓ_x^Λ operators are defined on different lattice sites, they commute and the dynamics of the whole system is just the product dynamics, that is

$$\mathbf{S}(t) = \bigotimes_{x\in\mathbf{Z}^d} \mathbf{S}_x(t),$$

$\mathbf{S}_x(t)$ being generated by ℓ_x^Λ. Making use of the spectral representation of the semigroup one gets immediately

$$\langle\alpha|\,\mathbf{S}(t)\,|\alpha\rangle = e^{-|\alpha|t} \qquad \text{for all } \alpha \subset \mathbf{Z}^d.$$

3. Interacting diffusion

We will write the generator of the Glauber dynamics of an interacting system in our notation. This will probably seem unfamiliar since it looks quite different from the usual one (see e.g. [4]). The reader may convince himself that they are the same by computing the Dirichlet form for a cylinder function for both.

Definition 3.1. (See [1].) The interaction for a spin system is realized through the Hamiltonian operator

$$\mathbf{H}_\Lambda = \sum_{\emptyset\subseteq\alpha\subseteq\Lambda} J_\alpha \sigma_\alpha,$$

where for all $\alpha \subseteq \Lambda$, J_α is a real function on the set of subsets of Λ. Moreover,

$$\overline{\mathbf{H}}_\Lambda = \mathbf{U}_\Lambda \mathbf{H}_\Lambda \mathbf{U}_\Lambda = \sum_{\emptyset\subseteq\alpha\subseteq\Lambda} H_\alpha\,|\alpha\rangle\,\langle\alpha|_\Lambda\,,$$

$$H_\alpha = \sum_{\emptyset\subseteq\gamma\subseteq\Lambda} J_\gamma (-1)^{|\alpha\cap\gamma|}.$$

Here we assume taht J_α at least satisfy the stability condition for the existence of the equilibrium thermodynamic limit (see e.g. [11]).

Definition 3.2. For all $\Lambda \subset \mathbf{Z}^d : |\Lambda| < \infty$, the generator of the stochastic dynamics of an interacting system is

$$\mathcal{L}_\Lambda(\beta) := e^{\beta \mathrm{H}_\Lambda} \sum_{x\in\Lambda} \ell_x^\Lambda e^{-\beta \mathrm{H}_\Lambda} \ell_x^\Lambda\,, \quad \mathcal{L}_\Lambda(\beta) \in \mathcal{U}_\Lambda,$$

$$\mathcal{L}_\Lambda\,(\beta = 0) := \mathcal{L}_\Lambda.$$

The right and left eigenvectors corresponding to the 0 eigenvalue are respectively $|\emptyset\rangle_\Lambda$ and

$$\left\langle \frac{e^{-\beta \mathrm{H}_\Lambda}}{Z_\Lambda^{(d)}(\beta)} \right| = \langle\emptyset| \, \frac{e^{-\beta \mathrm{H}_\Lambda}}{Z_\Lambda^{(d)}(\beta)} = \sum_{\emptyset\subseteq\alpha\subseteq\Lambda} \Omega_\alpha^\Lambda(\beta)_\Lambda\langle\alpha|, \tag{3.1}$$

where

$$Z_\Lambda^{(d)}(\beta) = \sum_{\emptyset\subseteq\alpha\subseteq\Lambda} e^{-\beta H_\alpha}$$

is the partition function [1] and

$$\Omega_\alpha^\Lambda(\beta) = \sum_{\emptyset\subseteq\gamma\subseteq\Lambda} \left(\frac{e^{-\beta H_\gamma}(-1)^{|\alpha\cap\gamma|}}{Z_\Lambda^{(d)}(\beta)\, 2^{|\Lambda|}} \right)$$

for all $\alpha \subseteq \Lambda$.

Our purpose is to give a lower bound for the spectral gap of the generator. We will rewrite the process in a reversible form. This can be done considering the isometry

$$e^{-\frac{\beta}{2}\mathrm{H}_\Lambda} : \mathcal{H}_\Lambda \to \mathcal{H}_\Lambda^G,$$

where $\mathcal{H}_\Lambda^G$ is the Hilbert space of square summable functions on $\mathbf{Z}_2^\Lambda$ with respect to the Gibbs measure, associated with the invariant vector state (3.1) (see [3]). This map is obtained applying $\exp\{-\frac{\beta}{2}\mathbf{H}_\Lambda\}$ to any vector in $\mathcal{H}_\Lambda$. Such isometry can be lifted to an automorphism on $\mathcal{U}_\Lambda$ preserving the Gibbs state, that is for all $\mathbf{A}_\Lambda \in \mathcal{U}_\Lambda$

$$\begin{aligned} \mathbf{A}_\Lambda \to e^{-\frac{\beta}{2}\mathrm{H}_\Lambda}\mathbf{A}_\Lambda e^{\frac{\beta}{2}\mathrm{H}_\Lambda} &:= \mathbf{A}_\Lambda(\beta) \in \mathcal{U}_\Lambda, \\ \omega_\Lambda^0(\rho_\Lambda^{(d)}(\beta)\,\mathbf{A}_\Lambda) = \omega_\Lambda^\beta(\mathbf{A}_\Lambda) &= \omega_\Lambda^0(\mathbf{A}_\Lambda(\beta)) \end{aligned} \tag{3.2}$$

with

$$\bar{\rho}_\Lambda^{(d)}(\beta) = \sum_{\emptyset\subseteq\alpha\subseteq\Lambda} \frac{e^{-\beta H_\alpha}}{Z_\Lambda^{(d)}(\beta)} \, |\alpha\rangle\,\langle\alpha|_\Lambda$$

the Gibbs density operator [1].

From this follows that the action of $\mathcal{L}_\Lambda(\beta) : \mathcal{H}_\Lambda \to \mathcal{H}_\Lambda$ is equivalent to that of $\mathcal{L}_\Lambda^s(\beta) : \mathcal{H}_\Lambda^G \to \mathcal{H}_\Lambda^G$, where

$$\mathcal{L}_\Lambda^s(\beta) := e^{-\frac{\beta}{2}\mathrm{H}_\Lambda}\mathcal{L}_\Lambda(\beta)e^{\frac{\beta}{2}\mathrm{H}_\Lambda} = e^{\frac{\beta}{2}\mathrm{H}_\Lambda}\sum_{x\in\Lambda}\ell_x e^{-\beta\mathrm{H}_\Lambda}\ell_x e^{\frac{\beta}{2}\mathrm{H}_\Lambda}.$$

This operator is selfadjoint and has a unique eigenvector associated to the 0 eigenvalue that is

$$|\Omega_\Lambda^\beta\rangle \quad = \quad \frac{e^{-\frac{\beta}{2}\mathrm{H}_\Lambda}}{(Z_\Lambda^{(d)}(\beta))^{1/2}}|\emptyset\rangle_\Lambda = \sum_{\emptyset\subseteq\alpha\subseteq\Lambda}\omega_\alpha^\Lambda(\beta)|\alpha\rangle_\Lambda,$$

$$\omega_\alpha^\Lambda(\beta) \quad = \quad \sum_{\emptyset\subseteq\gamma\subseteq\Lambda} \left(\frac{e^{-\frac{\beta}{2}H_\gamma}(-1)^{|\alpha\cap\gamma|}}{(Z_\Lambda^{(d)}(\beta))^{\frac{1}{2}}2^{|\Lambda|}} \right).$$

Let $\mathbf{S}_\Lambda^\beta(t)$ be the semigroup generated by $\mathcal{L}_\Lambda^s(\beta)$. Consider its action on the element of $\mathcal{C}_\Lambda$

$$\mathbf{A}_\Lambda \to \mathbf{S}_\Lambda^\beta(t)\mathbf{A}_\Lambda\mathbf{S}_\Lambda^\beta(-t) := \mathbf{A}_\Lambda(t) \in \mathcal{U}_\Lambda \tag{3.3}$$

for all $\mathbf{A}_\Lambda \in \mathcal{C}_\Lambda$.

This automorphism obviously preserves the Gibbs state.

$$\langle\Omega_\Lambda^\beta|\mathbf{A}_\Lambda|\Omega_\Lambda^\beta\rangle = \langle\Omega_\Lambda^\beta|\mathbf{A}_\Lambda(t)|\Omega_\Lambda^\beta\rangle.$$

In the following $|\Omega_\Lambda^\beta\rangle\langle\Omega_\Lambda^\beta|$ denotes the projector on the eigenspace with eigenvalue 0. From (3.2), (3.3) the generic time correlation function now reads

$$\begin{aligned}
&\omega_\Lambda^0\left(\rho_\Lambda^{(d)}(\beta)\,\sigma_\alpha\left[e^{-t\mathcal{L}_\Lambda(\beta)} - |\Omega_\Lambda^\beta\rangle\langle\Omega_\Lambda^\beta|\right]\sigma_\gamma\right)\\
&= 2^{|\Lambda|}\,\langle\emptyset|\,\frac{e^{-\beta\mathrm{H}_\Lambda}}{Z_\Lambda^{(d)}(\beta)}\sigma_\alpha\left[e^{-t\mathcal{L}_\Lambda(\beta)} - |\Omega_\Lambda^\beta\rangle\langle\Omega_\Lambda^\beta|\right]\sigma_\gamma\,|\emptyset\rangle_\Lambda\\
&= 2^{|\Lambda|}\langle\Omega_\Lambda^\beta|\sigma_\alpha\left[\mathbf{S}_\Lambda^\beta(t) - |\Omega_\Lambda^\beta\rangle\langle\Omega_\Lambda^\beta|\right]\sigma_\gamma|\Omega_\Lambda^\beta\rangle\\
&= 2^{|\Lambda|}\langle\overline{\Omega}_\Lambda^\beta|\mathbf{f}_\alpha\left[e^{-t\overline{\mathcal{L}}_\Lambda^s(\beta)} - |\Omega_\Lambda^\beta\rangle\langle\Omega_\Lambda^\beta|\right]\mathbf{f}_\gamma|\overline{\Omega}_\Lambda^\beta\rangle\\
&= 2^{|\Lambda|}\langle 0|\frac{e^{-\frac{\beta}{2}\overline{\mathrm{H}}_\Lambda}}{\left(Z_\Lambda^{(d)}(\beta)\right)^{\frac{1}{2}}}\mathbf{f}_\alpha\left[e^{-t\overline{\mathcal{L}}_\Lambda^s(\beta)} - |\Omega_\Lambda^\beta\rangle\langle\Omega_\Lambda^\beta|\right]\mathbf{f}_\gamma\frac{e^{-\frac{\beta}{2}\overline{\mathrm{H}}_\Lambda}}{\left(Z_\Lambda^{(d)}(\beta)\right)^{\frac{1}{2}}}\,|0\rangle_\Lambda\\
&= \frac{\sum\limits_{\emptyset\subseteq\alpha_0,\gamma_0\subseteq\Lambda} e^{-\frac{\beta}{2}\left(H_{\alpha_0}+H_{\gamma_0}\right)}(-1)^{|\alpha_0\cap\alpha|+|\gamma_0\cap\gamma|}}{\sum\limits_{\emptyset\subseteq\alpha\subseteq\Lambda} e^{-\beta H_\alpha}}\\
&\quad\times\langle\alpha_0|\left[e^{-t\overline{\mathcal{L}}_\Lambda^s(\beta)} - |\Omega_\Lambda^\beta\rangle\langle\Omega_\Lambda^\beta|\right]|\gamma_0\rangle_\Lambda\,,
\end{aligned}$$

where

$$|\overline{\Omega}_\Lambda^\beta\rangle = \mathbf{U}_\Lambda|\Omega_\Lambda^\beta\rangle = 2^{-\frac{|\Lambda|}{2}}\sum_{\emptyset\subseteq\alpha\subseteq\Lambda} e^{-\beta H_\alpha}\,|\alpha\rangle_\Lambda\,.$$

Now rewrite $\overline{\mathcal{L}}_\Lambda^s(\beta)$ in the following form

$$\overline{\mathcal{L}}_\Lambda^s(\beta) = \sum_{x\in\Lambda}\bar{\ell}_x^\Lambda(-\beta)\bar{\ell}_x^\Lambda(\beta)\,,$$

where from (3.2)

$$\ell_x^\Lambda(\beta) = e^{-\frac{\beta}{2}\mathrm{H}_\Lambda}\ell_x e^{\frac{\beta}{2}\mathrm{H}_\Lambda} = \ell_x^\Lambda + \sum_{n=1}^\infty\frac{1}{n!}\left(\frac{\beta}{2}\right)^n[\ldots[\ell_x^\Lambda,\mathbf{H}_\Lambda]_1\ldots]\ldots]]_n.$$

Given any $x \in \Lambda$,

$$\begin{aligned}\bar{\ell}_x^{\Lambda}(-\beta)\bar{\ell}_x^{\Lambda}(\beta) &= \frac{1}{4}\Big\{\Big[\mathbf{I}_\Lambda - e^{-\frac{\beta}{2}\bar{\mathrm{H}}_\Lambda}\sigma_x e^{\frac{\beta}{2}\bar{\mathrm{H}}_\Lambda} - e^{\frac{\beta}{2}\bar{\mathrm{H}}_\Lambda}\sigma_x e^{-\frac{\beta}{2}\bar{\mathrm{H}}_\Lambda} \\ &\quad + e^{\frac{\beta}{2}\bar{\mathrm{H}}_\Lambda}\sigma_x e^{-\beta\bar{\mathrm{H}}_\Lambda}\sigma_x e^{\frac{\beta}{2}\bar{\mathrm{H}}_\Lambda}\Big]\Big\}.\end{aligned}$$

Since

$$\begin{aligned}\sigma_x|\alpha\rangle_\Lambda &= \delta(x\in\alpha)\,|\alpha\backslash\{x\}\rangle_\Lambda + \delta(x\notin\alpha)\,|\alpha\cup\{x\}\rangle_\Lambda \\ &= \delta(x\in\alpha)\sigma_x|\alpha\rangle_\Lambda + \delta(x\notin\alpha)\sigma_x|\alpha\rangle_\Lambda,\end{aligned}$$

we have, considering any basis vector $|\alpha\rangle_\Lambda \in \mathcal{H}_\Lambda$,

$$\begin{aligned}\bar{\ell}_x^{\Lambda}(-\beta)\bar{\ell}_x^{\Lambda}(\beta)|\alpha\rangle_\Lambda &= \frac{1}{4}\Big\{\delta(x\in\alpha)\Big[\mathbf{I}_\Lambda - \sigma_x\Big(e^{\frac{\beta}{2}(H_\alpha - H_{\alpha\backslash\{x\}})} \\ &\quad + e^{-\frac{\beta}{2}(H_\alpha - H_{\alpha\backslash\{x\}})}\Big) + \mathbf{I}_\Lambda e^{\beta(H_\alpha - H_{\alpha\backslash\{x\}})}\Big] \\ &\quad + \delta(x\notin\alpha)\Big[\mathbf{I}_\Lambda - \sigma_x\Big(e^{\frac{\beta}{2}(H_\alpha - H_{\alpha\cup\{x\}})} \\ &\quad + e^{-\frac{\beta}{2}(H_\alpha - H_{\alpha\cup\{x\}})}\Big) \\ &\quad + \mathbf{I}_\Lambda e^{\beta\left(H_\alpha - H_{\alpha\cup\{x\}}\right)}\Big]\Big\}|\alpha\rangle_\Lambda .\end{aligned}$$

Thus

$$\begin{aligned}\bar{\ell}_x^{\Lambda}(-\beta)\bar{\ell}_x^{\Lambda}(\beta)|\alpha\rangle_\Lambda &= \frac{1}{4}\Big\{\delta(x\in\alpha)\Big[\mathbf{I}_\Lambda\Big(1+e^{\beta(H_\alpha - H_{\alpha\backslash\{x\}})}\Big) \\ &\quad - 2\sigma_x\cosh\frac{\beta}{2}\Big(H_\alpha - H_{\alpha\backslash\{x\}}\Big)\Big] \\ &\quad + \delta(x\notin\alpha)\Big[\mathbf{I}_\Lambda\Big(1+e^{\beta(H_\alpha - H_{\alpha\cup\{x\}})}\Big) \\ &\quad - 2\sigma_x\cosh\frac{\beta}{2}\left(H_\alpha - H_{\alpha\cup\{x\}}\right)\Big]\Big\}|\alpha\rangle_\Lambda,\end{aligned}$$

from which follows

$$\begin{aligned}\overline{\mathcal{L}}_\Lambda^s(\beta)|\alpha\rangle_\Lambda &= \sum_{x\in\Lambda}\frac{1}{4}\Big\{\delta(x\in\alpha)\Big[\mathbf{I}_\Lambda\Big(1+\sinh\beta\left(H_\alpha - H_{\alpha\backslash\{x\}}\right) \\ &\quad + \cosh\beta\left(H_\alpha - H_{\alpha\backslash\{x\}}\right)\Big) - \sigma_x 2\cosh\frac{\beta}{2}\left(H_\alpha - H_{\alpha\backslash\{x\}}\right)\Big] \\ &\quad + \delta(x\notin\alpha)\Big[\mathbf{I}_\Lambda\Big(1+\sinh\beta\left(H_\alpha - H_{\alpha\cup\{x\}}\right) \\ &\quad + \cosh\beta\left(H_\alpha - H_{\alpha\cup\{x\}}\right)\Big) \\ &\quad - \sigma_x 2\cosh\frac{\beta}{2}\left(H_\alpha - H_{\alpha\cup\{x\}}\right)\Big]\Big\}|\alpha\rangle_\Lambda.\end{aligned}$$

Since

$$\begin{aligned} 1+\cosh\beta\Big(H_\alpha - H_{\alpha\left\{\substack{\setminus\{x\}\\ \cup\{x\}}\right.}\Big) &= 2\cosh^2\frac{\beta}{2}\Big(H_\alpha - H_{\alpha\left\{\substack{\setminus\{x\}\\ \cup\{x\}}\right.}\Big), \\ \sinh\beta\Big(H_\alpha - H_{\alpha\left\{\substack{\setminus\{x\}\\ \cup\{x\}}\right.}\Big) &= 2\cosh\frac{\beta}{2}\Big(H_\alpha - H_{\alpha\left\{\substack{\setminus\{x\}\\ \cup\{x\}}\right.}\Big) \\ &\quad \times\sinh\frac{\beta}{2}\Big(H_\alpha - H_{\alpha\left\{\substack{\setminus\{x\}\\ \cup\{x\}}\right.}\Big), \end{aligned}$$

we have

$$\begin{aligned} \overline{\mathcal{L}}^s_\Lambda(\beta)|\alpha\rangle_\Lambda &= \sum_{x\in\Lambda}\frac{1}{2}\Big\{\delta(x\in\alpha)\Big[\mathbf{I}_\Lambda\Big(\cosh^2\frac{\beta}{2}(H_\alpha - H_{\alpha\setminus\{x\}}) \\ &\quad + \cosh\frac{\beta}{2}(H_\alpha - H_{\alpha\setminus\{x\}})\sinh\frac{\beta}{2}(H_\alpha - H_{\alpha\setminus\{x\}})\Big) \\ &\quad - \sigma_x\cosh\frac{\beta}{2}(H_\alpha - H_{\alpha\setminus\{x\}})\Big] \\ &\quad + \delta(x\notin\alpha)\Big[\mathbf{I}_\Lambda\Big(\cosh^2\frac{\beta}{2}(H_\alpha - H_{\alpha\cup\{x\}}) \\ &\quad + \cosh\frac{\beta}{2}(H_\alpha - H_{\alpha\cup\{x\}})\sinh\frac{\beta}{2}(H_\alpha - H_{\alpha\cup\{x\}})\Big) \\ &\quad -\sigma_x\cosh\frac{\beta}{2}\left(H_\alpha - H_{\alpha\cup\{x\}}\right)\Big]\Big\}\,|\alpha\rangle_\Lambda\,. \end{aligned}$$

Remembering the definition of $\bar\ell^\Lambda_x$ the last expression reduces to

$$\begin{aligned} \overline{\mathcal{L}}^s_\Lambda(\beta)|\alpha\rangle_\Lambda &= \sum_{x\in\Lambda}\frac{1}{2}\Big\{\delta(x\in\alpha)\cosh\frac{\beta}{2}\left(H_\alpha - H_{\alpha\setminus\{x\}}\right) \\ &\quad \times\left[\mathbf{I}_\Lambda\left(e^{\frac{\beta}{2}\left(H_\alpha - H_{\alpha\setminus\{x\}}\right)} - 1\right) + 2\bar\ell^\Lambda_x\right] \\ &\quad + \delta(x\notin\alpha)\cosh\frac{\beta}{2}\left(H_\alpha - H_{\alpha\cup\{x\}}\right) \\ &\quad \times\left[\mathbf{I}_\Lambda\left(e^{\frac{\beta}{2}\left(H_\alpha - H_{\alpha\cup\{x\}}\right)} - 1\right) + 2\bar\ell^\Lambda_x\right]\Big\}\,|\alpha\rangle_\Lambda \\ &= \sum_{x\in\Lambda}\Big\{\delta(x\in\alpha)\cosh\frac{\beta}{2}\left(H_\alpha - H_{\alpha\setminus\{x\}}\right) \\ &\quad \times\Big[\frac{\tanh\frac{\beta}{4}\left(H_\alpha - H_{\alpha\setminus\{x\}}\right)}{1-\tanh\frac{\beta}{4}\left(H_\alpha - H_{\alpha\setminus\{x\}}\right)}\mathbf{I}_\Lambda + \bar\ell^\Lambda_x\Big] \\ &\quad + \delta(x\notin\alpha)\cosh\frac{\beta}{2}\left(H_\alpha - H_{\alpha\cup\{x\}}\right) \\ &\quad \times\Big[\frac{\tanh\frac{\beta}{4}\left(H_\alpha - H_{\alpha\cup\{x\}}\right)}{1-\tanh\frac{\beta}{4}\left(H_\alpha - H_{\alpha\cup\{x\}}\right)}\mathbf{I}_\Lambda + \bar\ell^\Lambda_x\Big]\Big\}\,|\alpha\rangle_\Lambda\,, \end{aligned}$$

where we have used the identity

$$\exp\frac{\beta}{2}\Big(H_\alpha - H_{\alpha\left\{\substack{\setminus\{x\}\\ \cup\{x\}}\right.}\Big) - 1 = \frac{2\tanh\frac{\beta}{4}\Big(H_\alpha - H_{\alpha\left\{\substack{\setminus\{x\}\\ \cup\{x\}}\right.}\Big)}{1-\tanh\frac{\beta}{4}\Big(H_\alpha - H_{\alpha\left\{\substack{\setminus\{x\}\\ \cup\{x\}}\right.}\Big)}.$$

Let us consider now the generic matrix element of $\mathcal{L}_\Lambda^s(\beta)$ acting on $\mathcal{H}_\Lambda$. By definition for all $\gamma,\alpha\subseteq\Lambda$

$$\begin{aligned}
\langle\gamma|\,\mathcal{L}_\Lambda^s(\beta)\,|\alpha\rangle_\Lambda &= \sum_{\emptyset\subseteq\gamma_1,\eta\subseteq\Lambda}\langle\gamma|\,\mathbf{U}_\Lambda\,|\gamma_1\rangle\,\langle\gamma_1|_\Lambda\,\overline{\mathcal{L}}_\Lambda^s(\beta)\,|\eta\rangle\,\langle\eta|_\Lambda\,\mathbf{U}_\Lambda\,|\alpha\rangle_\Lambda \\
&= \langle\gamma|\sum_{\emptyset\subseteq\eta\subseteq\Lambda}\sum_{x\in\Lambda}\Bigg\{\delta(x\in\eta)\cosh\frac{\beta}{2}\left(H_\eta - H_{\eta\setminus\{x\}}\right) \\
&\quad\times\Big[\frac{\tanh\frac{\beta}{4}\left(H_\eta - H_{\eta\setminus\{x\}}\right)}{1-\tanh\frac{\beta}{4}\left(H_\eta - H_{\eta\setminus\{x\}}\right)}\mathbf{I}_\Lambda + \ell_x^\Lambda\Big] \\
&\quad+\delta\left(x\notin\eta\right)\cosh\frac{\beta}{2}\left(H_\eta - H_{\eta\cup\{x\}}\right) \\
&\quad\times\Big[\frac{\tanh\frac{\beta}{4}\left(H_\eta - H_{\eta\cup\{x\}}\right)}{1-\tanh\frac{\beta}{4}\left(H_\eta - H_{\eta\cup\{x\}}\right)}\mathbf{I}_\Lambda + \ell_x^\Lambda\Big]\Bigg\} \\
&\quad\times\mathbf{U}_\Lambda\,|\eta\rangle\,\langle\eta|_\Lambda\,\mathbf{U}_\Lambda\,|\alpha\rangle_\Lambda\,.
\end{aligned}$$

This implies, since $\mathbf{I}_\Lambda \geq \ell_x^\Lambda$,

$$\begin{aligned}
\langle\gamma|\,\mathcal{L}_\Lambda^s(\beta)\,|\alpha\rangle_\Lambda &\geq \langle\gamma|\sum_{\emptyset\subseteq\eta\subseteq\Lambda}\sum_{x\in\Lambda}\Bigg\{\delta\left(x\in\eta\right)\cosh\frac{\beta}{2}\left(H_\eta - H_{\eta\setminus\{x\}}\right) \\
&\quad\times\Big[\frac{\tanh\frac{\beta}{4}\left(H_\eta - H_{\eta\setminus\{x\}}\right)}{1-\tanh\frac{\beta}{4}\left(H_\eta - H_{\eta\setminus\{x\}}\right)} + 1\Big]\ell_x^\Lambda \\
&\quad+\delta\left(x\notin\eta\right)\cosh\frac{\beta}{2}\left(H_\eta - H_{\eta\cup\{x\}}\right) \\
&\quad\times\Big[\frac{\tanh\frac{\beta}{4}\left(H_\eta - H_{\eta\cup\{x\}}\right)}{1-\tanh\frac{\beta}{4}\left(H_\eta - H_{\eta\cup\{x\}}\right)} + 1\Big]\ell_x^\Lambda\Bigg\} \\
&\quad\times\mathbf{U}_\Lambda\,|\eta\rangle\,\langle\eta|_\Lambda\,\mathbf{U}_\Lambda\,|\alpha\rangle_\Lambda \\
&\geq \langle\gamma|\sum_{\emptyset\subseteq\eta\subseteq\Lambda}\sum_{x\in\Lambda}\Bigg\{\delta\left(x\in\eta\right)\cosh\frac{\beta}{2}\left(H_\eta - H_{\eta\setminus\{x\}}\right) \\
&\quad\times\Big[1-\frac{\tanh\frac{\beta}{4}\left|H_\eta - H_{\eta\setminus\{x\}}\right|}{1+\tanh\frac{\beta}{4}\left|H_\eta - H_{\eta\setminus\{x\}}\right|}\Big]\ell_x^\Lambda
\end{aligned}$$

$$
\begin{aligned}
& + \delta\left(x \notin \eta\right) \cosh \frac{\beta}{2}\left(H_{\eta} - H_{\eta \cup \{x\}}\right) \\
& \left. \times \left[1 - \frac{\tanh \frac{\beta}{4}\left|H_{\eta} - H_{\eta \cup \{x\}}\right|}{1 + \tanh \frac{\beta}{4}\left|H_{\eta} - H_{\eta \cup \{x\}}\right|} \right] \ell_x^{\Lambda} \right\} \\
& \times \mathbf{U}_{\Lambda} \left|\eta\right\rangle \left\langle \eta\right|_{\Lambda} \mathbf{U}_{\Lambda} \left|\alpha\right\rangle_{\Lambda} \\
\geq \quad & \left\langle \gamma\right| \sum_{x \in \Lambda} \ell_x^{\Lambda} \left\{ 1 - \max_{\eta \subseteq \Lambda} \max \left[\max_{x \in \eta} \cosh \frac{\beta}{2}\left(H_{\eta} - H_{\eta \setminus \{x\}}\right) \right. \right. \\
& \times \frac{\tanh \frac{\beta}{4}\left|H_{\eta} - H_{\eta \setminus \{x\}}\right|}{1 + \tanh \frac{\beta}{4}\left|H_{\eta} - H_{\eta \setminus \{x\}}\right|}; \\
& \max_{x \notin \eta} \cosh \frac{\beta}{2}\left(H_{\eta} - H_{\eta \cup \{x\}}\right) \\
& \left. \left. \times \frac{\tanh \frac{\beta}{4}\left|H_{\eta} - H_{\eta \cup \{x\}}\right|}{1 + \tanh \frac{\beta}{4}\left|H_{\eta} - H_{\eta \cup \{x\}}\right|} \right] \right\} \left|\alpha\right\rangle_{\Lambda}.
\end{aligned}
$$

Hence we have the following

Theorem 3.1. *For all* $\Lambda \subset \mathbf{Z}^d$, *for all* $\beta \in [0, \bar{\beta}\Lambda^{(d)}]$, $\inf[\mathrm{spec}(\overline{\mathcal{L}}^s_{\Lambda}(\beta)) \setminus \{0\}]$ *is bounded from below by*

$$
\begin{aligned}
\overline{g}_{\Lambda}^{(d)}(\beta) \quad := \quad & 1 - \max_{\alpha \subseteq \Lambda} \max \left[\max_{x \in \alpha} \cosh \frac{\beta}{2}\left(H_{\alpha} - H_{\alpha \setminus \{x\}}\right) \right. \\
& \times \frac{\tanh \frac{\beta}{4}\left|H_{\alpha} - H_{\alpha \setminus \{x\}}\right|}{1 + \tanh \frac{\beta}{4}\left|H_{\alpha} - H_{\alpha \setminus \{x\}}\right|}; \\
& \left. \max_{x \notin \alpha} \cosh \frac{\beta}{2}\left(H_{\alpha} - H_{\alpha \cup \{x\}}\right) \frac{\tanh \frac{\beta}{4}\left|H_{\alpha} - H_{\alpha \cup \{x\}}\right|}{1 + \tanh \frac{\beta}{4}\left|H_{\alpha} - H_{\alpha \cup \{x\}}\right|} \right].
\end{aligned}
$$

Proof. From the last inequality it follows that

$$
\left\langle \alpha\right| \mathbf{I}_{\Lambda} + \varepsilon \mathcal{L}^s_{\Lambda}(\beta) \left|\gamma\right\rangle_{\Lambda} \geq \left\langle \alpha \mid \mathbf{I}_{\Lambda} + \varepsilon \mathbf{D}_{\Lambda}\left(\beta\right) \mid \gamma\right\rangle_{\Lambda}, \tag{3.4}
$$

where by definition

$$
\begin{aligned}
\mathbf{D}_{\Lambda}(\beta) \quad := \quad & \sum_{\emptyset \subseteq \alpha \subseteq \Lambda} \left|\alpha\right| \left\{ 1 - \max_{\alpha \subseteq \Lambda} \max \left[\max_{x \in \alpha} \cosh \frac{\beta}{2}\left(H_{\alpha} - H_{\alpha \setminus \{x\}}\right) \right. \right. \\
& \times \frac{\tanh \frac{\beta}{4}\left|H_{\alpha} - H_{\alpha \setminus \{x\}}\right|}{1 + \tanh \frac{\beta}{4}\left|H_{\alpha} - H_{\alpha \setminus \{x\}}\right|}; \\
& \max_{x \notin \alpha} \cosh \frac{\beta}{2}\left(H_{\alpha} - H_{\alpha \cup \{x\}}\right) \\
& \left. \left. \times \frac{\tanh \frac{\beta}{4}\left|H_{\alpha} - H_{\alpha \cup \{x\}}\right|}{1 + \tanh \frac{\beta}{4}\left|H_{\alpha} - H_{\alpha \cup \{x\}}\right|} \right] \right\} \left|\alpha\right\rangle \left\langle \alpha\right|_{\Lambda}.
\end{aligned}
$$

Let us set $\bar{g}_{\Lambda}^{(d)}(\beta)$ to be inf [spec $(\mathbf{D}_{\Lambda}(\beta))\setminus\{0\}$],

$$\begin{aligned}\bar{g}_{\Lambda}^{(d)}(\beta) &= 1-\max_{\alpha\subseteq\Lambda}\max\Big[\max_{x\in\alpha}\cosh\frac{\beta}{2}\left(H_{\alpha}-H_{\alpha\setminus\{x\}}\right)\\ &\quad\times\frac{\tanh\frac{\beta}{4}\left|H_{\alpha}-H_{\alpha\setminus\{x\}}\right|}{1+\tanh\frac{\beta}{4}\left|H_{\alpha}-H_{\alpha\setminus\{x\}}\right|};\\ &\quad\max_{x\notin\alpha}\cosh\frac{\beta}{2}\left(H_{\alpha}-H_{\alpha\cup\{x\}}\right)\frac{\tanh\frac{\beta}{4}\left|H_{\alpha}-H_{\alpha\cup\{x\}}\right|}{1+\tanh\frac{\beta}{4}\left|H_{\alpha}-H_{\alpha\cup\{x\}}\right|}\Big].\end{aligned}$$

Of course, $\mathbf{D}_{\Lambda}(\beta)$ is selfadjoint and will stay positive for β ranging from 0 to a certain value $\bar{\beta}_{\Lambda}^{(d)}$ such that $\bar{g}_{\Lambda}^{(d)}(\bar{\beta}_{\Lambda}^{(d)})=0$. Inequality (3.4) implies

$$\langle\alpha|\left(\mathbf{I}_{\Lambda}+\varepsilon\mathcal{L}_{\Lambda}^{s}(\beta)\right)^{-1}|\gamma\rangle_{\Lambda}\leq\langle\alpha|\left(\mathbf{I}_{\Lambda}+\varepsilon\mathbf{D}_{\Lambda}(\beta)\right)^{-1}|\gamma\rangle_{\Lambda}.$$

Iterating n times the last inequality, we get

$$\langle\alpha|\left(\mathbf{I}_{\Lambda}+\varepsilon\mathcal{L}_{\Lambda}^{s}(\beta)\right)^{-n}|\gamma\rangle_{\Lambda}\leq\langle\alpha|\left(\mathbf{I}_{\Lambda}+\varepsilon\mathbf{D}_{\Lambda}(\beta)\right)^{-n}|\gamma\rangle_{\Lambda}.$$

Setting $\varepsilon=t/n$, the last expression will be valid at least for all $t\leq n^{\alpha}$, $\alpha<1$. By the Hille−Yosida theorem, we obtain

$$\langle\alpha|\left(e^{-t\mathcal{L}_{\Lambda}^{s}(\beta)}-|\Omega_{\Lambda}^{\beta}\rangle\langle\Omega_{\Lambda}^{\beta}|\right)|\gamma\rangle_{\Lambda}\leq\langle\alpha|\left(e^{-t\mathrm{D}_{\Lambda}(\beta)}-|\Omega_{\Lambda}^{\beta}\rangle\langle\Omega_{\Lambda}^{\beta}|\right)|\gamma\rangle_{\Lambda}\leq e^{-t\bar{g}_{\Lambda}^{(d)}(\beta)}$$

which proves the theorem. □

Since $\bar{g}_{\Lambda}^{(d)}(\beta)$ involves the difference of two cylindric functions whose supports differ by just one lattice site, it is clear that if for all $\alpha\subseteq\Lambda$, $H_{\alpha}\propto|\Lambda|$ and also $\max_{\alpha\subseteq\Lambda}\max_{x\in\alpha}\left(H_{\alpha}-H_{\alpha\setminus\{x\}}\right)$ $\left(\max_{\alpha\subseteq\Lambda}\max_{x\notin\alpha}\left(H_{\alpha}-H_{\alpha\cup\{x\}}\right)\right)$ do not depend on α, we would obtain a volume independent expression for the spectral gap of $\mathbf{D}_{\Lambda}(\beta)$; that is $\bar{g}_{\Lambda}^{(d)}(\beta)\to\bar{g}^{(d)}(\beta)$. A sufficient condition for this argument to be valid is to assume $\mathbf{H}_{\Lambda}(\beta)$ to be traslation invariant. This proves the following

Proposition 3.1. *The spectral gap for the Glauber dynamics of an interacting spin system of the type described above is bounded from below by $\bar{g}^{(d)}(\beta)$ for all $\beta\in\left[0,\bar{\beta}^{(d)}\right]$, where $\bar{\beta}^{(d)}:=\inf\left\{\beta\geq0:\bar{g}^{(d)}\left(\bar{\beta}^{(d)}\right)=0\right\}$.*

Now we will apply these results to some selected models. For the description of the models in our framework see [1] and reference therein. We will always assume periodic boundary conditions, but it is clear that in the thermodynamic limit, for small β, the estimate for the gap will not be sensitive to the boundary conditions.

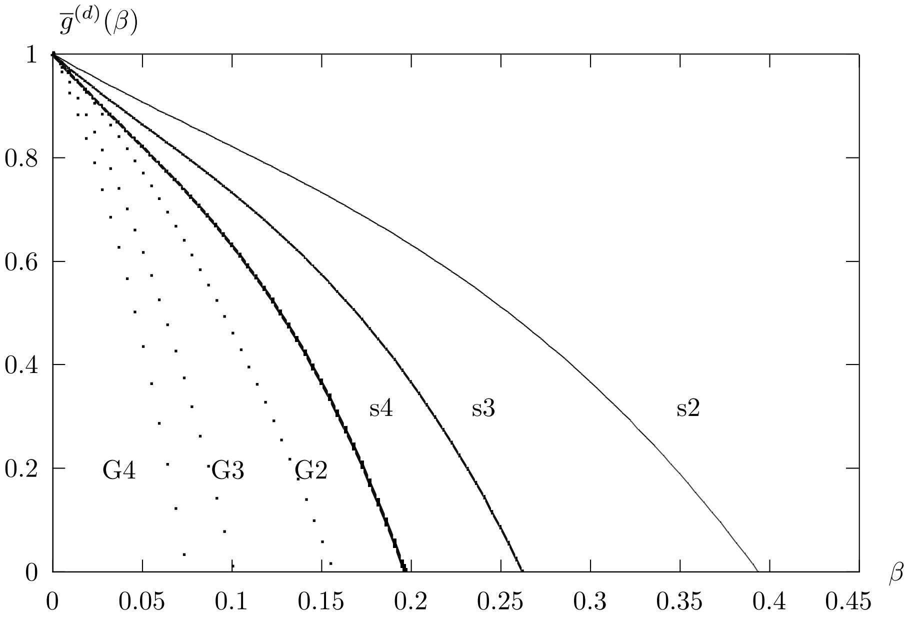


Figure 1.

$$\mathrm{G}d: \quad \overline{g}_G^{(d)}(\beta) = 1 - de^{2d\beta}(1 - e^{-2\beta});$$

$$\mathrm{s}d: \quad \overline{g}_I^{(d)}(\beta) = 1 - \frac{\tanh d\beta}{1 + \tanh d\beta} \cosh 2d\beta;$$

$$d = 2, 3, 4.$$

Example 3.1. Nearest neighbour Ising model without external source. The eigenvalues of the Hamiltonian operator

$$\mathbf{H}_\Lambda^I := - \sum_{\emptyset \subseteq \alpha \subseteq \Lambda} \delta\left(d\left(\alpha\right) = 1\right) \sigma_\alpha$$

are

$$H_\alpha^I = -d\left|\Lambda\right| + 2\left|\partial\alpha\right| \qquad \text{for all } \alpha \subseteq \Lambda,$$

where $|\partial\alpha|$ is the area of the surface bounding α (the length of the contour surrounding α if $d = 2$). Then

$$\begin{aligned} H_\alpha^I - H_{\alpha \setminus \{x\}}^I &= 2\left(\left|\partial\alpha\right| - \left|\partial\left(\alpha \setminus \{x\}\right)\right|\right), \\ H_\alpha^I - H_{\alpha \cup \{x\}}^I &= 2\left(\left|\partial\alpha\right| - \left|\partial\left(\alpha \cup \{x\}\right)\right|\right), \end{aligned}$$

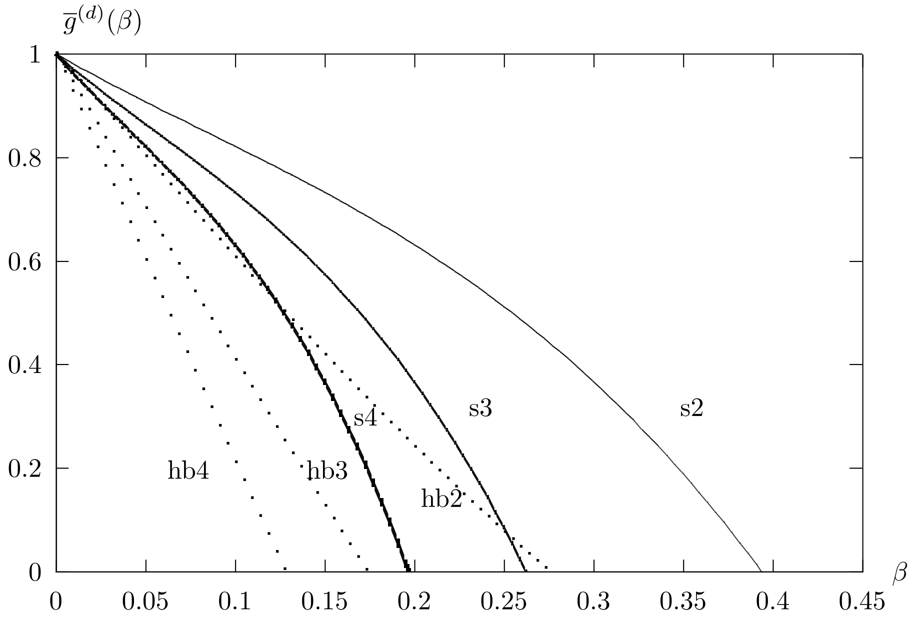


Figure 2.

$$\text{hb}d: \quad \overline{g}_{hb}^{(d)}(\beta) = 1 - d\tanh 2\beta;$$

$$\text{s}d: \quad \overline{g}_{I}^{(d)}(\beta) = 1 - \frac{\tanh d\beta}{1+\tanh d\beta}\cosh 2d\beta;$$

$$d = 2, 3, 4.$$

with the right-hand sides of the above expressions ranging from $-4d$ to $4d$. Thus

$$\overline{g}_{I}^{(d)}(\beta) = 1 - \cosh 2\beta d \frac{\tanh \beta d}{1+\tanh \beta d}.$$

Figures 1 and 2 show a sketch of $\overline{g}_{I}^{(d)}(\beta)$ as a function of β ($d = 2, 3, 4$), in comparison to equivalent estimates for Glauber ($\overline{g}_{G}^{(d)}(\beta)$) and heat bath ($\overline{g}_{hb}^{(d)}(\beta)$) dynamics [4]. In dimension 2 the critical value of β is $\frac{\ln\left(1+\sqrt{2}\right)}{2} \simeq 0.440$. Our estimate is $\overline{\beta}_{I}^{(2)} \simeq 0.394$.

Example 3.2. Ising model with second neighbours antiferromagnetic interaction in the presence of an external source.

$$\mathbf{H}_\Lambda(h,J) := -\sum_{\emptyset\subseteq\alpha\subseteq\Lambda}\left[\delta\left(d\left(\alpha\right)=1\right)+J_\alpha+h\delta\left(d\left(\alpha\right)=0\right)\right]\sigma_\alpha,$$

$$J_\alpha := -J\delta\left(\alpha=\alpha_1\cup\alpha_2 : d\left(\alpha_1\right)=d\left(\alpha_2\right)=0\ ;\ dist\left(\alpha_1,\alpha_2\right)=2\right),$$

$$H_\gamma(h,J) = \left(-d\left|\Lambda\right|+2\left|\partial\gamma\right|\right)(1-J)-h\left|\Lambda\right|+2h\left|\gamma\right|-2JY_\gamma,$$

$$Y_\gamma := \sum_{\emptyset\subseteq\alpha\subseteq\Lambda}\delta\left(\alpha=\alpha_1\cup\alpha_2 : d\left(\alpha_1\right)=d\left(\alpha_2\right)=0\ ;\ dist\left(\alpha_1,\alpha_2\right)=2\right)$$
$$\times\left[\delta(\alpha_1\subset\subset\gamma)\delta(\alpha_2\cap\gamma=\emptyset)+\delta(\alpha_2\subset\subset\gamma)\delta(\alpha_1\cap\gamma=\emptyset)\right]$$

with $h\in\mathbf{R}, J\geq 0$ and $\delta\left(\alpha_i\subset\subset\gamma\right)=\delta\left(\alpha_i\subset\gamma\right)\delta\left(\alpha_i\cap\partial\gamma=\emptyset\right)$, $i=1,2$. Then

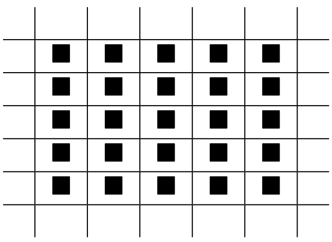
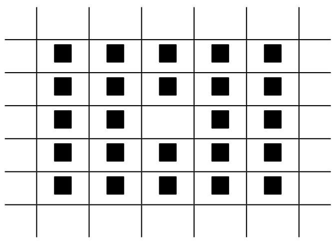
$$\begin{aligned} &H_\alpha(h,J)-H_{\alpha\backslash\{x\}}\left(h,J\right)\\ &\quad= 2\left(\left|\partial\alpha\right|-\left|\partial\left(\alpha\backslash\{x\}\right)\right|\right)(1-J)+2J\left(Y_{\alpha\backslash\{x\}}-Y_\alpha\right)+2h, \end{aligned} \tag{3.5}$$

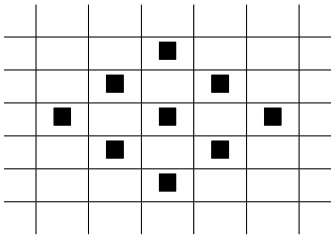
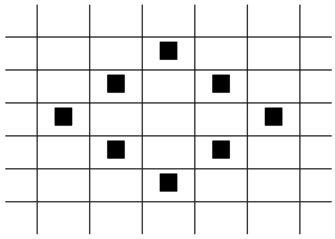
$$\begin{aligned} &H_\alpha(h,J)-H_{\alpha\cup\{x\}}\left(h,J\right)\\ &\quad= 2\left(\left|\partial\alpha\right|-\left|\partial\left(\alpha\cup\{x\}\right)\right|\right)(1-J)+2J\left(Y_{\alpha\cup\{x\}}-Y_\alpha\right)-2h, \end{aligned} \tag{3.6}$$

where it easily follows from the definition of Y_γ that

$$\left|Y_{\left\{\substack{\alpha\backslash\{x\}\\ \alpha\cup\{x\}}\right.}-Y_\alpha\right|\leq 2d.$$

The maximum value of the right-hand side of the above expressions is realized, for example, considering the sets sketched below ($d=2$). If $J>1$, we set $\alpha=\gamma_1$, $\alpha\backslash\{x\}=\gamma_2$ and $\alpha=\gamma_2$, $\alpha\cup\{x\}=\gamma_1$, respectively in (3.5) and in (3.6). If $J<1$, we set $\alpha=\gamma_3$, $\alpha\backslash\{x\}=\gamma_4$ in (3.5) and $\alpha=\gamma_4$, $\alpha\cup\{x\}=\gamma_3$ in (3.6)

$\gamma_1 =$  $\gamma_2 =$ 

$\gamma_3 =$  $\gamma_4 =$ 

So we get

$$\overline{g}^{(d)}_{h,J}(\beta) = 1 - \cosh\beta[2d(|J-1|+J)+h]\frac{\tanh\beta\left|d(|J-1|+J)+\frac{h}{2}\right|}{1+\tanh\beta\left|d(|J-1|+J)+\frac{h}{2}\right|}.$$

Finally, if $J=1$, we have

$$\overline{g}^{(d)}_{h,J}(\beta) = 1 - \cosh\beta[2d+h]\frac{\tanh\beta\left|d+\frac{h}{2}\right|}{1+\tanh\beta\left|d+\frac{h}{2}\right|}.$$

Example 3.3. Dobrushin − Gertsik model. The lattice dimension is $d=2$ and the Hamiltonian operator involves a nearest-neighbour interaction term with coupling J_1, a next-nearest neighbour interaction term with coupling J_2 and an external field h.

$$\begin{aligned}
&\mathbf{H}^{DG}_{\Lambda}(h,J_1,J_2) := -\sum_{\emptyset\subseteq\alpha\subseteq\Lambda}\left[J_1\delta\left(d\left(\alpha\right)=1\right)+J_\alpha+h\delta\left(d\left(\alpha\right)=0\right)\right]\sigma_\alpha,\\
&J_\alpha := J_2\delta\left(\alpha=\alpha_1\cup\alpha_2 : d\left(\alpha_1\right)=d\left(\alpha_2\right)=0\ ;\ dist\left(\alpha_1,\alpha_2\right)=\sqrt{2}\right),\\
&H^{DG}_{\gamma}(h,J_1,J_2) = \left(-2\left|\Lambda\right|+2\left|\partial\gamma\right|\right)J_1+\left(-2\left|\Lambda\right|+2\left|\not{\partial}\gamma\right|\right)J_2\\
&\qquad\qquad\qquad +\left(-\left|\Lambda\right|+2\left|\gamma\right|\right)h,\\
&\left|\not{\partial}\gamma\right| := \sum_{\emptyset\subseteq\alpha\subseteq\Lambda}\delta\left(\alpha=\alpha_1\cup\alpha_2 : d\left(\alpha_1\right)=d\left(\alpha_2\right)=0\ ;\ dist\left(\alpha_1,\alpha_2\right)=\sqrt{2}\right)\\
&\qquad\qquad \times\delta\left(\left|\alpha\cap\gamma\right|=1\right),
\end{aligned}$$

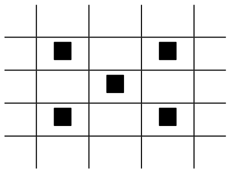
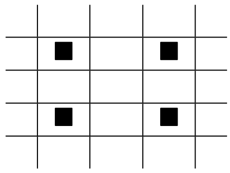
$h, J_i \in \mathbf{R}$, $i=1,2$ and

$$\begin{aligned}
&H^{DG}_{\alpha}(h,J_1,J_2) - H^{DG}_{\alpha\backslash\{x\}}(h,J_1,J_2)\\
&\quad = 2J_1(|\partial\alpha| - |\partial(\alpha\backslash\{x\})|) + 2J_2(|\not{\partial}\alpha| - |\not{\partial}(\alpha\backslash\{x\})|) + 2h, \qquad (3.7)\\
&H^{DG}_{\alpha}(h,J_1,J_2) - H^{DG}_{\alpha\cup\{x\}}(h,J_1,J_2)\\
&\quad = 2J_1\left(|\partial\alpha| - \left|\partial\left(\alpha\cup\{x\}\right)\right|\right) + 2J_2\left(|\not{\partial}\alpha| - \left|\not{\partial}\left(\alpha\cup\{x\}\right)\right|\right) - 2h, \quad (3.8)
\end{aligned}$$

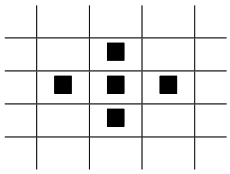
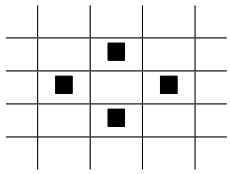
with

$$-4 \le \left(|\not{\partial}\alpha| - \left|\not{\partial}\left(\alpha\backslash\{x\}\right)\right|\right), \left(|\not{\partial}\alpha| - \left|\not{\partial}\left(\alpha\cup\{x\}\right)\right|\right) \le 4.$$

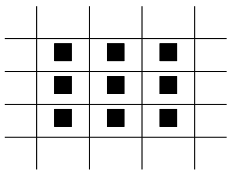
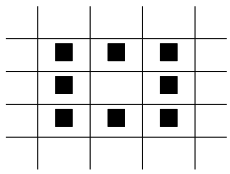
The maximum value of the previous expressions is obtained considering, for example, the sets of the following type. If $J_1 > 0$, $J_2 < 0$, we set $\alpha = \gamma_1$, $\alpha \backslash \{x\} = \gamma_2$ in (3.7) and $\alpha = \gamma_2$, $\alpha \cup \{x\} = \gamma_1$ in (3.8)

$\gamma_1 =$  $\gamma_2 =$ 

If $J_1 < 0$, $J_2 > 0$, we set $\alpha = \gamma_3$, $\alpha \backslash \{x\} = \gamma_4$ in (3.7) and $\alpha = \gamma_4$, $\alpha \cup \{x\} = \gamma_3$ in (3.8)

$\gamma_3 =$  $\gamma_4 =$ 

If $J_1, J_2 < 0$, we set $\alpha = \gamma_5$, $\alpha \backslash \{x\} = \gamma_6$ in (3.7) and $\alpha = \gamma_6$, $\alpha \cup \{x\} = \gamma_5$ in (3.8)

$\gamma_5 =$  $\gamma_6 =$ 

while if $J_1, J_2 > 0$, we take α and $\alpha \cup \{x\}$ as a singleton in both (3.7) and (3.8). Finally, we obtain

$$\overline{g}^{(d)}_{DG}(\beta, h, J_1, J_2) = 1 - \cosh \beta(4(|J_1| + |J_2|) + h) \frac{\tanh \beta \left| 2\left(|J_1| + |J_2|\right) + \frac{h}{2} \right|}{1 + \tanh \beta \left| 2\left(|J_1| + |J_2|\right) + \frac{h}{2} \right|}.$$

4. Conclusions

Even if we stated our results in terms of bounds for the gap, what we really get is bounds for the matrix elements of the generator. Applying the general result to specific models may require some combinatorial manipulation as in Examples 3.1–3.3. Models which are not traslation invariant (in particular we plan to investigate models with random interacions) require some additional work.

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