Decay of Correlations for one-dimensional Kac rotators

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Abstract

We study the correlation function of two block-spins of size $\delta \geq \gamma$, with γ the Kac parameter, for 1-dimensional Kac rotators and prove, in the limit $\gamma \downarrow 0$ and for values of the temperature below the mean field critical one, the decay to be exponential with a rate dependent of the inverse temperature β.

1 Introduction and notations

We analyse the behaviour of spin-spin and two block-spins correlation functions of a system of planar rotators in dimension one, interacting through a ferromagnetic Kac potential. We first give an upper bound for such quantities making use of a suitably modified version of the McBrian and Spencer approach [McBS]. Then we will provide a lower bound for the two block-spins correlation function based in part on the study of the two point correlation function of the Villain model, which can be obtained as a limit of our model when the size of the block-spins is very large (see the Appendix and [Gi]), and in part on large deviation estimates for the Gibbs measure of our model given in [BPi].

1.1 The model

Given $\gamma \in (0,1]$, and Λ a bounded subset of $\mathbb{R}$ $(\Lambda \subset\subset \mathbb{R})$, we set

$$\Lambda_\gamma := \{n \in \mathbb{Z} : \gamma n \in \Lambda\}\,. \tag{1}$$

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To each site of the lattice $\mathbb{Z}$ we attach a spin variable

$$\mathbb{Z} \ni i \longmapsto \sigma_i \in S^1 \tag{2}$$

and, denoting by ν the Haar mesure on S^1, we consider the probability space $(S, \mathcal{B}(S), \mu)$, where $S := (S^1)^{\mathbb{Z}}$ is the configuration space, $\mu := \bigotimes_{i\in\mathbb{Z}} \nu_i$ and $\mathcal{B}(S)$ is the σalgebra of subsets of S generated by the finite-dimensional cylinders. We also take, with abuse of notation, σ_i to be the projection of the configuration $\sigma \in S$ on the site i. Therefore, for any $\Lambda \subset \mathbb{R}$, $\sigma_{\Lambda_\gamma} := \{\sigma_i\}_{i\in\Lambda_\gamma}$ denotes the restriction of the configuration $\sigma \in S$ to Λ_γ and S_{Λ_γ} the set of the spin configurations on Λ_γ.

For $\gamma \in (0,1]$, the interaction among the σ's in a finite region Λ_γ, $\Lambda \subset\subset \mathbb{R}$, with fixed boundary condition $\sigma_{\Lambda_\gamma^c} \in S_{\Lambda_\gamma^c}$, is defined by the Hamiltonian

$$H^{J_\gamma}\left(\sigma_{\Lambda_\gamma} | \sigma_{\Lambda_\gamma^c}\right) := -\left[\sum_{i,j\in\Lambda_\gamma} \frac{1}{2}(1-\delta_{i,j}) + \sum_{\substack{i\in\Lambda_\gamma \\ j\in\Lambda_\gamma^c}}\right] J_\gamma(i,j)\, \sigma_i \cdot \sigma_j \,, \tag{3}$$

where $u \cdot v$ denotes the scalar product of the vectors $u, v \in \mathbb{R}^2$ and

$$J_\gamma(i,j) := \gamma J(\gamma |i-j|) \quad i,j \in \mathbb{Z} \tag{4}$$

is the Kac interaction matrix associated to the function

$$\mathbb{R}^+ \ni x \longmapsto J(x) \in \mathbb{R}^+, \tag{5}$$

which satisfies the following conditions:

- is compactly supported;
- $\left|\left|\frac{dJ}{dx}\right|\right|_\infty < \infty$;
- $\int_{\mathbb{R}} dx J(|x|) = 1$.

For technical convenience, in the rest of the paper we will also assume J to satisfy the inequalities

$$\mathbf{1}_{[0,\frac{1}{2}-\delta]}(\delta |k|) \leq J(\delta |k|) \leq \mathbf{1}_{[0,\frac{1}{2}+\delta]}(\delta |k|)\,, \tag{6}$$

for any $\delta \in (0,1]$, $\delta \geq \gamma$.

The Gibbs measure at the temperature β^{-1} in a finite region Λ_γ, $\Lambda \subset\subset \mathbb{R}$ with boundary condition $\sigma_{\Lambda_\gamma^c} \in S_{\Lambda_\gamma^c}$ is

$$\mu_{\Lambda_\gamma}^{\beta,J_\gamma}\left(d\sigma | \sigma_{\Lambda_\gamma^c}\right) := \frac{e^{-\beta H^{J_\gamma}\left(\sigma_{\Lambda_\gamma}|\sigma_{\Lambda_\gamma^c}\right)}}{Z_{\Lambda_\gamma}\left(\beta, \gamma | \sigma_{\Lambda_\gamma}\right)} \bigotimes_{i\in\Lambda_\gamma} \nu(d\sigma_i)\,, \tag{7}$$

where $Z_{\Lambda_\gamma}\left(\beta,\gamma|\sigma_{\Lambda^c_\gamma}\right) := \mu\left(e^{-\beta H\left(\sigma_{\Lambda_\gamma}|\sigma_{\Lambda^c_\gamma}\right)}\right)$. We denote by μ^{β,J_γ} the Gibbs state specified by $\mu^{\beta,J_\gamma}_{\Lambda_\gamma}(d\sigma|\cdot)$ that is

$$\mu^{\beta,J_\gamma}\left(d\sigma_{\Lambda_\gamma}|\sigma_{\Lambda^c_\gamma}\right) = \mu^{\beta,J_\gamma}_{\Lambda_\gamma}\left(d\sigma|\sigma_{\Lambda^c_\gamma}\right) \quad \mu^{\beta,J_\gamma} - a.s. \quad \forall\Lambda_\gamma,\ \Lambda \subset\subset \mathbb{R}. \tag{8}$$

We remark that μ^{β,J_γ} satisfies the hypothesis of Theorem 8.39 in [Ge], hence is unique and therefore is left invariant by lattice translation and rotations of the spins. Rotational invariance of μ^{β,J_γ} can also be directly proven ([Gi] Therem 302) arguing in a similar fashion to [Pi1] and [Pf] (see also [BC] for the case of random interactions).

Let $\mathbf{B} := \{v \in \mathbb{R}^2 : ||v|| \leq 1\}$ and $\mathcal{M}$ be the space λ-measurable maps $\mathbb{R} \ni x \longmapsto m(x) \in \mathbf{B}$ endowed with the weak topology with respect to the functions $L^2_{loc}(\mathbb{R};\mathbb{R}^2)$. $\mathcal{M}$ is easily seen to be a convex and compact space. For any $\Lambda \subset \mathbb{R}$, we denote by m_Λ the restriction of $m \in \mathcal{M}$ to Λ, and by $\mathcal{M}_\Lambda := \{v \in \mathcal{M} : v = m_\Lambda\,;\, m \in \mathcal{M}\}$.

Defining, for any $\gamma \in (0,1]$, the continuous injective map

$$S \ni \sigma \longmapsto \iota_\gamma(\sigma) = \sigma_\gamma := \sum_{i\in\mathbb{Z}} \sigma_i \mathbf{1}_{[i\gamma,(i+1)\gamma)} \in \mathcal{M}, \tag{9}$$

we denote by $A_\gamma := \{\sigma \in S : \sigma_\gamma \in A\} \in \mathcal{B}(S)$ the image of any $A \in \mathcal{B}(\mathcal{M})$ via the map ι^{-1}_γ and by $\bar{\mu}^\gamma$ the image of any probability measure $\bar{\mu}$ on $(S,\mathcal{B}(S))$ in the set of probability measures on $(\mathcal{M},\mathcal{B}(\mathcal{M}))$ through the probability kernel

$$\mathcal{B}(\mathcal{M}) \times S \ni (A,\sigma) \longmapsto \mathbf{1}_A \circ \iota_\gamma(\sigma) = \mathbf{1}_{\{\sigma\in S\,:\,\sigma_\gamma\in A\}}. \tag{10}$$

Hence $\forall A \in \mathcal{B}(\mathcal{M})$, $\gamma \in (0,1]$, $\bar{\mu}^\gamma(A) := \bar{\mu}(A_\gamma)$. In particular we set $\mu^{\beta,\gamma} := \left(\mu^{\beta,J_\gamma}\right)^\gamma$.

Moreover, if $f : S \longmapsto \mathbb{R}$ is a measurable function, we denote by $f_\gamma := f \circ \iota^{-1}_\gamma$ the image of f in the set of measurable real functions defined on $(\mathcal{M},\mathcal{B}(\mathcal{M}))$.

For any $\Lambda \subset\subset \mathbb{R}$, with $\lambda(\Lambda) \geq \lambda(suppJ)$ and any fixed $m_{\Lambda^c} \in \mathcal{M}_{\Lambda^c}$, let

$$\mathcal{M}_\Lambda \ni m_\Lambda \longmapsto E_J(m_\Lambda|m_{\Lambda^c}) := -\frac{1}{2}\langle m_\Lambda, \mathbf{J}m_\Lambda\rangle - \langle m_\Lambda, \mathbf{J}m_{\Lambda^c}\rangle \in \mathbb{R} \tag{11}$$

be the *energy functional* with boundary condition m_{Λ^c}, where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $L^2(\mathbb{R};\mathbb{R}^2)$ and $\mathbf{J}m := (J \circ ||\cdot||) * m \in \mathcal{M}$. We have

$$\begin{aligned} E_J(m_\Lambda|m_{\Lambda^c}) &= -\frac{1}{2}\int_\Lambda dx \int_\Lambda dy J(|x-y|)\, m(x)\, m(y) + \\ &- \int_\Lambda dx \int_{\Lambda^c} dy J(|x-y|)\, m(x)\, m(y) \\ &= U^J(m_\Lambda) + W_J(m_\Lambda|m_{\Lambda^c}) - W^J(m_{\Lambda^c}) - \frac{1}{2}||m_\Lambda||^2 \\ &- \frac{1}{2}\int_\Lambda dx \int_{\Lambda^c} dy J(|x-y|)\, |m(x)|^2\ , \end{aligned} \tag{12}$$

where

$$\mathcal{M}_\Lambda \ni m_\Lambda \longmapsto U^J(m_\Lambda) := \frac{1}{4}\int_\Lambda dx \int_\Lambda dy J(|x-y|)\,|m(x)-m(y)|^2 \in \mathbb{R}\,, \tag{13}$$

$$\mathcal{M}_\Lambda \ni m_\Lambda \longmapsto W_J(m_\Lambda|m_{\Lambda^c}) := \frac{1}{2}\int_\Lambda dx \int_{\Lambda^c} dy J(|x-y|)\,|m(x)-m(y)|^2 \in \mathbb{R} \tag{14}$$

and $W^J(m_{\Lambda^c}) := W_J(0|m_{\Lambda^c})$.

1.2 Coarse graining

For any $\delta \geq \gamma \in (0,1]$, let $\mathcal{Q}_\delta$ be the partition of $\mathbb{R}$, whose atoms are

$$Q_n^{(\delta)} := \{x \in \mathbb{R} : x \in [\delta n, \delta(n+1))\,;\, n \in \mathbb{Z}\}\,. \tag{15}$$

Then, considering the measurable space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, where λ denotes the Lebesgue measure, for any λ-mesurable function $f : \mathbb{R} \longmapsto \mathbb{R}^q$, $q = 1, 2$, we denote by $f^{(\delta)} = \mathbb{E}_\delta(f) := \mathbb{E}(f|\mathcal{Q}_\delta)$ its conditional expectation with respect to the σalgebra generated by $\mathcal{Q}_\delta$. Therefore, δ-measurable functions are those functions f such that $f^{(\delta)} = f$. In particular, if this occurs for $f = \mathbf{1}_\Lambda$, with $\Lambda \subset \mathbb{R}$, Λ will be called δ-measurable.

For any $\delta \geq \gamma \in (0,1]$, the map

$$\mathcal{M} \ni m \longmapsto \mathbb{E}_\delta(m) := m^{(\delta)} \in \mathcal{M}\,, \tag{16}$$

is called *coarse graining* at the scale δ and $\sigma_\gamma^{(\delta)}$ is called block spin of size δ. We also set $\mathcal{M}^{(\delta)} := \mathbb{E}_\delta \mathcal{M}$.

In the following, to simplify the computations, we will choose the Kac parameter γ and any coarse graining parameter $\delta \geq \gamma \in (0,1]$ to be diadic numbers, since, as it will clear from the sequel, this assumption will not affect the results.

Given $\delta \geq \gamma$ and any δ-measurable $\Lambda \subset\subset \mathbb{R}$, for any fixed $\xi' \in \mathcal{M}_{\Lambda^c}^{(\delta)}$ we define the block Hamiltonian of size δ to be the functional

$$\mathcal{M}_\Lambda^{(\delta)} \ni \xi_\Lambda \longmapsto H_\delta(\xi_\Lambda|\xi'_{\Lambda^c}) := \left[\sum_{\substack{n\in\Lambda_\delta\\ k\in\Lambda_\delta}} \frac{1}{2} J_\delta(n,k)\,\xi(n\delta)\cdot\xi(k\delta) + \sum_{\substack{n\in\Lambda_\delta\\ k\in\Lambda_\delta^c}} J_\delta(n,k)\,\xi(n\delta)\cdot\xi'(k\delta)\right] \in \mathbb{R} \tag{17}$$

with $J_\delta(n,k) := \delta J(\delta|n-k|)$. Notice that by (3) for any fixed $\bar{\sigma}_{\Lambda_\gamma^c} \in S_{\Lambda_\gamma^c}$

$$H_\gamma\left((\sigma_\gamma)_\Lambda \,|\, (\bar{\sigma}_\gamma)_\Lambda\right) = H^{J_\gamma}\left(\sigma_{\Lambda_\gamma}|\bar{\sigma}_{\Lambda_\gamma^c}\right) - \frac{1}{2}\gamma\,|\Lambda_\gamma|\,. \tag{18}$$

Moreover, there exist two positive constans $b_1(J), b_2(J)$ such that

$$\left| H^{J_\gamma}\left(\sigma_{\Lambda_\gamma}|\bar{\sigma}_{\Lambda_\gamma^c}\right) - \frac{\delta}{\gamma} H_\gamma\left(\left(\sigma_\gamma^{(\delta)}\right)_\Lambda | \left(\bar{\sigma}_\gamma^{(\delta)}\right)_\Lambda\right)\right| \leq b_1(J)\frac{\delta}{\gamma}\lambda(\Lambda) , \tag{19}$$

$$\left| E_J\left(\left(\sigma_\gamma^{(\delta)}\right)_\Lambda | \left(\bar{\sigma}_\gamma^{(\delta)}\right)_\Lambda\right) - \delta H_\gamma\left(\left(\sigma_\gamma^{(\delta)}\right)_\Lambda | \left(\bar{\sigma}_\gamma^{(\delta)}\right)_\Lambda\right)\right| \leq b_2(J)\lambda(\Lambda) . \tag{20}$$

Therefore, a Kac model can be interpreted as a discretized version of a model, whose configuration space is $\mathcal{M}$, described by the Hamiltonian $\gamma^{-1}E_J\left(\left(\sigma_\gamma^{(\delta)}\right)_\Lambda | \left(\bar{\sigma}_\gamma^{(\delta)}\right)_\Lambda\right)$.

The Lebowitz-Penrose theorem [LP] (see [BPi] Theorem 2.1 and [Gi] Theorem 4.2.1 for this particular model and [TS] for its the Grand-Canonical version) states that, in the thermodynamic limit, for any value of the temperature and of the lattice dimesion d, the thermodynamic potentials of the block-spin model derived by a Kac one are very well approximated by the convex envelope of their mean field equivalents with an error proportional to the size of the block $\delta(\gamma) > \gamma$ tending to zero as $\gamma \downarrow 0$. Furthermore, for $d = 1$ and for any value of the inverse temperature β, the sequence of probability masures $\{\mu^{\beta,\gamma}\}_{\gamma\in(0,1]}$ satisfies the large deviation principle with rate function $\beta\mathcal{F}^J$ ([BPi] Theorem 2.7), $\mathcal{F}^J$ being the *excess of free energy* functional

$$\mathcal{M} \ni m \longmapsto \mathcal{F}^J(m) := U^J(m) + F(m) \in \overline{\mathbb{R}}^+, \tag{21}$$

where

$$\mathcal{M} \ni m \longmapsto U^J(m) := \frac{1}{4}\int dx \int dy J(||x-y||)|m(x) - m(y)|^2 \in \overline{\mathbb{R}}^+, \tag{22}$$

$$\mathcal{M} \ni m \longmapsto F(m) := \int dx \left[\bar{f}_\beta(m(x)) - \bar{f}_\beta(m_\beta)\right] \in \overline{\mathbb{R}}^+, \tag{23}$$

with $\mathbf{B} \ni u \longmapsto \bar{f}_\beta(|u|) = f_\beta(u) \in \mathbb{R}$ the mean field free energy density and $m_\beta \geq 0$ the solution of the mean field equation $\frac{d}{dx}\bar{f}_\beta(x) = 0$.

We also recall that

$$\bar{f}_\beta(|u|) = f_\beta(u) := -\frac{|u|^2}{2} + \beta^{-1}\bar{I}(|u|) \in \mathbb{R}, \tag{24}$$

where $\bar{I}(|w|) := \sup_{t\geq 0}\{t|w| - \ln\bar{\varphi}(t)\} = I(w)$,

$$\mathbb{R}^2 \ni w \longmapsto I(w) := \sup_{h\in\mathbb{R}^2}\{h\cdot w - \ln\varphi(h)\} \in \overline{\mathbb{R}}^+, \tag{25}$$

is the entropy of the measure ν, that is the Legendre transform of the generating function of the cumulants of ν, $\log\varphi(h)$, with

$$\bar{\varphi}(|h|) = \varphi(h) := \int_{S^1} v(ds)\, e^{h\cdot s} \quad h \in \mathbb{R}^2. \tag{26}$$

For β larger than the mean field critical value $\beta_{mf} = 1$, m_β is strictly larger than 0 and for $d = 1$, $\mu^{\beta,\gamma}$ weakly converges to the product measure ν^β on $\mathcal{B}(S)$, which is a.c. with respect to ν with density $\frac{d\nu^\beta}{d\nu}(s) = \frac{e^{h_\beta \cdot s}}{\varphi(h_\beta)}$, such that $\nu^\beta(\sigma_i) = m_\beta s$ ([BPi] Theorem 2.3). In the following, unless differently specified, we will restrict ourselves to this range of temperatures.

2 Correlation functions

We will study the behaviour of the correlation function of two block-spins of size $\delta \geq \gamma$ when the distance between the variables is very large compared to the size of the blocks. To do this, we will first give upper bounds for correlation functions of two spins and of two block spins and then provide a lower bound of such quantities when the distance between the variables diverges in the limit $\gamma \downarrow 0$.

2.1 Upper bound

First we give an estimate from above of the correlation function of two spins. To this aim we make use of a modified version of the strategy proposed originally by McBrian and Spencer [McBS] and successively developed by Messager et al. [MMSR] and Picco [Pi2]. For further details we address the reader to [Gi] Chapter 6.

In the sequel we will assume $\gamma \in \left(0, \frac{1}{2}\right]$. In fact, if $\gamma \in \left(\frac{1}{2}, 1\right]$, the interaction among the spins involves at most those whose mutual distance is smaller than or equal to $\sqrt{2}$. Hence, by the second Griffiths' inequality [Si], the spin-spin correlation function relative to Kac potential is dominated by the one relative to the standard nearest-neighbor potential. In this case, we adress the reader to [McBS] and to the Appendix.

Theorem 1 *For any $\gamma \in \left(0, \frac{1}{2}\right]$ let $L(\gamma) \geq 1$. Then there exist two constants $c, \tau \geq 1$ such that, $\forall p > 0$ and $\beta > \frac{\gamma^{1+p}}{1+\tau}$,*

$$\mu^{\beta,J_\gamma}\left[\cos\left(\theta_0 - \theta_{L(\gamma)}\right)\right] \leq \exp\left\{-\frac{\gamma^{2+p} L(\gamma)}{c\beta}(1-\gamma^p) + \frac{\gamma^{1+2p}}{8(1+\tau)\beta}\right\}. \tag{27}$$

Corollary 2 *Assuming the hypotesys of the preceding theorem, $\forall p > 0,\ L(\gamma) = [L\gamma^{-2-p}]$ and $\mathbb{N} \ni L > 0$, we have that there exist two constants $c, \tau \geq 1$ such that $\forall \gamma \in \left(0, \frac{1}{2}\right]$ and $\beta > \frac{\gamma^{1+p}}{1+\tau}$*

$$\mu^{\beta,J_\gamma}\left[\cos\left(\theta_0 - \theta_{L(\gamma)}\right)\right] \leq \exp\left\{-\frac{L-1}{c\beta}(1-\gamma^p) + \frac{\gamma^{1+2p}}{8(1+\tau)\beta}\right\}. \tag{28}$$

The proofs of the above results lie on the following lemma which will also provide the proofs of analogous results for the two-dimensional case [Gi1].

Lemma 3 *Let $d = 1, 2$. For any $\gamma \in (0, 1)$, Let $L_d(\gamma) > 1$ and $\Lambda \subset\subset \mathbb{R}^d$ such that Λ_γ contains the segment of lenght $2L_d(\gamma) + 1$ centred in the origin. Assuming free b.c., $\forall \beta > 0$, we have that there exists a convex functional $f(\cdot; \gamma L_d(\gamma), \gamma, \beta)$ on the space of non negative real valued γ-measurable summable functions on $[1, +\infty)$, $L^{1,+}_\gamma([1, +\infty))$, such that, uniformly in Λ_γ,*

$$\mu^{\beta, J_\gamma}_{\Lambda_\gamma}\left[\cos\left(\theta_0 - \theta_{L_d(\gamma)}\right)\right] \leq \exp\left\{\inf_{b \in L^{1,+}_\gamma([1,+\infty))} f(b; \gamma L_d(\gamma), \gamma, \beta)\right\}. \tag{29}$$

Proof. We have

$$\begin{aligned}
&\mu^{\beta, J_\gamma}_{\Lambda_\gamma}[\sigma_0 \cdot \sigma_{L_d(\gamma)}] = \mu^{\beta, J_\gamma}_{\Lambda_\gamma}\left[\cos\left(\theta_0 - \theta_{L_d(\gamma)}\right)\right] \\
&= (Z_{\Lambda_\gamma}(\beta, \gamma))^{-1} \mu_{\Lambda_\gamma}\left[e^{i\left(\theta_0 - \theta_{L_d(\gamma)}\right)} \times \right. \\
&\left. \times \exp\left\{\beta\gamma^d \sum_{i,j \in \Lambda_\gamma} \frac{1 - \delta_{i,j}}{2} J(\gamma\,||i - j||) \cos(\theta_i - \theta_j)\right\}\right].
\end{aligned} \tag{30}$$

Integrating over the the variables $\theta_j,\ j \in \Lambda_\gamma$, along the positively oriented closed contour

$$\begin{aligned}
\Gamma_j \equiv &\{z \in \mathbb{C} : \Re z \in [-\pi, \pi]; \Im z = 0\} \cup \{z \in \mathbb{C} : \Re z = \pi; \Im z \in [0, a_j]\} \\
&\cup \{z \in \mathbb{C} : \Re z \in [-\pi, \pi]; \Im z \in [-a_j, a_j]\} \cup \{z \in \mathbb{C} : \Re z = -\pi; \Im z \in [-a_j, 0]\},
\end{aligned} \tag{31}$$

with

$$\Lambda_\gamma \ni j \longmapsto a_j = a_j(\beta, \gamma) \in \mathbb{R}^+ \tag{32}$$

we get

$$\begin{aligned}
\mu^{\beta, J_\gamma}_{\Lambda_\gamma}\left[\cos\left(\theta_0 - \theta_{L_d(\gamma)}\right)\right] &= \left(Z^{(d)}_{\Lambda\gamma}(\beta, \gamma)\right)^{-1} e^{-\left(a_0 - a_{L_d(\gamma)}\right)} \times \\
&\times \mu^{(d)}_{\Lambda_\gamma}\left[\exp\left\{\beta\gamma^d \sum_{i,j \in \Lambda_\gamma} \frac{1 - \delta_{i,j}}{2} J(\gamma\,||i - j||) \times \right.\right. \\
&\times [\cos(\theta_i - \theta_j)\cosh(a_i - a_j) + \\
&\left.\left. + i \sin(\theta_i - \theta_j)\sinh(a_i - a_j)]\right\} e^{i\left(\theta_0 - \theta_{L_d(\gamma)}\right)}\right] \\
&\leq e^{-\left(a_0 - a_{L_d(\gamma)}\right)} \left(Z^{(d)}_{\Lambda\gamma}(\beta, \gamma)\right)^{-1} \times \\
&\times \mu_{\Lambda_\gamma}\left[\exp\left\{\beta\gamma^d \sum_{i,j \in \Lambda_\gamma} \frac{1 - \delta_{i,j}}{2} J(\gamma\,||i - j||) \times \right.\right. \\
&\left.\left. \times \cosh(a_i - a_j)\cos(\theta_i - \theta_j)\right\}\right].
\end{aligned} \tag{33}$$

By the Jensen inequality

$$Z_{\Lambda_\gamma}^{(d)}(\beta,\gamma)=\mu_{\Lambda_\gamma}\left[\exp\left\{\beta\gamma^d\sum_{i,j\in\Lambda_\gamma}\frac{1-\delta_{i,j}}{2}J\left(\gamma\left|\left|i-j\right|\right|;\gamma,d\right)\cos\left(\theta_i-\theta_j\right)\right\}\right] \tag{34}$$

$$\geq\exp\left\{\beta\gamma^d\sum_{i,j\in\Lambda_\gamma}\frac{1-\delta_{i,j}}{2}J\left(\gamma\left|\left|i-j\right|\right|;\gamma,d\right)\mu_{\Lambda_\gamma}^{(d)}\left[\cos\left(\theta_i-\theta_j\right)\right]\right\}.$$

Hence

$$\mu_{\Lambda_\gamma}^{\beta,J_\gamma}\left[\cos\left(\theta_0-\theta_{L_d(\gamma)}\right)\right] \tag{35}$$

$$\leq e^{-\left(a_0-a_{L_d(\gamma)}\right)}\mu_{\Lambda_\gamma}\left[\exp\left\{\beta\gamma^d\sum_{i,j\in\Lambda_\gamma}\frac{1-\delta_{i,j}}{2}J\left(\gamma\left|\left|i-j\right|\right|\right)\left[\cosh\left(a_i-a_j\right)-1\right]\times\right.\right.$$

$$\times\cos\left(\theta_i-\theta_j\right)\}]$$

$$\leq e^{-\left(a_0-a_{L_d(\gamma)}\right)}\exp\left\{\beta\gamma^d\sum_{i,j\in\Lambda_\gamma}\frac{1-\delta_{i,j}}{2}J\left(\gamma\left|\left|i-j\right|\right|\right)\left[\cosh\left(a_i-a_j\right)-1\right]\right\}.$$

Let us set $\forall i\in\Lambda_\gamma,\ C_k:=\left\{i\in\Lambda_\gamma:|i|=k\right\},\ \forall k\in\left[0,L\left(\gamma\right)\right]\cap\mathbb{N}$. Then

$$\sum_{i,j\in\Lambda_\gamma}=\sum_{k\geq0}\sum_{h\geq0}\sum_{i\in C_k}\sum_{j\in C_h}\left[\left(1-\delta_{k.h}\right)+\delta_{k,h}\left(1-\delta_{i,j}\right)\right] \tag{36}$$

$$=\left(\sum_{k\geq0}\sum_{h\geq k}+\sum_{h\geq0}\sum_{k\geq h}\right)\sum_{i\in C_k}\sum_{j\in C_h}\left[\left(1-\delta_{k.h}\right)+\frac{1}{2}\delta_{k,h}\left(1-\delta_{i,j}\right)\right].$$

We also set

$$a_i=a_k\mathbf{1}_{\{i\in\Lambda_\gamma:i\in C_k\}}\mathbf{1}_{\{k\in\mathbb{N}:0\leq k\leq L_d(\gamma)\}} \tag{37}$$

$$a_k:=\sum_{l\geq0}b_l\mathbf{1}_{\{l\in\mathbb{N}:k+1\leq l\leq L_d(\gamma)+1\}} \tag{38}$$

$$b_l:=b_l\left(\beta,\gamma\right) \tag{39}$$

Therefore,

$$\mu_{\Lambda_\gamma}^{\beta,J_\gamma}\left[\cos\left(\theta_0-\theta_{L_d(\gamma)}\right)\right] \tag{40}$$

$$\leq\exp\left\{-\sum_{l=1}^{L_d(\gamma)}b_l+\beta\gamma^d\sum_{k=0}^{L_d(\gamma)-1}\sum_{r=1}^{L_d(\gamma)-k}\sum_{i\in C_k}\sum_{j\in C_{k+r}}J\left(\gamma\left|i-j\right|\right)\times\right.$$

$$\left.\times\left[\cosh\left(\sum_{l=k+1}^{k+r}b_l\right)-1\right]\right\}.$$

We now consider the case $d=1$. For the two-dimensional case we refer the reader to [Gi]. Since by definition of the Kac potential $J\left(\gamma\left|i-j\right|\right)\leq\mathbf{1}\left(\gamma\left|i-j\right|\leq\frac{1}{2}+\gamma\right)$, setting $t=k+1,\ s=t+r-1,\ r=s-t+1,$ we obtain

$$\begin{aligned}&\mu_{\Lambda_\gamma}^{\beta,J_\gamma}\left[\cos\left(\theta_0-\theta_{L(\gamma)}\right)\right] \\ &=\exp\left\{-\sum_{l=1}^{L(\gamma)}b_l+8\beta\gamma\sum_{t=1}^{L(\gamma)}\sum_{s=t}^{L(\gamma)\wedge t+\frac{1}{2\gamma}}\left[\cosh\left(\sum_{l=t}^{s}b_l\right)-1\right]\right\}.\end{aligned}\tag{41}$$

Let

$$b\left(x;\beta,\gamma\right):=\sum_{k\in\mathbb{Z}}b_k\left(\beta,\gamma\right)\mathbf{1}_{[\gamma k,\gamma(k+1))}\left(x\right),\tag{42}$$

then

$$\begin{aligned}\mu_{\Lambda_\gamma}^{\beta,J_\gamma}\left[\cos\left(\theta_0-\theta_{L(\gamma)}\right)\right]\leq\exp\Bigg\{&-\int_{\gamma}^{\gamma(L(\gamma)+1)}\frac{dz}{\gamma}b\left(z;\beta,\gamma\right)+\frac{8\beta}{\gamma}\Bigg[\left(\gamma L\left(\gamma\right)-\frac{1}{2}-\gamma\right)\left(\frac{1}{2}+\gamma\right)+ \\ &+\int_{\gamma L(\gamma)-\frac{1}{2}}^{\gamma(L(\gamma)+1)}dx\int_{x}^{\gamma(L(\gamma)+1)}dy\Bigg]\left[\cosh\left(\int_x^{y+\gamma}\frac{dz}{\gamma}b\left(z;\beta,\gamma\right)\right)-1\right]\Bigg\}.\end{aligned}\tag{43}$$

Since b is non-negative, $\left|\int_x^{y+\gamma}\frac{dz}{\gamma}b\left(z;\beta,\gamma\right)\right|$ is an increasing function of y. Therefore, setting $m=\left(x+\frac{1}{2}+\gamma\right),\ \gamma\left(L\left(\gamma\right)+1\right)$ we get

$$\begin{aligned}&\int_x^m dy\left\{\cosh\left[\int_x^{y+\gamma}\frac{dz}{\gamma}b\left(z;\beta,\gamma\right)\right]-1\right\} \\ &\leq\left(m-x\right)\left\{\cosh\left[\int_x^{m+\gamma}\frac{dz}{\gamma}b\left(z;\beta,\gamma\right)\right]-1\right\} \\ &=\left(m-x\right)\left\{\int_x^{m+\gamma}\frac{1}{m-x+\gamma}dz\cosh\left[\frac{m-x+\gamma}{\gamma}b\left(z;\beta,\gamma\right)\right]-1\right\} \\ &=\frac{\left(m-x\right)}{m-x+\gamma}\int_x^{m+\gamma}dz\left\{\cosh\left[\frac{m-x+\gamma}{\gamma}b\left(z;\beta,\gamma\right)\right]-1\right\} \\ &\leq\int_x^{m+\gamma}dz\left\{\cosh\left[\frac{m-x+\gamma}{\gamma}b\left(z;\beta,\gamma\right)\right]-1\right\}.\end{aligned}\tag{44}$$

Thus defining

$$
\begin{aligned}
f\left(b;\gamma L\left(\gamma\right),\gamma,\beta\right) := & -\int_{\gamma}^{\gamma(L(\gamma)+1)} \frac{dz}{\gamma} b\left(z;\beta,\gamma\right) + \\
& + \frac{8\beta}{\gamma^d}\left[\left(\gamma L\left(\gamma\right)-\frac{1}{2}-\gamma\right)\left(\frac{1}{2}+\gamma\right)\times\right. \\
& \times\left\{\cosh\left[\frac{\left(\frac{1}{2}+2\gamma\right)}{\gamma} b\left(z;\beta,\gamma\right)\right]-1\right\}+ \\
& + \int_{\gamma L(\gamma)-\frac{1}{2}}^{\gamma(L(\gamma)+1)} dx \int_{x}^{\gamma(L(\gamma)+1)} dz \times \\
& \left.\times\left\{\cosh\left[\frac{\gamma\left(L\left(\gamma\right)+1\right)-x}{\gamma} b\left(z;\beta,\gamma\right)\right]-1\right\}\right],
\end{aligned} \tag{45}
$$

which is convex on $L_{\gamma}^{1,+}\left(\left[1,+\infty\right)\right)$, we get

$$
\mu_{\Lambda_\gamma}^{\beta,J_\gamma}\left[\cos\left(\theta_0-\theta_{L_d(\gamma)}\right)\right] \leq \exp\left\{f\left(b_d;\gamma L_d\left(\gamma\right),\gamma,\beta,d\right)\right\}. \tag{46}
$$

■

Proof of the Theorem. Let us set $b\left(z;\beta,\gamma\right) := B\left(\beta,\gamma\right)$ in f we obtain

$$
\begin{aligned}
\mu_{\Lambda_\gamma}^{\beta,J_\gamma}\left[\cos\left(\theta_0-\theta_{L(\gamma)}\right)\right] & \leq \exp\left\{\inf_{b\in L_{\gamma}^{1,+}([1,+\infty))} f\left(b;\gamma L\left(\gamma\right),\gamma,\beta\right)\right\} \\
& \leq \exp\left\{-B\left(\beta,\gamma\right)L\left(\gamma\right)+\frac{8\beta}{\gamma}\left[\gamma L\left(\gamma\right)-\left(\frac{1}{2}+\gamma\right)\right]\times\right. \\
& \times\left(\frac{1}{2}+2\gamma\right)\left\{\cosh\left[\left(\frac{1}{2}+2\gamma\right)\frac{B\left(\beta,\gamma\right)}{\gamma}\right]-1\right\} \\
& \left.+\frac{8\beta}{\gamma}\left(\frac{1}{2}+\gamma\right)^2\left\{\cosh\left[\left(\frac{1}{2}+\gamma\right)\frac{B\left(\beta,\gamma\right)}{\gamma}\right]-1\right\}\right\}.
\end{aligned} \tag{47}
$$

Since $\forall\gamma\in\left(0,\frac{1}{2}\right]$ there exists $\tau\left(\gamma\right)>0$ such that, if $\gamma^{-1}B\left(\beta,\gamma\right)<1$,

$$
\cosh\left[\left(\frac{1}{2}+\gamma\right)\gamma^{-1}B\left(\beta,\gamma\right)\right]-1 \leq \frac{1+\tau\left(\gamma\right)}{2}\left(\frac{1}{2}+2\gamma\right)^2\gamma^{-2}B^2\left(\beta,\gamma\right), \tag{48}
$$

setting $\eta(\gamma) := \left(\frac{1}{2}+\gamma\right) \in \left[\frac{1}{2},1\right]$ we have

$$\begin{aligned}
&\exp\left\{-B(\beta,\gamma)L(\gamma)+\frac{8\beta}{\gamma}\left[\left[\gamma L(\gamma)-\left(\frac{1}{2}+\gamma\right)\right]\times\right.\right. \\
&\times\left\{\cosh\left[\left(\frac{1}{2}+2\gamma\right)\frac{B(\beta,\gamma)}{\gamma}\right]-1\right\}+ \\
&\left.\left.+\left(\frac{1}{2}+\gamma\right)^2\left\{\cosh\left[\left(\frac{1}{2}+\gamma\right)\frac{B(\beta,\gamma)}{\gamma}\right]-1\right\}\right]\right\} \\
&\leq \exp\left\{-B(\beta,\gamma)L(\gamma)\left\{1-\frac{8\beta\gamma^{-3}B^2(\beta,\gamma)}{B(\beta,\gamma)L(\gamma)}\times\right.\right. \\
&\left.\times\frac{[\gamma L(\gamma)-\eta(\gamma)](1+\tau(\gamma))}{2}(\eta(\gamma)+\gamma)^2\right\}+ \\
&\left.+2\beta\eta^4(\gamma)(1+\tau(\gamma))\gamma^{-3}B^2(\beta,\gamma)\right\} \\
&=\exp\left\{-B(\beta,\gamma)L(\gamma)\left\{1-4\beta\gamma^{-2}B(\beta,\gamma)\times\right.\right. \\
&\left.\times\left(1-\frac{\eta(\gamma)}{\gamma L(\gamma)}\right)(1+\tau(\gamma))(\eta(\gamma)+\gamma)^2\right\}+ \\
&\left.+2\beta\eta^4(\gamma)(1+\tau(\gamma))\gamma^{-3}B^2(\beta,\gamma)\right\}.
\end{aligned} \tag{49}$$

Choosing, $\forall p>0$,

$$B(\beta,\gamma)=\frac{\gamma^{2+p}}{4(1+\tau(\gamma))(\eta(\gamma)+\gamma)^2\beta} \tag{50}$$

with $\beta>\frac{\gamma^{1+p}}{1+\tau}$ and

$$\tau := \min_{\gamma\in\left[0,\frac{1}{2}\right]}\tau(\gamma)\ , \tag{51}$$

we obtain

$$\begin{aligned}
&\inf_{b\in L_\gamma^{1,+}([1,+\infty))} f(b;\gamma L(\gamma),\gamma,\beta) \\
&<\exp\left\{-\frac{\gamma^{2+p}L(\gamma)}{4(1+\tau(\gamma))(\eta(\gamma)+\gamma)^2\beta}\left[1-\gamma^p\left(1-\frac{\eta(\gamma)}{\gamma L(\gamma)}\right)\right]+\right. \\
&\left.+2\beta\eta(\gamma)^4(1+\tau(\gamma))\gamma^{-3}\frac{\gamma^{4+2p}}{16(1+\tau(\gamma))^2(\eta(\gamma)+\gamma)^4\beta^2}\right\} \\
&<\left\{-\frac{\gamma^{2+p}L(\gamma)}{4(1+\tau(\gamma))(\eta(\gamma)+\gamma)^2\beta}[1-\gamma^p]+\right. \\
&\left.+\gamma^{1+2p}\frac{\eta(\gamma)^4}{8(1+\tau(\gamma))(\eta(\gamma)+\gamma)^4\beta}\right\}
\end{aligned} \tag{52}$$

uniformly in Λ_γ. Setting

$$c := \max_{\gamma\in\left[0,\frac{1}{2}\right]} (1+\tau(\gamma))(1+4\gamma)^2 \tag{53}$$

and passing to the thermodynamic limit, we get the result. ■

Proof of the Corollary. From (52), setting $\forall p>0$, $L(\gamma)=[L\gamma^{-2-p}]$ with $L\geq 1$, we obtain

$$\inf_{b\in L_\gamma^{1,+}([1,+\infty))} f(b;\gamma L(\gamma),\gamma,\beta) < \exp\left\{-\frac{\gamma^{2+\varepsilon}\left(L\gamma^{-2-p}-1\right)}{c\beta}\left(1-\gamma^p\right)+\frac{\gamma^{1+2p}}{8(1+\tau)\beta}\right\}. \tag{54}$$

■

Since the scalar product of two block-spins is a linear combination of scalar products of the single spins, it is possible to apply the estimates given in the preceding theorems to each term of this linear combination and obtain an upper bound of the correlation function of two empirical magnetizations. Anyway, making use of Griffiths' inequalities, these results can also be reproduced in the case in which we consider, instead of single spins, their empirical mean on lattice blocks of size $\delta>\gamma$ with $\delta,\gamma\in(0,1)$.

Let $\delta,\gamma\in(0,1)$ such that $\delta>\gamma$ and $\Lambda\subset\subset\mathbb{R}$, δ-measurable. Assuming free b.c. let us denote by $\left(J_\gamma^+(i,j)\right)_{i,j\in\mathbb{Z}}$ the interaction matrix $J_\gamma^+(i,j):=\left(\mathbf{1}\left(\gamma|i-j|\leq\left(\frac{1}{2}+\gamma\right)\right)\right)_{i,j\in\mathbb{Z}}$. If

$$Q^{+,\gamma}(i\gamma):=\left\{x\in\mathbb{R}:|x-i\gamma|\leq\left(\frac{1}{2}+\gamma\right)\right\} \tag{55}$$

and

$$Q_\gamma^{+,\gamma}(i\gamma)=\left\{j\in\mathbb{Z}:\gamma|i-j|\leq\left(\frac{1}{2}+\gamma\right)\right\}, \tag{56}$$

then

$$\begin{aligned}-H^{J_\gamma^+}\left(\sigma_{\Lambda_\gamma}\right)+\frac{1}{2}\gamma|\Lambda_\gamma| &:= \frac{\gamma}{2}\sum_{\substack{i\gamma\in\Lambda\\ j\gamma\in Q^{+,\gamma}(i\gamma)\cap\Lambda}}\sigma_i\cdot\sigma_j \\ &= \frac{\gamma}{2}\sum_{k\delta\in\Lambda}\sum_{i\in Q_\gamma^{(\delta)}(k\delta)}\sum_{n\delta\in Q^{+,\gamma}(i\gamma)\cap\Lambda}\sum_{j\in Q_\gamma^{(\delta)}(n\delta)}\sigma_i\cdot\sigma_j\,.\end{aligned} \tag{57}$$

But for $i\in Q_\gamma^{(\delta)}(k\delta)$

$$\mathbf{1}_{Q^{+,\gamma}(i\gamma)}(n\delta)\leq\mathbf{1}_{Q^{+,\delta}(k\delta)}(n\delta)\ , \tag{58}$$

and the positivity of the interaction matrix allow us to make use of the second Griffiths' inequality, that is to bound from above the correlation function of two empirical magnetizations of size δ for the Gibbs measure associated to $H^{J_\gamma^+}\left(\sigma_{\Lambda_\gamma}\right)$, with the one relative to the Gibbs measure associated to

$$
\begin{aligned}
& -H^{J_\gamma^{+,\delta}}\left(\sigma_{\Lambda_\gamma}\right)+\frac{1}{2}\gamma\left|\Lambda_\gamma\right| \\
& :=\frac{\gamma}{2}\sum_{k\delta\in\Lambda}\sum_{i\in Q_\gamma^{(\delta)}(k\delta)}\sum_{n\delta\in Q^{+,\delta}(k\delta)\cap\Lambda}\sum_{j\in Q_\gamma^{(\delta)}(n\delta)}\sigma_i\cdot\sigma_j \\
& =\frac{\gamma}{2}\sum_{k\delta\in\Lambda}\sum_{n\delta\in Q^{+,\delta}(k\delta)\cap\Lambda}\left(\frac{\delta}{\gamma}\right)^2\sigma_\gamma^{(\delta)}(k\delta)\cdot\sigma_\gamma^{(\delta)}(n\delta) \\
& =\frac{1}{2}\left(\frac{\delta}{\gamma}\right)\delta\sum_{k\delta\in\Lambda}\sum_{n\delta\in\Lambda}\mathbf{1}\left(|n\delta-k\delta|\leq\frac{1}{2}+\delta\right)\times \\
& \times\sigma_\gamma^{(\delta)}(k\delta)\cdot\sigma_\gamma^{(\delta)}(n\delta)\,.
\end{aligned}
\tag{59}
$$

We now consider, $\forall\beta>0$ and any fixed value $\delta>\gamma\in(0,1)$, the Gibbs measure $\mu^{+,\beta}$ associated to the interaction matrix

$$
J_\gamma^{+,\delta}(i,j):=\gamma\mathbf{1}_{Q^{+,\delta}(k\delta)}(j\gamma)\,\mathbf{1}\left(Q^{(\delta)}(k\delta)\ni i\gamma\right)\quad i,j\in\mathbb{Z} \tag{60}
$$

where $\mathbf{1}\left(Q^{(\delta)}(k\delta)\ni i\gamma\right)=\mathbf{1}_{Q^{(\delta)}(k\delta)}(\gamma i)$. Defining $\nu_n^{\left(\frac{\delta}{\gamma}\right)}(dm_n):=\mu\left\{\frac{\gamma}{\delta}\sum_{j\in Q_\gamma^{(\delta)}(n\delta)}\sigma_j\in dm_n\right\}$, since

$$
\sigma_\gamma^{(\delta)}(\delta n)=\left\{\begin{array}{l} m_n\cos\theta_n \\ m_n\sin\theta_n \end{array}\right.\qquad n\in\mathbb{Z}\,, \tag{61}
$$

for $\delta L_d(\delta,\gamma) \in \Lambda$, we have

$$\mu_\Lambda^{\beta,\gamma}\left[\sigma_\gamma^{(\delta)}(\delta 0)\cdot\sigma_\gamma^{(\delta)}(\delta L(\delta,\gamma))\right] \le \mu_\Lambda^{+,\beta}\left[\sigma_\gamma^{(\delta)}(\delta 0)\cdot\sigma_\gamma^{(\delta)}(\delta L(\delta,\gamma))\right] \tag{62}$$
$$= \mu_\Lambda^{+,\beta}\left[m_0 m_{L(\delta,\gamma)}\cos\left(\theta_0-\theta_{L(\delta,\gamma)}\right)\right]$$
$$= \left\{\int_{[0,1]^{|\Lambda_\delta|}}\prod_{n\in\Lambda_\delta} dm_n \prod_{n\in\Lambda_\delta}\frac{\nu_n^{\left(\frac{\delta}{\gamma}\right)}(dm_n)}{dm_n}\int_{[-\pi,\pi]^{|\Lambda_\delta|}}\prod_{n\in\Lambda_\delta}\frac{d\theta_n}{2\pi}\times\right.$$
$$\left.\times e^{\frac{\beta}{2}\left(\frac{\delta}{\gamma}\right)\delta c_d(\gamma)\sum_{n,k\in\Lambda_\delta}\mathbf{1}\left(\delta|n-k|\le\frac{1}{2}+\delta\right)m_n m_k\cos(\theta_n-\theta_k)}\right\}^{-1}$$
$$\left[\int_{[0,1]^{|\Lambda_\delta|}}\prod_{n\in\Lambda_\delta} dm_n \prod_{n\in\Lambda_\delta}\frac{\nu_n^{\left(\frac{\delta}{\gamma}\right)}(dm_n)}{dm_n}\int_{[-\pi,\pi]^{|\Lambda_\delta|}}\prod_{n\in\Lambda_\delta}\frac{d\theta_n}{2\pi}\times\right.$$
$$\times\, e^{\frac{\beta}{2}\left(\frac{\delta}{\gamma}\right)\delta c_d(\gamma)\sum_{n,k\in\Lambda_\delta}\mathbf{1}\left(\delta|n-k|\le\frac{1}{2}+\delta\right)m_n m_k\cos(\theta_n-\theta_k)}\times$$
$$\left.\times m_0 m_{L(\delta,\gamma)}\cos\left(\theta_0-\theta_{L(\delta,\gamma)}\right)\right]$$
$$= \left\{\int_{[0,1]^{|\Lambda_\delta|}}\prod_{n\in\Lambda_\delta} dm_n \prod_{n\in\Lambda_\delta} e^{\left(\frac{\delta}{\gamma}\right)\left[-\bar{I}(m_n)+\bar{\varepsilon}\left(m_n;\left(\frac{\delta}{\gamma}\right)\right)\right]}\int_{[-\pi,\pi]^{|\Lambda_\delta|}}\prod_{n\in\Lambda_\delta}\frac{d\theta_n}{2\pi}\times\right.$$
$$\left.\times e^{\frac{\beta}{2}\left(\frac{\delta}{\gamma}\right)\delta\sum_{n,k\in\Lambda_\delta}\mathbf{1}\left(\delta|n-k|\le\frac{1}{2}+\delta\right)m_n m_k\cos(\theta_n-\theta_k)}\right\}^{-1}\times$$
$$\left[\int_{[0,1]^{|\Lambda_\delta|}}\prod_{n\in\Lambda_\delta} dm_n \prod_{n\in\Lambda_\delta} e^{\left(\frac{\delta}{\gamma}\right)\left[-\bar{I}(m_n)+\bar{\varepsilon}\left(m_n;\left(\frac{\delta}{\gamma}\right)\right)\right]}\int_{[-\pi,\pi]^{|\Lambda_\delta|}}\prod_{n\in\Lambda_\delta}\frac{d\theta_n}{2\pi}\times\right.$$
$$\times\, e^{\frac{\beta}{2}\left(\frac{\delta}{\gamma}\right)\delta\sum_{n,k\in\Lambda_\delta}\mathbf{1}\left(\delta|n-k|\le\frac{1}{2}+\delta\right)m_n m_k\cos(\theta_n-\theta_k)}\times$$
$$\left.\times m_0 m_{L(\delta,\gamma)}\cos\left(\theta_0-\theta_{L(\delta,\gamma)}\right)\right],$$

with $\bar{\varepsilon}\left(m_n;\left(\frac{\delta}{\gamma}\right)\right)$ such that, $\forall r\in(0,1)$ and $m_n<r$, there exist $c(r)>0$ such that $\left|\bar{\varepsilon}\left(m_n;\left(\frac{\delta}{\gamma}\right)\right)\right|\le c(r)\frac{\gamma}{\delta}\log\frac{\delta}{\gamma}$ ([BPi] Theorem 2.2). Since $\forall n\in\Lambda_\delta$, $m_n\in[0,1]$, we can still make use of the second Griffiths' inequality because the interaction among the block-spins angular variables is given by a ferromagnetic random potential induced by the moduli of the block-spin. Hence,

$$\begin{aligned}
&\mu_{\Lambda}^{\beta,\gamma}\left[\sigma_{\gamma}^{(\delta)}(\delta 0)\cdot\sigma_{\gamma}^{(\delta)}(\delta L(\delta,\gamma))\right] \qquad (63)\\
&\leq \left\{\int_{[0,1]^{|\Lambda_\delta|}}\prod_{n\in\Lambda_\delta} dm_n \prod_{n\in\Lambda_\delta} e^{\left(\frac{\delta}{\gamma}\right)\left[-\bar{I}(m_n)+\bar{\varepsilon}\left(m_n;\left(\frac{\delta}{\gamma}\right)\right)\right]}\int_{[-\pi,\pi]^{|\Lambda_\delta|}}\prod_{n\in\Lambda_\delta}\frac{d\theta_n}{2\pi}\times\right.\\
&\left.\times e^{\frac{\beta}{2}\left(\frac{\delta}{\gamma}\right)\delta\sum_{n,k\in\Lambda_\delta}\mathbf{1}\left(\delta|n-k|\leq\frac{1}{2}+\delta\right)\cos(\theta_n-\theta_k)}\right\}^{-1}\times\\
&\left[\int_{[0,1]^{|\Lambda_\delta|}}\prod_{n\in\Lambda_\delta} dm_n \prod_{n\in\Lambda_\delta} e^{\left(\frac{\delta}{\gamma}\right)\left[-\bar{I}(m_n)+\bar{\varepsilon}\left(m_n;\left(\frac{\delta}{\gamma}\right)\right)\right]}\int_{[-\pi,\pi]^{|\Lambda_\delta|}}\prod_{n\in\Lambda_\delta}\frac{d\theta_n}{2\pi}\times\right.\\
&\left.\times e^{\frac{\beta}{2}\left(\frac{\delta}{\gamma}\right)\delta\sum_{n,k\in\Lambda_\delta}\mathbf{1}\left(\delta|n-k|\leq\frac{1}{2}+\delta\right)\cos(\theta_n-\theta_k)}\cos\left(\theta_0-\theta_{L(\delta,\gamma)}\right)\right]\\
&=\frac{\mu_{\Lambda_\delta}\left[\prod_{n\in\Lambda_\delta} e^{\frac{\beta}{2}\left(\frac{\delta}{\gamma}\right)\delta\sum_{n,k\in\Lambda_\delta}\mathbf{1}\left(\delta|n-k|\leq\frac{1}{2}+\delta\right)\cos(\theta_n-\theta_k)}\cos\left(\theta_0-\theta_{L(\delta,\gamma)}\right)\right]}{\mu_{\Lambda_\delta}\left[\prod_{n\in\Lambda_\delta} e^{\frac{\beta}{2}\left(\frac{\delta}{\gamma}\right)\delta\sum_{n,k\in\Lambda_\delta}\mathbf{1}\left(\delta|n-k|\leq\frac{1}{2}+\delta\right)\cos(\theta_n-\theta_k)}\right]}.
\end{aligned}$$

Then, proceding as in the case of spin-spin correlation, we obtain:

Theorem 4 *For $\gamma > 0$ sufficiently small and $\delta \in \left(\gamma, \frac{1}{2}\right]$, given $L(\delta,\gamma) \geq 1$, there exist two constants $c, \tau > 1$ such that, $\forall p > 0$ and $\beta > \frac{\delta^{1+p}}{1+\tau}$,*

$$\mu^{\beta,\gamma}\left[\sigma_{\gamma}^{(\delta)}(\delta 0)\cdot\sigma_{\gamma}^{(\delta)}(\delta L(\delta,\gamma))\right] \leq \exp\left\{-\frac{\delta^{1+p}\gamma L(\delta,\gamma)}{c\beta}(1-\delta^p)+\frac{\delta^{2p}\gamma}{8(1+\tau)\beta}\right\}. \qquad (64)$$

Proof. Changing β for $\beta\frac{\delta}{\gamma}$ and δ for γ in the proof of the preceding theorem, we derive an expression analogous to (49). Then, substituting into (50)

$$B(\beta,\delta,\gamma) = B\left(\beta\frac{\delta}{\gamma},\delta\right) = \frac{\delta^{2+p}\gamma}{4(1+\tau(\delta))(\eta(\delta)+\delta)^2\beta\delta}, \qquad (65)$$

we get the result. ■

Corollary 5 *Assuming the hypotesis of the preceding theorem and setting $\forall p > 0$, $L(\delta,\gamma) = \left[\frac{L}{\delta^{1+p}\gamma}\right]$, where L is a positive integer, there exist two positive constants c, $\tau \geq 1$ such that, for $\beta > \frac{\delta^{1+p}}{1+\tau}$,*

$$\begin{aligned}
\mu^{\beta,\gamma}\left[\sigma_{\gamma}^{(\delta)}(\delta 0)\cdot\sigma_{\gamma}^{(\delta)}(\delta L_d(\delta,\gamma))\right] &\leq \exp\left\{-\frac{L-1}{c\beta}(1-\delta^p)+\frac{\delta^{2p}\gamma}{8(1+\tau)\beta}\right\} \qquad (66)\\
&\leq \exp\left\{-\frac{L-1}{c\beta}(1-\delta^p)+\frac{\delta^{2p+1}}{8(1+\tau)\beta}\right\}.
\end{aligned}$$

Proof. Follows directly from (64). ■

Furthermore, it is possible to prove, by means of a polymer expansion, that, if the temperature of the system is high enough, the decay of the truncated correlation function of two block-spins, when their mutual distance diverges, is at most exponential.

Theorem 6 *For any $\gamma \in (0,1)$ and $\forall d, q \geq 1$, if $\beta > 0$ is small enough, $\forall \delta \geq \gamma$ there exists a constant $C > 0$ such that*

$$\begin{aligned} &\mu^{\beta,\gamma} \left| \left[\sigma_\gamma^{(\delta)} (\delta 0) \cdot \sigma_\gamma^{(\delta)} (\delta L_d (\delta)) \right] - \mu^{\beta,\gamma} \left[\sigma_\gamma^{(\delta)} (\delta 0) \right] \cdot \mu^{\beta,\gamma} \left[\sigma_\gamma^{(\delta)} (\delta L_d (\delta)) \right] \right| \\ &:= \left| \mu^{\beta,\gamma} \left[\sigma_\gamma^{(\delta)} (\delta 0) ; \sigma_\gamma^{(\delta)} (\delta L_d (\delta)) \right] \right| \leq C e^{-\xi(\delta,\beta)\delta L_d(\delta)} \end{aligned} \tag{67}$$

where $0 < \xi (\delta, \beta) = O (\delta |\ln \beta|)$.

For the proof of this result we address the reader to [PS] and for this specific model to [Gi] Theorem 6.2.3.

2.2 Lower bound

To have a more precise characterization of the asymptotic behaviour of the correlation function of two block-spins, it is not enough to give an estimate from above, since this would imply only that the decay in space of such function is not faster than the decay of its upper bound.

To get a lower bound estimate of block-spin correlation functions we will follow a strategy which is in general valid for d-dimensional Kac rotators models with $d \geq 1$. In particular, for $d = 2$ and for temperature sufficiently lower than the mean field critical one, this argument will allow us to show the decay of two block-spin correlations to be polinomial with an exponent proportional to a function of β, implying the system to undergo a Berezinskij-Kosterlitz-Thouless phase transition [Gi], [FrS], [KT] and [Be].

For any $\beta > \beta_{mf} = 1$ we will procede along these lines:

1) First we will confine the system to a subset $\Delta \subset\subset \mathbb{R}^d$ sufficiently large to which we will eventually impose free b.c.. Since the angular part of the interaction among the $\sigma_\gamma^{(\delta)}$'s is subject to a random ferromagnetic potential induced by their moduli, we will restrict ourseves to the event in $\mathcal{B}(S)$ such that all the moduli of the empirical magnetization defined inside Δ are larger than a given strictly positive value;

2) second, we will consider a given subset Λ of Δ. Inside Λ we will replace the original Kac potential with a weaker one and we will replace the values assumed by the moduli of the empirical magnetization with the values given in the preceding point;

3) third, we will set to zero some of the elements of the interaction matrix, reducing our system to a system where the block-spins interact only with their nearest neighbours. The intensity of the new interaction in Λ will be set proportional to the minimum

value assumed by the moduli of the empirical magnetizations given in 1) and will be eventually set to zero outside Λ (which means assuming outside Λ free b.c.). A this point the estimate from below of the correlation function of two block-spins is reduced to the estimate from below of the same quantity for a nearest neighbour model;

4) last, we will estimate the Gibbs measure of the event in $\mathcal{B}(S)$ considered at point 1).

We remark that a similar argument has been used in [Pe1] to obtain an analogous lower bound estimate on the correlation function of this model and in [Pe2] to show that the finite dimensional marginals of the angular part of the empirical magnetizations of size γ^{-1} weakly converge to a Brownian motion on the circle.

2.2.1 The 1-dimensional case

In this case, due to large deviations estimates for the Gibbs measure of the model and the possibility to approximate the block-spin model with a Villain one [Vi], it is possible to deviate slightly from the stategy previously described towards a finer analysis of block-spin correlation functions. More precisely:

- for any $\beta > 1$, it is possible to make use of the large deviation results given in [BPi] to obtain sharp estimates on the probability that, below the mean field critical temperature, the moduli of the empirical magnetizations of any size $\delta > \gamma > 0$ in the limit $\gamma \downarrow 0$ are arbitrary close to m_β for any choice of the b.c..
- We can restrict ourseves to study the model of a rotator with nearest neighbour potential whose coupling constant is proportional to $\frac{\delta}{\gamma}$ which diverges when $\gamma \downarrow 0$. Under these hypotesis the model can be approximated by a Gaussian one, the Villain's model (see Appendix) proving the decay of correlations to be exponential.

Theorem 7 *Let $d = 1$, $\beta > \beta_{mf} = 1$, and m_β the solution of the mean field equation. Then, for any $\gamma, \delta \in \left(0, \frac{1}{4}\right]$ with $\delta > \gamma$ and $\zeta \in (0, m_\beta)$, there exits a function $R^-_{\frac{\delta}{\gamma}}\left(\beta\delta\left(m_\beta - \zeta\right)^2\right)$ such that the limit $\lim_{\gamma\downarrow 0}\left|R^-_{\frac{\delta}{\gamma}}\left(\beta\delta\left(m_\beta - \zeta\right)^2\right)\right|$ exists and is finite and*

$$\begin{aligned}
&\mu^{\beta,\gamma}\left[\sigma^{(\delta)}_\gamma(\delta 0)\cdot\sigma^{(\delta)}_\gamma(\delta l(\delta,\gamma))\right] \\
&\geq (m_\beta - \zeta)^2\left(1 + \left(\frac{\gamma}{\delta}\right)^2 R^-_{\frac{\delta}{\gamma}}\left(\beta\delta\left(m_\beta - \zeta\right)^2\right)\right)\times \\
&\times\left(1 - 2\frac{\delta l(\delta,\gamma)}{\gamma}e^{-\frac{c(\delta,\zeta)}{\gamma}}\right)e^{-\frac{2\beta\delta^2\left(m_\beta-\zeta\right)^2\pi^2}{\gamma l(\delta,\gamma)}}\times \\
&\times e^{-\frac{\gamma l(\delta,\gamma)}{2\beta\delta^2\left(m_\beta-\zeta\right)^2}}.
\end{aligned} \tag{68}$$

Proof. Let $\beta > 1$, $\gamma \in (0,1)$, $\delta > \gamma$ and $\zeta \in (0, m_\beta)$. Let also $\forall p > 0$

$$V'_{\delta,p}(m_\beta;\zeta) := \left\{ m \in \mathcal{M} : \exists x \, tale \, che \; |x| \le \gamma^{-p}, \; \left| m^{(\delta)}(x) \right| \le m_\beta - \zeta \right\} \tag{69}$$

and $\Delta^p \subset\subset \mathbb{R}^d$ a sufficiently large δ-measurable set. By [BPi] Theorem 2.3, there exists a positive constant $c(\delta,\zeta)$ such that $\mu^{\beta,\gamma}\left(V'_{\delta,p}(m_\beta;\zeta)\right) \le \frac{2}{\gamma^{p+1}} e^{-\frac{c(\delta,\zeta)}{\gamma}}$. Let $Q^p = Q^p(\delta 0) := \{x \in \mathbb{R} : |x| \le \gamma^{-p}\}$ which is δ-measurable, then

$$\left(V'_{\delta,p}\right)^{(\delta)}(m_\beta;\zeta) = \bigcup_{n \in Q^p_\delta} V'^{(\delta)}_{\delta,p,n}(m_\beta;\zeta) := V'^{(\delta)}_{\delta,p}(m_\beta;\zeta) \tag{70}$$

$$V'^{(\delta)}_{\delta,p,n}(m_\beta;\zeta) := \left\{ m^{(\delta)} \in \mathcal{M}^{(\delta)} : \left| m^{(\delta)}(n\delta) \right| = m_n \le m_\beta - \zeta \right\} \tag{71}$$

$$\left(\left(V'_{\delta,p}\right)^{(\delta)}(m_\beta;\zeta) \right)^c = V'^{(\delta),c}_{\delta,p}(m_\beta;\zeta) = \bigcap_{n \in Q^p_\delta} V'^{(\delta),c}_{\delta,p,n}(m_\beta;\zeta) \tag{72}$$

$$V'^{(\delta),c}_{\delta,p,n}(m_\beta;\zeta) = \left(V'^{(\delta)}_{\delta,p,n}(m_\beta;\zeta) \right)^c. \tag{73}$$

Let $\Lambda \subset\subset \mathbb{R}$ such that $\Lambda_\gamma := \{j \in \mathbb{Z} : j\gamma \in [-L_\gamma + \delta, L_\gamma - \delta]\}$, where $L_\gamma = \frac{L}{\gamma}$ with L an integer strictly larger than 1 and $\delta l(\delta,\gamma) \in \Lambda \subset \bigcup_{k=0}^{\gamma^p \delta l(\delta,\gamma)} Q^p(\delta k)$ with $Q^p(\delta k) = Q^p + \delta k$ and $p > 0$ sufficiently large such that the system confined in Λ do not interact with $\sigma_{(\Delta^p)^c_\gamma} \in S_{(\Delta^p)^c_\gamma}$. Then, setting

$$\begin{aligned} V'^{(\delta),c}_{\delta,\Lambda}(m_\beta;\zeta) &:= \bigcap_{n \in \Lambda_\delta} V'^{(\delta),c}_{\delta,p,n}(m_\beta;\zeta) \subseteq \bigcap_{k=0}^{\gamma^p \delta l(\delta,\gamma)} \bigcap_{n \in Q_\delta(\delta k)} V'^{(\delta),c}_{\delta,p,n}(m_\beta;\zeta) \qquad (74) \\ &= \bigcap_{k=0}^{\gamma^p \delta l(\delta,\gamma)} \left(\left(V'^{,(k)}_{\delta,p} \right)^{(\delta)} (m_\beta;\zeta) \right)^c \end{aligned}$$

$$\left(\left(V'^{,(k)}_{\delta,p} \right)^{(\delta)} (m_\beta;\zeta) \right)^c := \bigcap_{n \in Q_\delta(\delta k)} V'^{(\delta),c}_{\delta,p,n}(m_\beta;\zeta) \,, \tag{75}$$

by the second Griffiths' inequality,

$$\begin{aligned} 0 &\le \mu^{\beta,\gamma}\left[\sigma^{(\delta)}_\gamma(\delta 0) \cdot \sigma^{(\delta)}_\gamma(\delta l(\delta,\gamma)) \mathbf{1}_{V'_{\delta,p}(m_\beta;\zeta)} \right] \qquad (76) \\ &\le \mu^{\beta,\gamma}\left[\left|\sigma^{(\delta)}_\gamma(\delta 0)\right| \left|\sigma^{(\delta)}_\gamma(\delta l(\delta,\gamma))\right| \frac{\sigma^{(\delta)}_\gamma(\delta 0)}{\left|\sigma^{(\delta)}_\gamma(\delta 0)\right|} \cdot \frac{\sigma^{(\delta)}_\gamma(\delta l(\delta,\gamma))}{\left|\sigma^{(\delta)}_\gamma(\delta l(\delta,\gamma))\right|} \mathbf{1}_{V'_{\delta,p}(m_\beta;\zeta)} \right] \\ &\le (m_\beta - \zeta)^2 \mu^{\beta,\gamma}\left[\frac{\sigma^{(\delta)}_\gamma(\delta 0)}{\left|\sigma^{(\delta)}_\gamma(\delta 0)\right|} \cdot \frac{\sigma^{(\delta)}_\gamma(\delta l(\delta,\gamma))}{\left|\sigma^{(\delta)}_\gamma(\delta l(\delta,\gamma))\right|} \mathbf{1}_{V'_{\delta,p}(m_\beta;\zeta)} \right] \\ &\le (m_\beta - \zeta)^2 \frac{2}{\gamma^{p+1}} \gamma^p \delta l(\delta,\gamma) e^{-\frac{c(\delta,\zeta)}{\gamma}} = 2 (m_\beta - \zeta)^2 \frac{\delta l(\delta,\gamma)}{\gamma} e^{-\frac{c(\delta,\zeta)}{\gamma}}. \end{aligned}$$

Therefore we get

$$
\begin{aligned}
&\mu^{\beta,\gamma}\left[\sigma_\gamma^{(\delta)}(\delta 0)\cdot\sigma_\gamma^{(\delta)}(\delta l(\delta,\gamma))\right] \\
&\geq \mu^{\beta,\gamma}\left[\sigma_\gamma^{(\delta)}(\delta 0)\cdot\sigma_\gamma^{(\delta)}(\delta l(\delta,\gamma))\,\mathbf{1}_{V'^{(\delta),c}_{\delta,p}\left(m_\beta;\zeta\right)}\right] \qquad (77)\\
&= \int \mu^{\beta,\gamma}\left(d(\sigma_\gamma)_{\Lambda^c}\right)\frac{\mu_\Lambda^\gamma\left[e^{-\beta H_\gamma\left(\cdot|(\sigma_\gamma)_{\Lambda^c}\right)}\sigma_\gamma^{(\delta)}(\delta 0)\cdot\sigma_\gamma^{(\delta)}(\delta l(\delta,\gamma))\,\mathbf{1}_{V'^{(\delta),c}_{\delta,p}\left(m_\beta;\zeta\right)}\left((\cdot)_\Lambda(\sigma_\gamma)_{\Lambda^c}\right)\right]}{\mu_\Lambda^\gamma\left[e^{-\beta H_\gamma\left(\cdot|(\sigma_\gamma)_{\Lambda^c}\right)}\right]}\\
&= \int \mu^{\beta,\gamma}\left(d(\sigma_\gamma)_{\Lambda^c}\right)\left[\frac{\mu_\Lambda^\gamma\left[e^{-\beta H_\gamma\left(\cdot|(\sigma_\gamma)_{\Lambda^c}\right)}\sigma_\gamma^{(\delta)}(\delta 0)\cdot\sigma_\gamma^{(\delta)}(\delta l(\delta,\gamma))\,\mathbf{1}_{V'^{(\delta),c}_{\delta,\Lambda}\left(m_\beta;\zeta\right)}\right]}{\mu_\Lambda^\gamma\left[e^{-\beta H_\gamma\left(\cdot|(\sigma_\gamma)_{\Lambda^c}\right)}\mathbf{1}_{V'^{(\delta),c}_{\delta,\Lambda}\left(m_\beta;\zeta\right)}\right]}\times\right.\\
&\left.\times\frac{\mu_\Lambda^\gamma\left[e^{-\beta H_\gamma\left(\cdot|(\sigma_\gamma)_{\Lambda^c}\right)}\mathbf{1}_{V'^{(\delta),c}_{\delta,\Lambda}\left(m_\beta;\zeta\right)}\right]}{\mu_\Lambda^\gamma\left[e^{-\beta H_\gamma\left(\cdot|(\sigma_\gamma)_{\Lambda^c}\right)}\right]}\mathbf{1}_{V'^{(\delta),c}_{\delta,\Delta^p\setminus\Lambda}\left(m_\beta;\zeta\right)}\left((\sigma_\gamma)_{\Lambda^c}\right)\right]\\
&\geq (m_\beta-\zeta)^2\int \mu^{\beta,\gamma}\left(d(\sigma_\gamma)_{\Lambda^c}\right)\frac{\mu_\Lambda^\gamma\left[e^{-\beta H_\gamma\left(\cdot|(\sigma_\gamma)_{\Lambda^c}\right)}\frac{\sigma_\gamma^{(\delta)}(\delta 0)}{\left|\sigma_\gamma^{(\delta)}(\delta 0)\right|}\cdot\frac{\sigma_\gamma^{(\delta)}(\delta l(\delta,\gamma))}{\left|\sigma_\gamma^{(\delta)}(\delta l(\delta,\gamma))\right|}\mathbf{1}_{V'^{(\delta),c}_{\delta,\Lambda}\left(m_\beta;\zeta\right)}\right]}{\mu_\Lambda^\gamma\left[e^{-\beta H_\gamma\left(\cdot|(\sigma_\gamma)_{\Lambda^c}\right)}\mathbf{1}_{V'^{(\delta),c}_{\delta,\Lambda}\left(m_\beta;\zeta\right)}\right]}\times\\
&\times\frac{\mu_\Lambda^\gamma\left[e^{-\beta H_\gamma\left(\cdot|(\sigma_\gamma)_{\Lambda^c}\right)}\mathbf{1}_{V'^{(\delta),c}_{\delta,\Lambda}\left(m_\beta;\zeta\right)}\right]}{\mu_\Lambda^\gamma\left[e^{-\beta H_\gamma\left(\cdot|(\sigma_\gamma)_{\Lambda^c}\right)}\right]}\mathbf{1}_{V'^{(\delta),c}_{\delta,\Delta^p\setminus\Lambda}\left(m_\beta;\zeta\right)}\left((\sigma_\gamma)_{\Lambda^c}\right)\,.
\end{aligned}
$$

We now consider

$$
\begin{aligned}
&\frac{\mu_\Lambda^\gamma\left[e^{-\beta H_\gamma\left(\cdot|(\sigma_\gamma)_{\Lambda^c}\right)}\frac{\sigma_\gamma^{(\delta)}(\delta 0)}{\left|\sigma_\gamma^{(\delta)}(\delta 0)\right|}\cdot\frac{\sigma_\gamma^{(\delta)}(\delta l_d(\delta,\gamma))}{\left|\sigma_\gamma^{(\delta)}(\delta l_d(\delta,\gamma))\right|}\mathbf{1}_{V'^{(\delta),c}_{\delta,\Lambda}\left(m_\beta;\zeta\right)}\right]}{\mu_\Lambda^\gamma\left[e^{-\beta H_\gamma\left(\cdot|(\sigma_\gamma)_{\Lambda^c}\right)}\mathbf{1}_{V'^{(\delta),c}_{\delta,\Lambda}\left(m_\beta;\zeta\right)}\right]} \qquad (78)\\
&:= \mu^{\beta,\gamma}_{\Lambda,V'^{(\delta),c}_{\delta,\Lambda}\left(m_\beta;\zeta\right)}\left[\frac{\sigma_\gamma^{(\delta)}(\delta 0)}{\left|\sigma_\gamma^{(\delta)}(\delta 0)\right|}\cdot\frac{\sigma_\gamma^{(\delta)}(\delta l_d(\delta,\gamma))}{\left|\sigma_\gamma^{(\delta)}(\delta l_d(\delta,\gamma))\right|}|\left(\sigma_\gamma^{(\delta)}\right)_{\Lambda^c}\right]\,.
\end{aligned}
$$

Again by the second Griffiths' inequality we have

$$\mu^{\beta,\gamma}_{\Lambda,V'^{(\delta),c}_{\delta,\Lambda}(m_\beta;\zeta)}\left[\frac{\sigma^{(\delta)}_\gamma(\delta 0)}{\left|\sigma^{(\delta)}_\gamma(\delta 0)\right|}\cdot\frac{\sigma^{(\delta)}_\gamma(\delta l(\delta,\gamma))}{\left|\sigma^{(\delta)}_\gamma(\delta l(\delta,\gamma))\right|}|\left(\sigma^{(\delta)}_\gamma\right)_{\Lambda^c}\right] \tag{79}$$

$$\geq \mu^{-,\beta}_{\Lambda,V'^{(\delta),c}_{\delta,p}(m_\beta;\zeta)}\left[\frac{\sigma^{(\delta)}_\gamma(\delta 0)}{\left|\sigma^{(\delta)}_\gamma(\delta 0)\right|}\cdot\frac{\sigma^{(\delta)}_\gamma(\delta l(\delta,\gamma))}{\left|\sigma^{(\delta)}_\gamma(\delta l(\delta,\gamma))\right|}|\left(\sigma^{(\delta)}_\gamma\right)_{\Lambda^c}\right]$$

$$= \left\{\int_{[0,1]^{|\Lambda_\delta|}}\prod_{n\in\Lambda_\delta}dm_n\prod_{n\in\Lambda_\delta}\frac{\nu^{\left(\frac{\delta}{\gamma}\right)}(dm_n)}{dm_n}\left(\prod_{n\in\Lambda_\delta}\mathbf{1}_{(m_\beta-\zeta,1]}(m_n)\right)\int_{[-\pi,\pi]^{|\Lambda_\delta|}}\prod_{n\in\Lambda_\delta}\frac{d\theta_n}{2\pi}\times\right.$$

$$\left.\times e^{\beta\left(\frac{\delta}{\gamma}\right)\delta\left[\sum_{n,k\in\Lambda_\delta}\frac{1}{2}+\sum_{n\in\Lambda_\delta}\sum_{k\in\Lambda^c_\delta}\right]\mathbf{1}\left(\delta|n-k|\leq\frac{1}{2}-\delta\right)m_n m_k\cos(\theta_n-\theta_k)}\right\}^{-1}\times$$

$$\times\left[\int_{[0,1]^{|\Lambda_\delta|}}\prod_{n\in\Lambda_\delta}dm_n\prod_{n\in\Lambda_\delta}\frac{\nu^{\left(\frac{\delta}{\gamma}\right)}(dm_n)}{dm_n}\left(\prod_{n\in\Lambda_\delta}\mathbf{1}_{(m_\beta-\zeta,1]}(m_n)\right)\int_{[-\pi,\pi]^{|\Lambda_\delta|}}\prod_{n\in\Lambda_\delta}\frac{d\theta_n}{2\pi}\times\right.$$

$$\left.\times e^{\beta\left(\frac{\delta}{\gamma}\right)\delta\left[\sum_{n,k\in\Lambda_\delta}\frac{1}{2}+\sum_{n\in\Lambda_\delta}\sum_{k\in\Lambda^c_\delta}\right]\mathbf{1}\left(\delta|n-k|\leq\frac{1}{2}-\delta\right)m_n m_k\cos(\theta_n-\theta_k)}\cos\left(\theta_0-\theta_{l(\delta,\gamma)}\right)\right]$$

$$\geq \left\{\int_{[0,1]^{|\Lambda_\delta|}}\prod_{n\in\Lambda_\delta}dm_n\prod_{n\in\Lambda_\delta}\frac{\nu^{\left(\frac{\delta}{\gamma}\right)}(dm_n)}{dm_n}\left(\prod_{n\in\Lambda_\delta}\mathbf{1}_{(m_\beta-\zeta,1]}(m_n)\right)\int_{[-\pi,\pi]^{|\Lambda_\delta|}}\prod_{n\in\Lambda_\delta}\frac{d\theta_n}{2\pi}\times\right.$$

$$\left.\times e^{\beta\left(\frac{\delta}{\gamma}\right)\delta\left(m_\beta-\zeta\right)^2\left[\sum_{n,k\in\Lambda_\delta}\frac{1}{2}+\sum_{n\in\Lambda_\delta}\sum_{k\in\Lambda^c_\delta}\right]\mathbf{1}\left(\delta|n-k|\leq\frac{1}{2}-\delta\right)\cos(\theta_n-\theta_k)}\right\}^{-1}\times$$

$$\times\left[\int_{[0,1]^{|\Lambda_\delta|}}\prod_{n\in\Lambda_\delta}dm_n\prod_{n\in\Lambda_\delta}\frac{\nu^{\left(\frac{\delta}{\gamma}\right)}(dm_n)}{dm_n}\left(\prod_{n\in\Lambda_\delta}\mathbf{1}_{(m_\beta-\zeta,1]}(m_n)\right)\int_{[-\pi,\pi]^{|\Lambda_\delta|}}\prod_{n\in\Lambda_\delta}\frac{d\theta_n}{2\pi}\times\right.$$

$$\left.\times e^{\beta\left(\frac{\delta}{\gamma}\right)\delta\left(m_\beta-\zeta\right)^2\left[\sum_{n,k\in\Lambda_\delta}\frac{1}{2}+\sum_{n\in\Lambda_\delta}\sum_{k\in\Lambda^c_\delta}\right]\mathbf{1}\left(\delta|n-k|\leq\frac{1}{2}-\delta\right)\cos(\theta_n-\theta_k)}\cos\left(\theta_0-\theta_{l(\delta,\gamma)}\right)\right]$$

$$:= \mu^{-,\beta,\left(m_\beta-\zeta\right)^2}_{\Lambda}\left[\cos\left(\theta_0-\theta_{l(\delta,\gamma)}\right)|\{\theta_n\}_{n\in\Lambda^c_\delta}\right].$$

Applying the third part of the scheme given before we reduce ourselves to estimate the two spins correlation function of a model described by a nearest neighbour potential. Hence, setting $\Lambda_\delta := \{l\in\mathbb{Z} : \delta l\in[-L_\gamma+\delta, L_\gamma-\delta]\}$ with $\frac{L_\gamma}{\delta}=L_\delta(\gamma)$, we have

$$\overline{Z}_{\Lambda_\delta}\left(\beta,\gamma,\delta|\{\theta_n\}_{n\in\Lambda^c_\delta}\right) := \int_{[-\pi,\pi]^{|\Lambda_\delta|}}\prod_{n\in\Lambda_\delta}\frac{d\theta_n}{2\pi}e^{\beta\left(\frac{\delta}{\gamma}\right)\delta\left(m_\beta-\zeta\right)^2\sum_{n=-L_\delta(\gamma)}^{L_\delta(\gamma)-1}\cos(\theta_n-\theta_{n+1})} \tag{80}$$

$$= \int_{[-\pi,\pi]^{|\Lambda_1|}}\prod_{k=-L_\gamma+\frac{\delta}{\gamma}}^{L_\gamma-\frac{\delta}{\gamma}}\frac{d\theta_k}{2\pi}\prod_{k=-L_\gamma+\frac{\delta}{\gamma}}^{L_\gamma}\widetilde{Z}_{\frac{\delta}{\gamma}}\left(\beta\delta\left(m_\beta-\zeta\right)^2\frac{\delta}{\gamma}|\theta_{k-\frac{\delta}{\gamma}},\theta_k\right),$$

where $\Lambda_1 = \left[-\frac{L}{\gamma}+\frac{\delta}{\gamma}, \frac{L}{\gamma}-\frac{\delta}{\gamma}\right] \cap \mathbb{Z}$ and $\widetilde{Z}_{\frac{\delta}{\gamma}}\left(\beta\delta\left(m_\beta-\zeta\right)^2 \frac{\delta}{\gamma}|\theta_{k-\frac{\delta}{\gamma}}, \theta_k\right)$ is the Gibbs factor of the Villain model (see the Appendix). Thus

$$
\begin{aligned}
&\mu_\Lambda^{-,\beta,\left(m_\beta-\zeta\right)^2}\left[\cos\left(\theta_0-\theta_{l(\delta,\gamma)}\right) \mid \{\theta_n\}_{n\in\Lambda_\delta^c}\right] \qquad (81)\\
&\geq \left\{\int_{[0,1]^{|\Lambda_\delta|}} \prod_{n\in\Lambda_\delta} dm_n \prod_{n\in\Lambda_\delta} \frac{\nu^{\left(\frac{\delta}{\gamma}\right)}(dm_n)}{dm_n}\left(\prod_{n\in\Delta_\delta^p} \mathbf{1}_{\left(m_\beta-\zeta,1\right]}(m_n)\right)\int_{[-\pi,\pi]^{|\Lambda_\delta|}}\prod_{n\in\Lambda_\delta}\frac{d\theta_n}{2\pi}\times\right.\\
&\left.\times e^{\beta\left(\frac{\delta}{\gamma}\right)\delta\left(m_\beta-\zeta\right)^2\sum_{n=-L_\delta(\gamma)}^{L_\delta(\gamma)-1}\cos(\theta_n-\theta_{n+1})}\right\}^{-1}\times\\
&\times\left[\int_{[0,1]^{|\Lambda_\delta|}} \prod_{n\in\Lambda_\delta} dm_n \prod_{n\in\Lambda_\delta} \frac{\nu^{\left(\frac{\delta}{\gamma}\right)}(dm_n)}{dm_n}\left(\prod_{n\in\Delta_\delta^p} \mathbf{1}_{\left(m_\beta-\zeta,1\right]}(m_n)\right)\int_{[-\pi,\pi]^{|\Lambda_\delta|}}\prod_{n\in\Lambda_\delta}\frac{d\theta_n}{2\pi}\times\right.\\
&\left.\times e^{\beta\left(\frac{\delta}{\gamma}\right)\delta\left(m_\beta-\zeta\right)^2\sum_{n=-L_\delta(\gamma)}^{L_\delta(\gamma)-1}\cos(\theta_n-\theta_{n+1})}\cos\left(\theta_0-\theta_{l(\delta,\gamma)}\right)\right]\\
&=\left\{\int_{[0,1]^{|\Lambda_\delta|}} \prod_{n\in\Lambda_\delta} dm_n \prod_{n\in\Lambda_\delta} \frac{\nu^{\left(\frac{\delta}{\gamma}\right)}(dm_n)}{dm_n}\left(\prod_{n\in\Delta_\delta^p} \mathbf{1}_{\left(m_\beta-\zeta,1\right]}(m_n)\right)\overline{Z}_{\Lambda_\delta}\left(\beta,\gamma,\delta|\{\theta_n\}_{n\in\Lambda_\delta^c}\right)\right\}^{-1}\times\\
&\times\left[\int_{[0,1]^{|\Lambda_\delta|}} \prod_{n\in\Lambda_\delta} dm_n \prod_{n\in\Lambda_\delta} \frac{\nu^{\left(\frac{\delta}{\gamma}\right)}(dm_n)}{dm_n}\left(\prod_{n\in\Delta_\delta^p} \mathbf{1}_{\left(m_\beta-\zeta,1\right]}(m_n)\right)\overline{Z}_{\Lambda_\delta}\left(\beta,\gamma,\delta|\{\theta_n\}_{n\in\Lambda_\delta^c}\right)\times\right.\\
&\left.\times\int_{[-\pi,\pi]^{|\Lambda_\delta|}}\prod_{n\in\Lambda_\delta}\frac{d\theta_n}{2\pi}\frac{e^{\beta\left(\frac{\delta}{\gamma}\right)\delta\left(m_\beta-\zeta\right)^2\sum_{n=-L_\delta(\gamma)}^{L_\delta(\gamma)-1}\cos(\theta_n-\theta_{n+1})}}{\overline{Z}_{\Lambda_\delta}\left(\beta,\gamma,\delta|\{\theta_n\}_{n\in\Lambda_\delta^c}\right)}\cos\left(\theta_0-\theta_{l(\delta,\gamma)}\right)\right].
\end{aligned}
$$

From (9), setting $l(\delta,\gamma)=\frac{\delta}{\gamma}l_\gamma$ with $\mathbb{N}\ni l_\gamma\geq 1$, it follows

$$
\begin{aligned}
&\int_{[-\pi,\pi]^{|\Lambda_\delta|}}\prod_{n\in\Lambda_\delta}\frac{d\theta_n}{2\pi}\frac{e^{\beta\left(\frac{\delta}{\gamma}\right)\delta\left(m_\beta-\zeta\right)^2\sum_{n=-L_\delta(\gamma)}^{L_\delta(\gamma)-1}\cos(\theta_n-\theta_{n+1})}}{\overline{Z}_{\Lambda_\delta}\left(\beta,\gamma,\delta|\{\theta_n\}_{n\in\Lambda_\delta^c}\right)}\cos\left(\theta_0-\theta_{l(\delta,\gamma)}\right) \qquad (82)\\
&=\frac{\int_{[-\pi,\pi]^{|\Lambda_1|}}\prod_{k=-L_\gamma+\frac{\delta}{\gamma}}^{L_\gamma-\frac{\delta}{\gamma}}\frac{d\theta_k}{2\pi}\prod_{k=-L_\gamma+\frac{\delta}{\gamma}}^{L_\gamma}\widetilde{Z}_{\frac{\delta}{\gamma}}\left(\beta\delta\left(m-\zeta\right)^2\frac{\delta}{\gamma}|\theta_{k-\frac{\delta}{\gamma}},\theta_k\right)e^{\pm i\left(\theta_0-\theta_{l_\gamma}\right)}}{\int_{[-\pi,\pi]^{|\Lambda_1|}}\prod_{k=-L_\gamma+\frac{\delta}{\gamma}}^{L_\gamma-\frac{\delta}{\gamma}}\frac{d\theta_k}{2\pi}\prod_{k=-L_\gamma+\frac{\delta}{\gamma}}^{L_\gamma}\widetilde{Z}_{\frac{\delta}{\gamma}}\left(\beta\delta\left(m_\beta-\zeta\right)^2\frac{\delta}{\gamma}|\theta_{k-\frac{\delta}{\gamma}},\theta_k\right)}\\
&\geq e^{-\frac{\gamma l(\delta,\gamma)}{2\beta\delta^2\left(m_\beta-\zeta\right)^2}}e^{-\frac{\beta\delta\gamma\left(m_\beta-\zeta\right)^2\pi^2}{L}\left\{\frac{\left(\theta_{-L_\gamma}-\theta_{L_\gamma}\right)}{2\pi}\right\}^2}\times\\
&\times\left(1+\left(\frac{\gamma}{\delta}\right)^2R^*_{\frac{\delta}{\gamma}}\left(\beta\delta\left(m_\beta-\zeta\right)^2;\theta_{-L_\gamma},\theta_{L_\gamma}\right)\right)
\end{aligned}
$$

with $\{x\} = x - [x]$, $\forall x \in \mathbb{R}$, and by (9)

$$R^*_{\frac{\delta}{\gamma}}\left(\beta\delta\left(m_\beta-\zeta\right)^2;\theta_{-L_\gamma},\theta_{L_\gamma}\right) = R^*_{\frac{\delta}{\gamma}}\left(\beta\delta\left(m_\beta-\zeta\right)^2;\left(\frac{\sigma_\gamma^{(\delta)}}{\left|\sigma_\gamma^{(\delta)}\right|}\right)_{\partial\Lambda^c}\right). \tag{83}$$

Hence, since $L_\gamma \geq l_\gamma$,

$$\begin{aligned}
&\mu^{\beta,\gamma}_{\Lambda,V'^{(\delta),c}_{\delta,\Lambda}(m_\beta;\zeta)}\left[\frac{\sigma_\gamma^{(\delta)}(\delta 0)}{\left|\sigma_\gamma^{(\delta)}(\delta 0)\right|}\cdot\frac{\sigma_\gamma^{(\delta)}(\delta l(\delta,\gamma))}{\left|\sigma_\gamma^{(\delta)}(\delta l(\delta,\gamma))\right|}\mid\left(\sigma_\gamma^{(\delta)}\right)_{\Lambda^c}\right] \\
&\geq \mu_\Lambda^{-,\beta,\left(m_\beta-\zeta\right)^2}\left[\cos\left(\theta_0-\theta_{l(\delta,\gamma)}\right)\mid\{\theta_n\}_{n\in\partial\Lambda^c_\delta}\right] \\
&\geq e^{-\frac{\gamma l(\delta,\gamma)}{2\beta\delta^2\left(m_\beta-\zeta\right)^2}}e^{-\frac{\beta\delta\left(m_\beta-\zeta\right)^2\pi^2}{l_\gamma}\left\{\frac{\left(\theta_{-L_\gamma}-\theta_{L_\gamma}\right)}{2\pi}\right\}^2}\times \\
&\times\left(1+\left(\frac{\gamma}{\delta}\right)^2R^*_{\frac{\delta}{\gamma}}\left(\beta\delta\left(m_\beta-\zeta\right)^2;\{\theta_n\}_{n\in\partial\Lambda^c_\delta}\right)\right)
\end{aligned} \tag{84}$$

and from (77) we obtain

$$\begin{aligned}
&\mu^{\beta,\gamma}\left[\sigma_\gamma^{(\delta)}(\delta 0)\cdot\sigma_\gamma^{(\delta)}(\delta l(\delta,\gamma))\,\mathbf{1}_{V'^{(\delta),c}_{\delta,p}(m_\beta;\zeta)}\right] \\
&\geq\left(m_\beta-\zeta\right)^2\int\mu^{\beta,\gamma}\left(d\left(\sigma_\gamma\right)_{\Lambda^c}\right)\left[\frac{\mu_\Lambda^\gamma\left[e^{-\beta H_\gamma\left(\cdot|\left(\sigma_\gamma\right)_{\Lambda^c}\right)}\mathbf{1}_{V'^{(\delta),c}_{\delta,\Lambda}(m_\beta;\zeta)}\right]}{\mu_\Lambda^\gamma\left[e^{-\beta H_\gamma\left(\cdot|\left(\sigma_\gamma\right)_{\Lambda^c}\right)}\right]}\mathbf{1}_{V'^{(\delta),c}_{\delta,\Delta^p\setminus\Lambda}(m_\beta;\zeta)}\left(\left(\sigma_\gamma\right)_{\Lambda^c}\right)\times\right. \\
&\times e^{-\frac{\gamma l(\delta,\gamma)}{2\beta\delta^2\left(m_\beta-\zeta\right)^2}}e^{-\frac{2\beta\delta\left(m_\beta-\zeta\right)^2\pi^2}{l_\gamma}\left\{\frac{1}{2\pi}\left(\frac{\sigma_\gamma^{(\delta)}}{\left|\sigma_\gamma^{(\delta)}\right|}(-L_\delta(\gamma))-\frac{\sigma_\gamma^{(\delta)}}{\left|\sigma_\gamma^{(\delta)}\right|}(L_\delta(\gamma))\right)\right\}^2}\times \\
&\left.\times\left(1+\left(\frac{\gamma}{\delta}\right)^2R^*_{\frac{\delta}{\gamma}}\left(\beta\delta\left(m_\beta-\zeta\right)^2;\left(\frac{\sigma_\gamma^{(\delta)}}{\left|\sigma_\gamma^{(\delta)}\right|}\right)_{\partial\Lambda^c}\right)\right)\right] \\
&\geq\left(m_\beta-\zeta\right)^2\left(1+\left(\frac{\gamma}{\delta}\right)^2R^-_{\frac{\delta}{\gamma}}\left(\beta\delta\left(m_\beta-\zeta\right)^2\right)\right)e^{-\frac{2\beta\delta\left(m_\beta-\zeta\right)^2\pi^2}{l_\gamma}}\times \\
&\times e^{-\frac{\gamma l(\delta,\gamma)}{2\beta\delta^2\left(m_\beta-\zeta\right)^2}}\int\mu^{\beta,\gamma}\left(d\left(\sigma_\gamma\right)_{\Lambda^c}\right)\frac{\mu_\Lambda^\gamma\left[e^{-\beta H_\gamma\left(\cdot|\left(\sigma_\gamma\right)_{\Lambda^c}\right)}\mathbf{1}_{V'^{(\delta),c}_{\delta,\Lambda}(m_\beta;\zeta)}\right]}{\mu_\Lambda^\gamma\left[e^{-\beta H_\gamma\left(\cdot|\left(\sigma_\gamma\right)_{\Lambda^c}\right)}\right]}\mathbf{1}_{V'^{(\delta),c}_{\delta,\Delta^p\setminus\Lambda}(m_\beta;\zeta)}\left(\left(\sigma_\gamma\right)_{\Lambda^c}\right),
\end{aligned} \tag{85}$$

with

$$R^-_{\frac{\delta}{\gamma}}\left(\beta\delta\left(m_\beta-\zeta\right)^2\right):=\min_{\left\{\theta_{-L_\gamma},\theta_{L_\gamma}\right\}\in[-\pi,\pi]^2}R^*_{\frac{\delta}{\gamma}}\left(\beta\delta\left(m_\beta-\zeta\right)^2;\theta_{-L_\gamma},\theta_{L_\gamma}\right). \tag{86}$$

But

$$\int \mu^{\beta,\gamma}\left(d\left(\sigma_{\gamma}\right)_{\Lambda^{c}}\right)\frac{\mu_{\Lambda}^{\gamma}\left[e^{-\beta H_{\gamma}\left(\cdot|\left(\sigma_{\gamma}\right)_{\Lambda^{c}}\right)}\mathbf{1}_{V_{\delta,\Lambda}^{\prime(\delta),c}\left(m_{\beta};\zeta\right)}\right]}{\mu_{\Lambda}^{\gamma}\left[e^{-\beta H_{\gamma}\left(\cdot|\left(\sigma_{\gamma}\right)_{\Lambda^{c}}\right)}\right]}\mathbf{1}_{V_{\delta,\Delta^{p}\backslash\Lambda}^{\prime(\delta),c}\left(m_{\beta};\zeta\right)}\left(\left(\sigma_{\gamma}\right)_{\Lambda^{c}}\right) \tag{87}$$

$$= \mu^{\beta,\gamma}\left[V_{\delta,p}^{\prime(\delta),c}\left(m_{\beta};\zeta\right)\right] = \mu^{\beta,\gamma}\left[\bigcap_{k=0}^{\gamma^{p}l(\delta,\gamma)}\left(\left(V_{\delta,p}^{\prime,(k)}\right)^{(\delta)}\left(m_{\beta};\zeta\right)\right)^{c}\right]$$

$$\geq 1-2\frac{\delta l\left(\delta,\gamma\right)}{\gamma}e^{-\frac{c(\delta,\zeta)}{\gamma}}\,.$$

■

3 Appendix

Let

$$I\left(k;\beta N\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi}d\theta e^{N\beta\cos\theta}\cos k\theta \tag{88}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}d\theta e^{N\beta\cos\theta}e^{ik\theta}\quad k\in\mathbb{Z},\ N\in\mathbb{N}\,.$$

We set $I\left(k=0;\beta N\right):=I_{0}\left(\beta N\right)$.

3.1 The Villain model limit

The proofs of the following results follow from direct computations and therefore have been omitted. For further details we address the reader to Appendix A and B of [Gi].

Proposition 8 *Let $\Lambda:=\{1,..,N\}\cap\mathbb{Z}$. For any fixed $\sigma_{\Lambda^{c}}\in S_{\Lambda^{c}}$, that is any $\theta_{0},\theta_{N+1}\in[-\pi,\pi)$, in the limit $N\uparrow\infty$ the partition function of a nearest neighbour rotator model with coupling constant N writes*

$$Z_{\Lambda}^{(1)}\left(\beta N;1|\sigma_{\Lambda^{c}}\right) := Z_{N}\left(\beta N|\theta_{0},\theta_{N+1}\right) \tag{89}$$

$$= I_{0}^{N+1}\left(\beta N\right)\left[\sqrt{2\pi\beta}V_{\beta}\left(\theta_{0}-\theta_{N+1}\right)+\right.$$

$$\left.+O\left(\frac{1}{N^{2}}\right)\right]\,,$$

where $V_{\beta}\left(\left(\theta-\theta'\right)\right):=\sum_{k\in\mathbb{Z}}e^{-\beta\frac{\left(\left(\theta-\theta'\right)-2\pi k\right)^{2}}{2}}$ is the Gibbs factor relative to the Villain model [Vi].

3.2 2-spins correlation function of the Villain model

Proposition 9 *Let* $\beta, R > 0$,

$$\Lambda_\gamma := \{l \in \mathbb{Z} : \gamma l \in [-L+\gamma, L-\gamma]\} \subset \gamma\mathbb{Z} \tag{90}$$

with L *a positive integer and* $\Lambda_1 = [-L+1, L-1] \cap \mathbb{Z}$. *Then, for any* $1 < T \leq L-1$ *and* $\sigma_{\Lambda_1^c} \in S_{\Lambda_1^c}$, *there exists a function* $R^*_{\frac{1}{\gamma}}\left(\beta R; \sigma_{\Lambda_1^c}\right) = O(\gamma)$ *such that*

$$e^{-\frac{T}{\beta R}} e^{-\frac{\beta R \pi^2}{L}\left\{\frac{(\theta_{-L}-\theta_L)}{2\pi}\right\}^2}\left(1+\gamma^2 R^*_{\frac{1}{\gamma}}\left(\beta R; \sigma_{\partial\Lambda_1^c}\right)\right) \leq \mu^{\beta R \frac{1}{\gamma}}_{\Lambda_\gamma}\left(\sigma_{-T}\cdot\sigma_T | \sigma_{\Lambda_1^c}\right) \leq 1\,. \tag{91}$$

References

[Be] **V. L. Berezinskij** *Destruction of Long-Range Order in one-dimensional and two-dimensional systems having a continuous symmetry group I* Classical systems JEPT **32** No. 3, 493-616 (1971).

[BC] **C. A. Bonato, M. Campanino**, *Absence of symmetry breaking for systems of rotors with random interactions* J. Stat. Phys. **54**, 81-88 (1989).

[BPi] **P. Buttà, P. Picco** *Large-Deviation Principle for One-Dymensional Vector Spin Models with Kac Potentials* J. Stat. Phys. **92**, Nos. 1/2, 101-150 (1998).

[FrS] **J. Fröhlich, T. Spencer** *The Kostrelitz-Thouless transition in two dimensional abelian spin systems and the Coulomb gas* Comm. Math. Phys. **81**, 527-607 (1981).

[Ge] **H.-O. Georgii** *Gibbs measure and phase transitions* Walter de Gruyter (1988).

[Gi] **M. Gianfelice** *Transizione di fase di Kosterlitz-Thouless per un modello di Kac a simmetria continua* http://www.dm.unibo.it/~gianfeli/tesi/tesi.html (PhD thesis, in italian) (2002).

[Gi1] **M. Gianfelice** *On the Kosterlitz-Thouless phase transition for Kac rotators* (preprint).

[KT] **J. M. Kosterlitz, D. J. Thouless** *Ordering, metastability and phase transitions in two-dimensional systems* J. Phys. C **6**, 1181-1203 (1973).

[LP] **J. Lebowitz, O. Penrose** *Rigorous treatment of the van der Waals Maxwell theory of the liquid vapour transition.* J. Math. Phys. **7,** 98-113 (1966).

[MMSR] **A. Messager, S. Miracle Solé, J. Ruiz** *Upper bounds on the decay of correlations in* $SO(n)$*-symmetric spin sysytems with long range interactions* Ann. Inst. H. Poincaré Phys. Théo. **40**, 85-96 (1984).

[McBS] **O. McBryan, T. Spencer** *On the Decay of Correlations in the SO(n)-symmetric Ferromagnets* Comm. Math. Phys. **53**, 209-302 (1977).

[Pe1] **M. Peterman** *Correlation lenght of the one-dimensional plane rotator model with Kac potential* preprint.

[Pe2] **M. Peterman** *Diffusive behaviour at the critical lenght of the one-dimensional plane rotator model with Kac potential* preprint.

[Pi1] **P. Picco** *On the absence of breakdown of symmetry for the plane rotator model with long range random interactions* J. Stat. Phys. **32**, 627-647 (1983).

[Pi2] **P. Picco** *Upper bound on the decay of correlations in the plane rotator model with long range random interactions* J. Stat. Phys. **36**, 489-516 (1984).

[Pf] **C.-E. Pfister** *On the Symmetry of the Gibbs States in Two Dimensional Lattice Systems* Commun. Math. Phys. **79**, 181-188 (1981).

[PS] **A. Procacci, B. Scoppola** *On the decay of correlations for unbounded spin systems with arbitrary boundary conditions* J. Stat. Phys. **105** n.3-4, 453-482 (2001).

[Si] **B. Simon** *The statistical mechanics of lattice gases Vol. I* Princeton University Press (1993).

[TS] **C. J. Tompson, M. Silver** *The classical limit of n-vector spin models* Comm Math. Phys. **33**, 53-60 (1973).

[Vi] **J. Villain** *Theory of one- and two-dimensional magnets with an easy megnetizzation plane II. The planar, classical, two-dimensional magnet* J. Physique **36**, 581-590 (1975).