Erratum and Addenda to "Quantum Methods for Interacting Particle Systems II, Glauber Dynamics for Ising Spin Systems"

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Abstract

Here we correct an error we discovered in [1] and clarify a statement on the form of the generator.

We discovered that an estimate in Theorem 3.1 of our paper [1], to which we refer for notation and background, is incorrect.

In estimating the quadratic form $\langle\gamma|\,\mathcal{L}_{\Lambda}^{s}(\beta)\,|\alpha\rangle_{\Lambda}$ on page 419 a term was treated as positive, which it isn't. To get the correct estimate one should first produce a lower bound for $\langle\gamma|\,\mathcal{L}_{x}^{s}(\beta)\,|\alpha\rangle_{\Lambda}$. $\mathcal{L}_{x}^{s}(\beta)=\ell_{x}^{\Lambda}(-\beta)\,\ell_{x}^{\Lambda}(\beta)$ is a positive selfadjoint operator on $\mathcal{H}_{\Lambda}$ which annihilates $\left|\overline{\Omega}_{\Lambda}^{\beta}\right\rangle$ corresponding to the square root of the Gibbs density (see page 417 of [1]). Then by a standard minimax argument we get the following bound for the spectral gap

$$\overline{g}_{\Lambda}^{(d)}(\beta):=1-\max_{\alpha\subseteq\Lambda}\max\left\{\max_{x\in\alpha}\left[\left(1+\cosh\frac{\beta}{2}\left(H_{\alpha}-H_{\alpha\setminus\{x\}}\right)\right)\frac{\tanh\frac{\beta}{4}\left|H_{\alpha}-H_{\alpha\setminus\{x\}}\right|}{1+\tanh\frac{\beta}{4}\left|H_{\alpha}-H_{\alpha\setminus\{x\}}\right|}\right];\right.$$
$$\left.\max_{x\notin\alpha}\left[\left(1+\cosh\frac{\beta}{2}\left(H_{\alpha}-H_{\alpha\cup\{x\}}\right)\right)\frac{\tanh\frac{\beta}{4}\left|H_{\alpha}-H_{\alpha\cup\{x\}}\right|}{1+\tanh\frac{\beta}{4}\left|H_{\alpha}-H_{\alpha\cup\{x\}}\right|}\right]\right\}$$

The expressions that appear in the examples also need to be changed in the following way:

Nearest neighbor Ising model without external source

$$\overline{g}_{I}^{(d)}(\beta)=\frac{1-\cosh 2\beta d\tanh\beta d}{1+\tanh\beta d}$$

This gives an estimate of the critical value of β in dimension 2 is $\bar{\beta}_{I}^{(2)}\simeq 0.309$.

Ising model with second neighbors antiferromagnetic interaction in the presence of an external source

$$\overline{g}_{h,J}^{(d)}(\beta)=\frac{1-\cosh\beta\left[2d+h\right]\tanh\beta\left|d+\frac{h}{2}\right|}{1+\tan\beta\left|d+\frac{h}{2}\right|}$$

Dobrushin-Gertsik model

$$\overline{g}_{DG}^{(d)}\left(\beta,h,J_1,J_2\right)=\frac{1-\cosh\beta\left[4\left(|J_1|+|J_2|\right)+h\right]\tanh\beta\left|2\left(|J_1|+|J_2|\right)+\frac{h}{2}\right|}{1+\tanh\beta\left|2\left(|J_1|+|J_2|\right)+\frac{h}{2}\right|}$$

We also take the chance to clarify what are the rates for the dynamics we considered.

Computing the Dirichlet form for the generator $\overline{\mathcal{L}}_\Lambda$ on $\mathcal{H}_\Lambda^G$ we get

$$\begin{aligned}\left\langle\bar{\rho}_\Lambda^\beta\middle|\Phi\overline{\mathcal{L}}_\Lambda\left(\beta\right)\Phi\middle|0\right\rangle_\Lambda &= \left\langle 0\middle|\bar{\rho}_\Lambda^\beta\Phi\overline{\mathcal{L}}_\Lambda\left(\beta\right)\Phi\middle|0\right\rangle_\Lambda \\ &= \frac{1}{2}\left(Z_\Lambda^{(d)}\left(\beta\right)\right)^{-1}\sum_{\alpha\subseteq\Lambda}\sum_{x\in\Lambda}\left\{\delta\left(x\in\alpha\right)\left(\frac{e^{-\beta H_\alpha}+e^{-\beta H_{\alpha\setminus\{x\}}}}{4}\right)\left(\phi_\alpha-\phi_{\alpha\setminus\{x\}}\right)^2+\right. \\ &\left.+\delta\left(x\notin\alpha\right)\left(\frac{e^{-\beta H_\alpha}+e^{-\beta H_{\alpha\cup\{x\}}}}{4}\right)\left(\phi_\alpha-\phi_{\alpha\cup\{x\}}\right)^2\right\}\end{aligned}$$

where $\left[\bar{\rho}_\Lambda^\beta,\Phi\right]=0$, $\mathcal{H}_\Lambda^G\ni\Phi\left|0\right\rangle_\Lambda=\sum_{\alpha\subseteq\Lambda}\phi_\alpha\left|\alpha\right\rangle_\Lambda$. This allows us to identify the rates of our process with

$$w_x\left(\alpha\right)=\delta(x\in\alpha)\frac{1+exp\beta(H_\alpha-H_{\alpha\setminus\{x\}})}{4}+\delta(x\notin\alpha)\frac{1+exp\beta(H_\alpha-H_{\alpha\cup\{x\}})}{4}\quad x\in\Lambda,\ \alpha\subseteq\Lambda$$

These may be more easily looked at as the rates of a birth and death process for a lattice gas. Let us now compare these rates with those of the heat-bath dynamics. In our notation their expression is

$$w_x^{hb}\left(\alpha\right):=\delta(x\in\alpha)\frac{1}{1+e^{-\beta\left(H_\alpha-H_{\alpha\setminus\{x\}}\right)}}+\delta(x\notin\alpha)\frac{1}{1+e^{-\beta\left(H_\alpha-H_{\alpha\cup\{x\}}\right)}}\quad x\in\Lambda,\ \alpha\subseteq\Lambda$$

we get

$$\frac{w_x^{hb}\left(\alpha\right)}{w_x\left(\alpha\right)}=\delta(x\in\alpha)\left(\cosh\frac{\beta}{2}\left(H_\alpha-H_{\alpha\setminus\{x\}}\right)\right)^{-2}+\delta(x\notin\alpha)\left(\cosh\frac{\beta}{2}\left(H_\alpha-H_{\alpha\cup\{x\}}\right)\right)^{-2}<1$$

therefore the dynamics we have previously introduced reaches equilibrium faster than the heat-bath one.

The graphs in [1] confusingly compare different dynamics. We fix this here.

The generic matrix element of the generator of heat-bath dynamics $\overline{\mathcal{L}}_\Lambda^{hb,s}\left(\beta\right)=\sum_{x\in\Lambda}\overline{\mathcal{L}}_x^{hb,s}\left(\beta\right)$ acting on $\mathcal{H}_\Lambda$ is

$$\begin{aligned}\left\langle\gamma\right|\overline{\mathcal{L}}_\Lambda^{hb,s}\left(\beta\right)\left|\alpha\right\rangle_\Lambda &= \left\langle\gamma\right|\sum_{x\in\Lambda}\left\{\delta\left(x\in\alpha\right)\frac{\left[\bar{\ell}_x^\Lambda+\bar{\ell}_x^{\Lambda,\perp}\tanh\frac{\beta}{4}\left(H_\alpha-H_{\alpha\setminus\{x\}}\right)\right]}{\left(1-\tanh\frac{\beta}{4}\left(H_\alpha-H_{\alpha\setminus\{x\}}\right)\right)\cosh\frac{\beta}{2}\left(H_\alpha-H_{\alpha\setminus\{x\}}\right)}+\right. \\ &\left.+\delta\left(x\notin\alpha\right)\frac{\left[\bar{\ell}_x^\Lambda+\bar{\ell}_x^{\Lambda,\perp}\tanh\frac{\beta}{4}\left(H_\alpha-H_{\alpha\cup\{x\}}\right)\right]}{\left(1-\tanh\frac{\beta}{4}\left(H_\alpha-H_{\alpha\cup\{x\}}\right)\right)\cosh\frac{\beta}{2}\left(H_\alpha-H_{\alpha\cup\{x\}}\right)}\right\}\left|\alpha\right\rangle_\Lambda.\end{aligned}$$

Then, proceding as before, we estimate the generic matrix element of $\mathcal{L}_x^{hb,s} = \mathbf{U}_\Lambda \overline{\mathcal{L}}_x^{hb,s} \mathbf{U}_\Lambda$ and get

$$\begin{aligned}\langle\gamma|\,\mathcal{L}_x^{hb,s}(\beta)\,|\alpha\rangle_\Lambda \geq \langle\gamma|\,\ell_x^\Lambda \Bigg(\max_{\eta\subseteq\Lambda}\max\Bigg[&\left(1+\tanh\frac{\beta}{4}\left|H_\eta - H_{\eta\setminus\{x\}}\right|\right)\cosh\frac{\beta}{2}\left(H_\eta - H_{\eta\setminus\{x\}}\right);\\ &\left(1+\tanh\frac{\beta}{4}\left|H_\eta - H_{\eta\cup\{x\}}\right|\right)\cosh\frac{\beta}{2}\left(H_\eta - H_{\eta\cup\{x\}}\right)\Bigg]\Bigg)^{-1} + \\ &-\max_{\eta\subseteq\Lambda}\max\left[\frac{\tanh\frac{\beta}{4}\left|H_\eta - H_{\eta\setminus\{x\}}\right|}{1+\tanh\frac{\beta}{4}\left|H_\eta - H_{\eta\setminus\{x\}}\right|};\frac{\tanh\frac{\beta}{4}\left|H_\eta - H_{\eta\cup\{x\}}\right|}{1+\tanh\frac{\beta}{4}\left|H_\eta - H_{\eta\cup\{x\}}\right|}\right]\mathbf{I}_\Lambda\,|\alpha\rangle_\Lambda\end{aligned}$$

Thus our estimate for the spectral gap of this process is

$$\overline{g}_\Lambda^{hb,(d)}(\beta) := \frac{\frac{1}{\cosh\frac{\beta}{2}\Delta} - \tanh\frac{\beta}{4}\Delta}{\left(1+\tanh\frac{\beta}{4}\Delta\right)} = \frac{\overline{g}_\Lambda^{(d)}(\beta)}{\cosh\frac{\beta}{2}\Delta}$$

where $\Delta = \max_{\alpha\subseteq\Lambda}\max\left[\max_{x\in\alpha}\left|H_\alpha - H_{\alpha\setminus\{x\}}\right|;\max_{x\notin\alpha}\left|H_\alpha - H_{\alpha\cup\{x\}}\right|\right]$.

Acknoledgements

The referee to [1] tried to tell us that something was wrong, but we were too blockheaded to understand there was a problem.

References

[1] M. Gianfelice, M. Isopi: *Quantum Methods for Interacting Particle Systems II, Glauber Dynamics for Ising Spin Systems*. Markov Processes and Related Fields **4**, 411–428 (1998)

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