On the Kosterlitz-Thouless phase transition for Kac rotators

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Abstract

We study the correlation function of two block-spins of size $\delta > \gamma$, γ being the Kac parameter, for the model of two-dimensional planar rotator with ferromagnetic Kac potential. We prove that for sufficiently small values of the the temperature, below the mean field critical temperature, the decay of the two block-spins correlation function is polynomial with an exponent which is a function of β.

1 Introduction and notations

We analyse the behaviour of spin-spin and two block-spins correlation functions of a system of planar rotators in dimension two, interacting through a ferromagnetic Kac potential. We first give an upper bound of such quantities making use of a suitably modified version of the McBrian and Spencer approach [McBS]. Then we will provide a lower bound for the two block-spins correlation function based on the same renormalization procedure formerly introduced by Fröhlich and Spencer [FrS], [FrS1] to study the decay of two spins correlations for the planar rotator model with nearest neighbour interactions. These estimates will show that, for sufficiently small values of the temperature, below the mean field critical temperature, the decay of two block empirical magnetization of size $\delta \in \left(\gamma, \left(2\left(\sqrt{2}+1\right)\right)^{-1}\right]$, $\gamma \in (0,1)$ being the Kac parameter, is polynomial with an exponent which is a function of the inverse temperature β. Since for sufficiently high values of the temperature, performing a polymer

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expansion [PS], [G] it can be proved that the decay of spin-spin and block-spin pair correlations is instead exponential, this proves the system to undergo a Berezinskij-Kosterlitz-Thouless phase transition [Be], [KT] and [JKKN].

1.1 The model

Given $\gamma \in (0,1]$, and Λ a bounded subset of $\mathbb{R}^2$ ($\Lambda \subset\subset \mathbb{R}^2$), we set

$$\Lambda_\gamma := \left\{ n \in \mathbb{Z}^2 : \gamma n \in \Lambda \right\}. \tag{1}$$

To each site of the lattice $\mathbb{Z}^2$ we attach a spin variable

$$\mathbb{Z}^2 \ni i \longmapsto \sigma_i \in S^1 \tag{2}$$

and, denoting by ν the Haar mesure on S^1, we consider the probability space $(S, \mathcal{B}(S), \mu)$, where $S := (S^1)^{\mathbb{Z}^2}$ is the configuration space, $\mu := \bigotimes_{i \in \mathbb{Z}^2} \nu_i$ and $\mathcal{B}(S)$ is the σalgebra of subsets of S generated by the finite-dimensional cylinders. We also take, with an abuse of notation, σ_i to be the projection of the configuration $\sigma \in S$ on the site i. Therefore, for any $\Lambda \subset \mathbb{R}^2$, $\sigma_{\Lambda_\gamma} := \{\sigma_i\}_{i \in \Lambda_\gamma}$ denotes the restriction of the configuration $\sigma \in S$ to Λ_γ and S_{Λ_γ} the set of the spin configurations on Λ_γ.

For $\gamma \in (0,1]$, the interaction among the σ's in a finite region Λ_γ, $\Lambda \subset\subset \mathbb{R}^2$, with fixed boundary condition $\sigma_{\Lambda_\gamma^c} \in S_{\Lambda_\gamma^c}$, is defined through the Hamiltonian

$$H^{J_\gamma}\left(\sigma_{\Lambda_\gamma} | \sigma_{\Lambda_\gamma^c}\right) := -\left[\sum_{i,j \in \Lambda_\gamma} \frac{1}{2}(1 - \delta_{i,j}) + \sum_{\substack{i \in \Lambda_\gamma \\ j \in \Lambda_\gamma^c}} \right] J_\gamma(i,j)\, \sigma_i \cdot \sigma_j, \tag{3}$$

where $u \cdot v$ denotes the scalar product of the vectors $u, v \in \mathbb{R}^2$ and

$$J_\gamma(i,j) := \gamma^2 J(\gamma\, ||i - j||) \quad i, j \in \mathbb{Z}^2 \tag{4}$$

is the Kac interaction matrix associated to the function

$$\mathbb{R}^+ \ni x \longmapsto J(x) \in \mathbb{R}^+, \tag{5}$$

which satisfies the following conditions:

- is compactly supported;
- $\left|\left|\frac{dJ}{dx}\right|\right|_\infty < \infty$;
- $\int_{\mathbb{R}} dx J(|x|) = 1$.

For thechnical convenience, in the rest of the paper, we will choose J to assume the particular form

$$J(t;\gamma) := \frac{16}{\pi\left(1+4\gamma^2\right)}\left[\mathbf{1}_{\left[0,\frac{1}{2}-\gamma\right)}(t) + \bar{f}\left(t-\left(\frac{1}{2}-\gamma\right)\right)\mathbf{1}_{\left[\frac{1}{2}-\gamma,\frac{1}{2}+\gamma\right]}(t)\right], \tag{6}$$

where $\bar{f} \in C^1\left(\mathbb{R}^+;\mathbb{R}^+\right)$ with support $[0,2\gamma]$. Setting $f := \bar{f}\circ||\cdot||$, we also assume:

- $f(0) = 1$;
- $f(x)\, 1_{\{x\in\mathbb{R}^2:\,||x||=2\gamma\}} = 0$;
- $\lambda(f) = \frac{1}{2}\lambda\left(\mathbf{1}_{\{x\in\mathbb{R}^2:\,||x||\leq 2\gamma\}}\right)$.

Let, $\forall x \in \mathbb{R}^2$, $|x| := |x_1| \vee |x_2|$. Hence, since $J(\cdot;\gamma)$ is a non increasing function, we have $J(||x||;\gamma) \geq J(|x|;\gamma)$. Furthermore, $\forall\delta \geq \gamma$,

$$J(||\delta k||;\gamma) \geq \frac{16}{\pi\left(1+4\gamma^2\right)}\left[\mathbf{1}\left(|\delta k| \leq \left(\frac{1}{2}-\gamma\right)\frac{1}{\sqrt{2}}\right)\mathbf{1}_{\left(0,\frac{1}{2}\right)}(\gamma) + \delta_{k,0}\mathbf{1}_{\left[\frac{1}{2},1\right]}(\gamma)\right], \tag{7}$$

$$J(||\delta k||;\gamma) \leq \frac{16}{\pi\left(1+4\gamma^2\right)}\mathbf{1}\left(|\delta k| < \left(\frac{1}{2}+\gamma\right)\right), \tag{8}$$

where,

$$\mathbf{1}(|\delta k| \leq r) := \mathbf{1}_{\{k\in\mathbb{Z}^2:|\delta k|\leq r\}} \quad r \in \mathbb{R}^+. \tag{9}$$

The Gibbs measure at the temperature β^{-1} in a finite region Λ_γ, $\Lambda \subset\subset \mathbb{R}$ with boundary condition $\sigma_{\Lambda_\gamma^c} \in S_{\Lambda_\gamma^c}$ is

$$\mu_{\Lambda_\gamma}^{\beta,J_\gamma}\left(d\sigma|\sigma_{\Lambda_\gamma^c}\right) := \frac{e^{-\beta H^{J_\gamma}\left(\sigma_{\Lambda_\gamma}|\sigma_{\Lambda_\gamma^c}\right)}}{Z_{\Lambda_\gamma}\left(\beta,\gamma|\sigma_{\Lambda_\gamma}\right)}\bigotimes_{i\in\Lambda_\gamma}\nu(d\sigma_i), \tag{10}$$

where $Z_{\Lambda_\gamma}\left(\beta,\gamma|\sigma_{\Lambda_\gamma^c}\right) := \mu\left(e^{-\beta H\left(\sigma_{\Lambda_\gamma}|\sigma_{\Lambda_\gamma^c}\right)}\right)$. We denote by μ^{β,J_γ} the Gibbs state specified by $\mu_{\Lambda_\gamma}^{\beta,J_\gamma}(d\sigma|\cdot)$ that is

$$\mu^{\beta,J_\gamma}\left(d\sigma_{\Lambda_\gamma}|\sigma_{\Lambda_\gamma^c}\right) = \mu_{\Lambda_\gamma}^{\beta,J_\gamma}\left(d\sigma|\sigma_{\Lambda_\gamma^c}\right) \quad \mu^{\beta,J_\gamma} - a.s. \quad \forall\Lambda_\gamma,\ \Lambda \subset\subset \mathbb{R}^2. \tag{11}$$

The uniqueness of the Gibbs measure for plane rotator models is a long-standing open problem and is tightly related to the absence of continuous symmetry breaking for two-dimensional spin systems; we refer the reader to the bibliografical notes to section 9.2. of [Ge] and to [BPi] for a complete historical account on this problem. Anyway, we remark that uniqueness and extremality of the translation invariant Gibbs measure has been established for ferromagnetic translation invariant interaction in absence of an external field in [BrFL] and [MMSPf] by means of correlation inequalities (see also [FrPf] for nearest-neighbour classical XY model). Rotational invariance of μ^{β,J_γ} can also be directly proven ([G] Theorem 302) arguing in a similar fashion to [Pi1] and [Pf] (see also [BC] for the case of random interactions).

Let $\mathbf{B} := \{v \in \mathbb{R}^2 : ||v|| \leq 1\}$ and $\mathcal{M}$ be the space λ-measurable maps $\mathbb{R}^2 \ni x \longmapsto m(x) \in \mathbf{B}$ endowed with the weak topology with respect to the functions $L^2_{loc}(\mathbb{R}^2;\mathbb{R}^2)$. $\mathcal{M}$ is easily seen to be a convex and compact space. For any $\Lambda \subset \mathbb{R}^2$, we denote by m_Λ the restriction of $m \in \mathcal{M}$ to Λ, and by $\mathcal{M}_\Lambda := \{v \in \mathcal{M} : v = m_\Lambda ; m \in \mathcal{M}\}$.

Defining, for any $\gamma \in (0,1]$, the continuous injective map

$$S \ni \sigma \longmapsto \iota_\gamma(\sigma) = \sigma_\gamma := \sum_{i\in\mathbb{Z}^2} \sigma_i \mathbf{1}_{Q^{(\gamma)}(i\gamma)} \in \mathcal{M}, \tag{12}$$

where

$$Q^{(\gamma)}(i\gamma) := \left\{x \in \mathbb{R}^2 : x \in [\gamma i_i, \gamma(i_1+1)) \times [\gamma i_2, \gamma(i_2+1))\, ;\, i = (i_1,i_2) \in \mathbb{Z}^2\right\} \tag{13}$$

we denote by $A_\gamma := \{\sigma \in S : \sigma_\gamma \in A\} \in \mathcal{B}(S)$ the image of any $A \in \mathcal{B}(\mathcal{M})$ via the map ι_γ^{-1} and by $\bar{\mu}^\gamma$ the image of any probability measure $\bar{\mu}$ on $(S,\mathcal{B}(S))$ in the set of probability measures on $(\mathcal{M},\mathcal{B}(\mathcal{M}))$ through the probability kernel

$$\mathcal{B}(\mathcal{M}) \times S \ni (A,\sigma) \longmapsto \mathbf{1}_A \circ \iota_\gamma(\sigma) = \mathbf{1}_{\{\sigma\in S : \sigma_\gamma \in A\}}. \tag{14}$$

Hence $\forall A \in \mathcal{B}(\mathcal{M})$, $\gamma \in (0,1]$, $\bar{\mu}^\gamma(A) := \bar{\mu}(A_\gamma)$. In particular we set $\mu^{\beta,\gamma} := \left(\mu^{\beta,J_\gamma}\right)^\gamma$.

Moreover, if $f : S \longmapsto \mathbb{R}$ is a measurable function, we denote by $f_\gamma := f \circ \iota_\gamma^{-1}$ the image of f in the set of measurable real functions defined on $(\mathcal{M},\mathcal{B}(\mathcal{M}))$.

For any $\Lambda \subset\subset \mathbb{R}^2$, $\lambda(\Lambda) \geq \lambda(suppJ)$ and any fixed $m_{\Lambda^c} \in \mathcal{M}_{\Lambda^c}$, let

$$\mathcal{M}_\Lambda \ni m_\Lambda \longmapsto E_J(m_\Lambda|m_{\Lambda^c}) := -\frac{1}{2}\langle m_\Lambda, \mathbf{J}m_\Lambda\rangle - \langle m_\Lambda, \mathbf{J}m_{\Lambda^c}\rangle \in \mathbb{R} \tag{15}$$

be the *energy functional* with boundary condition m_{Λ^c}, where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $L^2(\mathbb{R}^2;\mathbb{R}^2)$ and $\mathbf{J}m := (J \circ ||\cdot||) * m \in \mathcal{M}$. We have

$$\begin{aligned} E_J(m_\Lambda|m_{\Lambda^c}) &= -\frac{1}{2}\int_\Lambda dx \int_\Lambda dy J(|x-y|)\, m(x)\, m(y) + \\ &\quad - \int_\Lambda dx \int_{\Lambda^c} dy J(|x-y|)\, m(x)\, m(y) \\ &= U^J(m_\Lambda) + W_J(m_\Lambda|m_{\Lambda^c}) - W^J(m_{\Lambda^c}) - \frac{1}{2}||m_\Lambda||^2 \\ &\quad - \frac{1}{2}\int_\Lambda dx \int_{\Lambda^c} dy J(|x-y|)\, |m(x)|^2 , \end{aligned} \tag{16}$$

where

$$\mathcal{M}_\Lambda \ni m_\Lambda \longmapsto U^J(m_\Lambda) := \frac{1}{4}\int_\Lambda dx \int_\Lambda dy J(|x-y|)\,|m(x)-m(y)|^2 \in \mathbb{R}\,, \tag{17}$$

$$\mathcal{M}_\Lambda \ni m_\Lambda \longmapsto W_J(m_\Lambda|m_{\Lambda^c}) := \frac{1}{2}\int_\Lambda dx \int_{\Lambda^c} dy J(|x-y|)\,|m(x)-m(y)|^2 \in \mathbb{R} \tag{18}$$

and $W^J(m_{\Lambda^c}) := W_J(0|m_{\Lambda^c})$.

1.2 Coarse graining

For any $\delta \geq \gamma \in (0,1]$, let $\mathcal{Q}_\delta$ be the partition of $\mathbb{R}^2$, whose atoms are

$$Q_n^{(\delta)} := \left\{x \in \mathbb{R}^2 : x \in [\delta n_1, \delta(n_1+1)) \times [\delta n_2, \delta(n_2+1))\,;\, n=(n_1,n_2)\in\mathbb{Z}^2\right\}. \tag{19}$$

Then, considering the measurable space $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2), \lambda)$, where λ denotes the Lebesgue measure, for any λ-mesurable function $f : \mathbb{R}^2 \longmapsto \mathbb{R}^q$, $q=1,2$, we denote by $f^{(\delta)} = \mathbb{E}_\delta(f) := \mathbb{E}(f|\mathcal{Q}_\delta)$ its conditional expectation with respect to the σalgebra generated by $\mathcal{Q}_\delta$. Therefore, δ-measurable functions are those functions f such that $f^{(\delta)} = f$. In particular, if this occurs for $f = \mathbf{1}_\Lambda$, with $\Lambda \subset \mathbb{R}^2$, Λ will be called δ-measurable.

For any $\delta \geq \gamma \in (0,1]$, the map

$$\mathcal{M} \ni m \longmapsto \mathbb{E}_\delta(m) := m^{(\delta)} \in \mathcal{M}\,, \tag{20}$$

is called *coarse graining* at the scale δ and $\sigma_\gamma^{(\delta)}$ is called block spin of size δ. We also set $\mathcal{M}^{(\delta)} := \mathbb{E}_\delta\mathcal{M}$.

In the following, to simplify the computations, we will choose the Kac parameter γ and any coarse graining parameter $\delta \geq \gamma \in (0,1]$ to be diadic numbers, since, as it will clear from the sequel, this assumption will not affect the results.

Given $\delta \geq \gamma$ and any δ-measurable $\Lambda \subset\subset \mathbb{R}^2$, for any fixed $\xi' \in \mathcal{M}_{\Lambda^c}^{(\delta)}$ we define the block Hamiltonian of size δ to be the functional

$$\mathcal{M}_\Lambda^{(\delta)} \ni \xi_\Lambda \longmapsto H_\delta(\xi_\Lambda|\xi'_{\Lambda^c}) := \left[\sum_{\substack{n\in\Lambda_\delta\\ k\in\Lambda_\delta}} \frac{1}{2} J_\delta(n,k)\,\xi(n\delta)\cdot\xi(k\delta) + \sum_{\substack{n\in\Lambda_\delta\\ k\in\Lambda_\delta^c}} J_\delta(n,k)\,\xi(n\delta)\cdot\xi'(k\delta)\right] \in \mathbb{R} \tag{21}$$

with $J_\delta(n,k) := \delta^2 J(\delta|n-k|)$. Notice that by (3) for any fixed $\bar{\sigma}_{\Lambda_\gamma^c} \in S_{\Lambda_\gamma^c}$

$$H_\gamma\left((\sigma_\gamma)_\Lambda \,|\, (\bar{\sigma}_\gamma)_\Lambda\right) = H^{J_\gamma}\left(\sigma_{\Lambda_\gamma}|\bar{\sigma}_{\Lambda_\gamma^c}\right) - \frac{1}{2}\gamma^2|\Lambda_\gamma|\,. \tag{22}$$

Moreover, there exist two positive constans $b_1(J), b_2(J)$ such that

$$\left| H^{J_\gamma}\left(\sigma_{\Lambda_\gamma}|\bar{\sigma}_{\Lambda_\gamma^c}\right) - \left(\frac{\delta}{\gamma}\right)^2 H_\gamma\left(\left(\sigma_\gamma^{(\delta)}\right)_\Lambda | \left(\bar{\sigma}_\gamma^{(\delta)}\right)_\Lambda\right) \right| \leq b_1(J)\left(\frac{\delta}{\gamma}\right)^2 \lambda(\Lambda) , \tag{23}$$

$$\left| E_J\left(\left(\sigma_\gamma^{(\delta)}\right)_\Lambda | \left(\bar{\sigma}_\gamma^{(\delta)}\right)_\Lambda\right) - \delta^2 H_\gamma\left(\left(\sigma_\gamma^{(\delta)}\right)_\Lambda | \left(\bar{\sigma}_\gamma^{(\delta)}\right)_\Lambda\right) \right| \leq b_2(J)\lambda(\Lambda) . \tag{24}$$

Therefore, a Kac model can be interpreted as a discretized version of a model, whose configuration space is $\mathcal{M}$, described by the Hamiltonian $\gamma^{-2}E_J\left(\left(\sigma_\gamma^{(\delta)}\right)_\Lambda | \left(\bar{\sigma}_\gamma^{(\delta)}\right)_\Lambda\right)$.

The Lebowitz-Penrose theorem [LP] (see [BPi] Theorem 2.1 and [G] Theorem 4.2.1 for this particular model and [TS] for its Grand-Canonical version) states that, in the thermodynamic limit, for any value of the temperature and of the lattice dimesion d, the thermodynamic potentials of the block spin model derived by a Kac one are very well approximated by the convex envelope of their mean field equivalents with an error proportional to the size of the block $\delta(\gamma) > \gamma$ tending to zero as $\gamma \downarrow 0$.

Furthermore, for $d = 1$ and for any value of the inverse temperature β, the sequence of probability masures $\{\mu^{\beta,\gamma}\}_{\gamma\in(0,1]}$ satisfies a large deviation principle with rate function $\beta\mathcal{F}^J$ [BPi], $\mathcal{F}^J$ being the *excess of free energy* functional

$$\mathcal{M} \ni m \longmapsto \mathcal{F}^J(m) := U^J(m) + F(m) \in \overline{\mathbb{R}}^+, \tag{25}$$

where

$$\mathcal{M} \ni m \longmapsto U^J(m) := \frac{1}{4}\int dx \int dy J(||x-y||)\left|m(x) - m(y)\right|^2 \in \overline{\mathbb{R}}^+, \tag{26}$$

$$\mathcal{M} \ni m \longmapsto F(m) := \int dx\left[\bar{f}_\beta(m(x)) - \bar{f}_\beta(m_\beta)\right] \in \overline{\mathbb{R}}^+, \tag{27}$$

with $\mathbf{B} \ni u \longmapsto \bar{f}_\beta(|u|) = f_\beta(u) \in \mathbb{R}$ the mean field free energy density and $m_\beta \geq 0$ the solution of the mean field equation $\frac{d}{dx}\bar{f}_\beta(x) = 0$.

We recall that

$$\bar{f}_\beta(|u|) = f_\beta(u) := -\frac{|u|^2}{2} + \beta^{-1}\bar{I}(|u|) \in \mathbb{R}, \tag{28}$$

where $\bar{I}(|w|) := \sup_{t\geq 0}\{t|w| - \ln\bar{\varphi}(t)\} = I(w)$,

$$\mathbb{R}^2 \ni w \longmapsto I(w) := \sup_{h\in\mathbb{R}^2}\{h\cdot w - \ln\varphi(h)\} \in \overline{\mathbb{R}}^+, \tag{29}$$

is the entropy of the mesure ν, that is the Legendre transform of the generating function of the cumulants of ν, $\log\varphi(h)$, with

$$\bar{\varphi}(|h|) = \varphi(h) := \int_{S^1} v(ds)\, e^{h\cdot s} \quad h \in \mathbb{R}^2. \tag{30}$$

For β larger than the mean field critical value $\beta_{mf} = 1$, m_β is strictly larger than 0 and for the one-dimensional case, as a consequence of the large deviation principle for the family of Gibbs field $\{\mu^{\beta,\gamma}\}_{\gamma\in(0,1]}$, $\mu^{\beta,\gamma}$ weakly converges to the product measure ν^β on $\mathcal{B}(S)$, which is a.c. with respect to ν with density $\frac{d\nu^\beta}{d\nu}(s) = \frac{e^{h_\beta \cdot s}}{\varphi(h_\beta)}$, such that $\nu^\beta(\sigma_i) = m_\beta s$ ([BPi] Theorem 2.3). For lattice dimension larger than or equal to 2 a large deviation principle for $\{\mu^{\beta,\gamma}\}_{\gamma\in(0,1]}$ is still lacking.

While for the two-dimensional ferromagnetic Kac-Ising model, for γ small but finite and in the limit $\gamma \downarrow 0$, the structure of the set of Gibbs states and of the typical configurations has been a subject of deep study, [CP], [BMP], [BZ] and [2], [BBP], [1], an analogous analysis for the ferromagnetic classical Kac $SO(q)$ spin models has not yet been carried on. In particular, for the two-dimensional rotor case $(d = q = 2)$, it would be interesting to understand if the equilibrium states of the model share the same features with those of the classical XY model with nearest-neighbour interaction [FrPf] and if the typical Gibbs configurations can be described in terms of spin-waves and vortices as suggested by the analysis of the decay of the two block-spins correlation function given in the next section. In other words, if taking the limit $\gamma \downarrow 0$ after the thermodynamic limit, the minimizers of the large deviation rate functional for the family of measures $\{\mu^{\beta,\gamma}\}_{\gamma\in(0,1]}$ are of the kind of those appearing in the theory of Ginzburg-Landau vortices [BBH].

2 Correlation functions

We will study the behaviour of the correlation function of two block-spins of size $\delta \geq \gamma$ when the distance between the variables is very large compared to the size of the blocks. To do this, we will first give upper bounds for correlation functions of two spins and of two block spins and then provide a lower bound for these quantities when the distance between the variables diverges in the limit $\gamma \downarrow 0$.

2.1 Upper bound

First we give an estimate from above of the correlation function of two spins. To do this we make use of a modified version of the strategy proposed originally by McBrian and Spencer [McBS] and successively developed by Messager et al. [MMSPf], [MMSR] and Picco [Pi2]. In the sequel we will assume $\gamma \in \left(0, \frac{1}{2}\right]$. In fact, if $\gamma \in \left(\frac{1}{2}, 1\right]$ the interaction among the spins involves at most those whose mutual distance is smaller than or equal to $\sqrt{2}$, hence, by the second Griffiths' inequality [S], the spin-spin correlation function relative to Kac potential is dominated by the one relative to the standard nearest-neighbor potential. In this case, we adress the reader to [McBS].

Theorem 1 *For any $\gamma \in \left(0, \frac{1}{2}\right]$, let $L(\gamma) \geq 2$. Then, there exist three constants $\alpha_1, \alpha_2, \tau > 1$*

such that, $\forall p > 0$ *and* $\beta > \frac{\pi}{16}\frac{\gamma^{1+p}}{1+\tau}$,

$$\mu^{\beta,J_\gamma}\left[\cos\left(\theta_0-\theta_{L(\gamma)}\right)\right]\leq\exp\left\{-\frac{\gamma^{1+p}}{2\alpha_2\beta}\ln\left(\frac{L(\gamma)+2}{2}\right)\left[1-\gamma^p\left(1+\frac{\ln 3}{\ln 2}\right)\right]+\frac{\gamma^{1+2p}}{4\alpha_1\beta}\right\}. \quad (31)$$

Corollary 2 *For any* $\gamma\in\left(0,\frac{1}{2}\right]$ *and* $p>0$, *let* $L\geq 2$ *and* $L(\gamma)=2\left[e^{\gamma^{-1-p}\log L}\right]$. *Then, there exist three constants* $\alpha_1,\alpha_2,\tau>1$ *such that, for any* $\beta>\frac{\pi}{16}\frac{\gamma^{1+p}}{1+\tau}$,

$$\mu^{\beta,J_\gamma}\left[\cos\left(\theta_0-\theta_{L(\gamma)}\right)\right]\leq\exp\left\{\frac{\ln L}{2\alpha_2\beta}\left[1-\gamma^p\left(1+\frac{\ln 3}{\ln 2}\right)\right]+\frac{\gamma^{1+2p}}{4\alpha_1\beta}\right\}. \quad (32)$$

The proof of these results are identical to the one-dimensional case and so we omit it. For further details we address the reader to [G1] and to [G] Chapter 6.

Since the scalar product of two block-spins is a linear combination of scalar products of the single spins, it is possible to apply the estimates given in the preceding theorems to each term of this linear combination and obtain an upper bound of the correlation function of two empirical magnetizations. Moreover, making use of Griffiths' inequalities, these results can be reproduced even in the case in which we consider, instead of single spins, their empirical mean on lattice blocks of size $\delta>\gamma$ with $\delta,\gamma\in(0,1)$.

Let $\delta,\gamma\in(0,1)$ such that $\delta>\gamma$ and $\Lambda\subset\subset\mathbb{R}^2$, δ-measurable, assuming free b.c. let us denote by $\left(J_\gamma^+(i,j)\right)_{i,j\in\mathbb{Z}^2}$ the interaction matrix $J_\gamma^+(i,j):=\left(c_2(\gamma)\mathbf{1}\left(\gamma|i-j|\leq\left(\frac{1}{2}+\gamma\right)\right)\right)_{i,j\in\mathbb{Z}^2}$. If

$$Q^{+,\gamma}(i\gamma):=\left\{x\in\mathbb{R}^2:|x-i\gamma|\leq\left(\frac{1}{2}+\gamma\right)\right\} \quad (33)$$

and

$$Q_\gamma^{+,\gamma}(i\gamma)=\left\{j\in\mathbb{Z}^2:\gamma|i-j|\leq\left(\frac{1}{2}+\gamma\right)\right\}, \quad (34)$$

then,

$$-H^{J_\gamma^+}\left(\sigma_{\Lambda_\gamma}\right)+\frac{1}{2}\gamma^2c_2(\gamma)|\Lambda_\gamma|:=\frac{\gamma^2}{2}c_2(\gamma)\sum_{\substack{i\gamma\in\Lambda\\ j\gamma\in Q^{+,\gamma}(i\gamma)\cap\Lambda}}\sigma_i\cdot\sigma_j \quad (35)$$

$$=\frac{\gamma^2}{2}c_2(\gamma)\sum_{k\delta\in\Lambda}\sum_{i\in Q_\gamma^{(\delta)}(k\delta)}\sum_{n\delta\in Q^{+,\gamma}(i\gamma)\cap\Lambda}\sum_{j\in Q_\gamma^{(\delta)}(n\delta)}\sigma_i\cdot\sigma_j\,.$$

with $c_2(\gamma):=\frac{16}{\pi(1+4\gamma^2)}$. But for $i\in Q_\gamma^{(\delta)}(k\delta)$

$$\mathbf{1}_{Q^{+,\gamma}(i\gamma)}(n\delta)\leq\mathbf{1}_{Q^{+,\delta}(k\delta)}(n\delta)\ , \quad (36)$$

and the positivity of the interaction matrix allow us to make use of the second Griffiths' inequality, that is to bound from above the correlation function of two empirical magnetizations of size δ for the Gibbs measure associated to $H^{J_{\gamma}^{+}}\left(\sigma_{\Lambda_{\gamma}}\right)$, with the one relative to the Gibbs measure associated to

$$
\begin{aligned}
& -H^{J_{\gamma}^{+,\delta}}\left(\sigma_{\Lambda_{\gamma}}\right)+\frac{1}{2}\gamma^{2}c_{2}\left(\gamma\right)\left|\Lambda_{\gamma}\right| \qquad (37)\\
& :=\frac{\gamma^{2}}{2}c_{2}\left(\gamma\right)\sum_{k\delta\in\Lambda}\sum_{i\in Q_{\gamma}^{(\delta)}(k\delta)}\sum_{n\delta\in Q^{+,\delta}(k\delta)\cap\Lambda}\sum_{j\in Q_{\gamma}^{(\delta)}(n\delta)}\sigma_{i}\cdot\sigma_{j}\\
& =\frac{\gamma^{2}}{2}c_{2}\left(\gamma\right)\sum_{k\delta\in\Lambda}\sum_{n\delta\in Q^{+,\delta}(k\delta)\cap\Lambda}\left(\frac{\delta}{\gamma}\right)^{2}\sigma_{\gamma}^{(\delta)}\left(k\delta\right)\cdot\sigma_{\gamma}^{(\delta)}\left(n\delta\right)\\
& =\frac{c_{2}\left(\gamma\right)}{2}\left(\frac{\delta}{\gamma}\right)^{2}\delta^{2}\sum_{k\delta\in\Lambda}\sum_{n\delta\in\Lambda}\mathbf{1}\left(\left|n\delta-k\delta\right|\leq\frac{1}{2}+\delta\right)\times\\
& \times\sigma_{\gamma}^{(\delta)}\left(k\delta\right)\cdot\sigma_{\gamma}^{(\delta)}\left(n\delta\right).
\end{aligned}
$$

We now consider, $\forall\beta>0$ and any fixed value $\delta>\gamma\in(0,1)$, the Gibbs measure $\mu^{+,\beta}$ associated to the interaction matrix

$$
J_{\gamma}^{+,\delta}\left(i,j\right):=\gamma^{2}c_{2}\left(\gamma\right)\mathbf{1}_{Q^{+,\delta}(k\delta)}\left(j\gamma\right)\mathbf{1}\left(Q^{(\delta)}\left(k\delta\right)\ni i\gamma\right)\quad i,j\in\mathbb{Z}^{2} \qquad (38)
$$

with $\mathbf{1}\left(Q^{(\delta)}\left(k\delta\right)\ni i\gamma\right)=\mathbf{1}_{Q^{(\delta)}(k\delta)}\left(\gamma i\right)$. Defining $\nu_{n}^{\left(\frac{\delta}{\gamma}\right)^{2}}\left(dm_{n}\right):=\mu\left\{\left(\frac{\gamma}{\delta}\right)^{2}\sum_{j\in Q_{\gamma}^{(\delta)}(n\delta)}\sigma_{j}\in dm_{n}\right\}$, since

$$
\sigma_{\gamma}^{(\delta)}\left(\delta n\right)=\left\{\begin{array}{l} m_{n}\cos\theta_{n}\\ m_{n}\sin\theta_{n}\end{array}\right.\qquad n\in\mathbb{Z}\,, \qquad (39)
$$

for $\delta L(\delta,\gamma)\in\Lambda$, we have

$$\mu_{\Lambda}^{\beta,\gamma}\left[\sigma_{\gamma}^{(\delta)}(\delta 0)\cdot\sigma_{\gamma}^{(\delta)}(\delta L(\delta,\gamma))\right]\leq\mu_{\Lambda}^{+,\beta}\left[\sigma_{\gamma}^{(\delta)}(\delta 0)\cdot\sigma_{\gamma}^{(\delta)}(\delta L(\delta,\gamma))\right] \tag{40}$$

$$=\mu_{\Lambda}^{+,\beta}\left[m_0 m_{L(\delta,\gamma)}\cos\left(\theta_0-\theta_{L(\delta,\gamma)}\right)\right]$$

$$=\left\{\int_{[0,1]^{|\Lambda_\delta|}}\prod_{n\in\Lambda_\delta}dm_n\prod_{n\in\Lambda_\delta}\frac{\nu_n^{\left(\frac{\delta}{\gamma}\right)^2}(dm_n)}{dm_n}\int_{[-\pi,\pi]^{|\Lambda_\delta|}}\prod_{n\in\Lambda_\delta}\frac{d\theta_n}{2\pi}\times\right.$$

$$\left.\times e^{\frac{\beta}{2}\left(\frac{\delta}{\gamma}\right)^2\delta^2 c_2(\gamma)\sum_{n,k\in\Lambda_\delta}\mathbf{1}\left(\delta|n-k|\leq\frac{1}{2}+\delta\right)m_n m_k\cos(\theta_n-\theta_k)}\right\}^{-1}$$

$$\left[\int_{[0,1]^{|\Lambda_\delta|}}\prod_{n\in\Lambda_\delta}dm_n\prod_{n\in\Lambda_\delta}\frac{\nu_n^{\left(\frac{\delta}{\gamma}\right)^2}(dm_n)}{dm_n}\int_{[-\pi,\pi]^{|\Lambda_\delta|}}\prod_{n\in\Lambda_\delta}\frac{d\theta_n}{2\pi}\times\right.$$

$$\times\ e^{\frac{\beta}{2}\left(\frac{\delta}{\gamma}\right)^2\delta^2 c_2(\gamma)\sum_{n,k\in\Lambda_\delta}\mathbf{1}\left(\delta|n-k|\leq\frac{1}{2}+\delta\right)m_n m_k\cos(\theta_n-\theta_k)}\times$$

$$\left.\times m_0 m_{L(\delta,\gamma)}\cos\left(\theta_0-\theta_{L(\delta,\gamma)}\right)\right]$$

$$=\left\{\int_{[0,1]^{|\Lambda_\delta|}}\prod_{n\in\Lambda_\delta}dm_n\prod_{n\in\Lambda_\delta}e^{\left(\frac{\delta}{\gamma}\right)^2\left[-\bar{I}(m_n)+\bar{\varepsilon}\left(m_n;\left(\frac{\delta}{\gamma}\right)^2\right)\right]}\int_{[-\pi,\pi]^{|\Lambda_\delta|}}\prod_{n\in\Lambda_\delta}\frac{d\theta_n}{2\pi}\times\right.$$

$$\left.\times e^{\frac{\beta}{2}\left(\frac{\delta}{\gamma}\right)^2\delta^2 c_2(\gamma)\sum_{n,k\in\Lambda_\delta}\mathbf{1}\left(\delta|n-k|\leq\frac{1}{2}+\delta\right)m_n m_k\cos(\theta_n-\theta_k)}\right\}^{-1}\times$$

$$\left[\int_{[0,1]^{|\Lambda_\delta|}}\prod_{n\in\Lambda_\delta}dm_n\prod_{n\in\Lambda_\delta}e^{\left(\frac{\delta}{\gamma}\right)^2\left[-\bar{I}(m_n)+\bar{\varepsilon}\left(m_n;\left(\frac{\delta}{\gamma}\right)^2\right)\right]}\int_{[-\pi,\pi]^{|\Lambda_\delta|}}\prod_{n\in\Lambda_\delta}\frac{d\theta_n}{2\pi}\times\right.$$

$$\times\ e^{\frac{\beta}{2}\left(\frac{\delta}{\gamma}\right)^2\delta^2 c_2(\gamma)\sum_{n,k\in\Lambda_\delta}\mathbf{1}\left(\delta|n-k|\leq\frac{1}{2}+\delta\right)m_n m_k\cos(\theta_n-\theta_k)}\times$$

$$\left.\times m_0 m_{L(\delta,\gamma)}\cos\left(\theta_0-\theta_{L(\delta,\gamma)}\right)\right]$$

Since $\forall n\in\Lambda_\delta,\ m_n\in[0,1]$, we can still make use of the second Griffiths' inequality since the part of the interactions among the block-spins relative to the angles is given by a ferromagnetic random potential induced by the moduli of the block-spin. Hence

$$\begin{aligned}
&\mu_{\Lambda}^{\beta,\gamma}\left[\sigma_{\gamma}^{(\delta)}(\delta 0)\cdot\sigma_{\gamma}^{(\delta)}(\delta L(\delta,\gamma))\right] \qquad (41)\\
&\leq\left\{\int_{[0,1]^{|\Lambda_\delta|}}\prod_{n\in\Lambda_\delta}dm_n\prod_{n\in\Lambda_\delta}e^{\left(\frac{\delta}{\gamma}\right)^2\left[-\bar{I}(m_n)+\bar{\varepsilon}\left(m_n;\left(\frac{\delta}{\gamma}\right)^2\right)\right]}\int_{[-\pi,\pi]^{|\Lambda_\delta|}}\prod_{n\in\Lambda_\delta}\frac{d\theta_n}{2\pi}\times\right.\\
&\left.\times e^{\frac{\beta}{2}\left(\frac{\delta}{\gamma}\right)^2\delta^2 c_2(\gamma)\sum_{n,k\in\Lambda_\delta}\mathbf{1}\left(\delta|n-k|\leq\frac{1}{2}+\delta\right)\cos(\theta_n-\theta_k)}\right\}^{-1}\times\\
&\left[\int_{[0,1]^{|\Lambda_\delta|}}\prod_{n\in\Lambda_\delta}dm_n\prod_{n\in\Lambda_\delta}e^{\left(\frac{\delta}{\gamma}\right)^2\left[-\bar{I}(m_n)+\bar{\varepsilon}\left(m_n;\left(\frac{\delta}{\gamma}\right)^2\right)\right]}\int_{[-\pi,\pi]^{|\Lambda_\delta|}}\prod_{n\in\Lambda_\delta}\frac{d\theta_n}{2\pi}\times\right.\\
&\left.\times e^{\frac{\beta}{2}\left(\frac{\delta}{\gamma}\right)^2\delta^2 c_2(\gamma)\sum_{n,k\in\Lambda_\delta}\mathbf{1}\left(\delta|n-k|\leq\frac{1}{2}+\delta\right)\cos(\theta_n-\theta_k)}\cos\left(\theta_0-\theta_{L(\delta,\gamma)}\right)\right]\\
&=\frac{\mu_{\Lambda_\delta}\left[\prod_{n\in\Lambda_\delta}e^{\frac{\beta}{2}\left(\frac{\delta}{\gamma}\right)^2\delta^2 c_2(\gamma)\sum_{n,k\in\Lambda_\delta}\mathbf{1}\left(\delta|n-k|\leq\frac{1}{2}+\delta\right)\cos(\theta_n-\theta_k)}\cos\left(\theta_0-\theta_{L(\delta,\gamma)}\right)\right]}{\mu_{\Lambda_\delta}\left[\prod_{n\in\Lambda_\delta}e^{\frac{\beta}{2}\left(\frac{\delta}{\gamma}\right)^2\delta^2 c_2(\gamma)\sum_{n,k\in\Lambda_\delta}\mathbf{1}\left(\delta|n-k|\leq\frac{1}{2}+\delta\right)\cos(\theta_n-\theta_k)}\right]}.
\end{aligned}$$

Then, proceding as in the case of spin-spin correlation, we obtain the following results, for the proofs of which we refer to [G1] and to [G] Chapter 6 being identical to the one-dimesional case.

Theorem 3 *For $\gamma>0$ sufficiently small and $\delta\in\left(\gamma,\frac{1}{2}\right]$, given $L(\delta,\gamma)\geq 2$, there exist three constants $\alpha_1,\alpha_2,\tau>1$ such that, $\forall p>0$ and $\beta>\frac{\pi}{16}\frac{\delta^{1+p}}{1+\tau}$,*

$$\mu^{\beta,\gamma}\left[\sigma_{\gamma}^{(\delta)}(\delta 0)\cdot\sigma_{\gamma}^{(\delta)}(\delta L(\delta,\gamma))\right]\leq\exp\left\{-\frac{\gamma^{1+p}}{2\alpha_2\beta}\ln\left(\frac{L(\delta,\gamma)+2}{2}\right)\times\right.$$
$$\left.\times\left[1-\gamma^p\left(1+\frac{\ln 3}{\ln 2}\right)\right]+\frac{\gamma^{1+2p}}{4\alpha_1\beta}\right\}. \qquad (42)$$

Corollary 4 *Assuming the hypotesys of the preceding theorem and setting $\forall p>0$, $L(\delta,\gamma)=2\left[e^{\delta\gamma^{-2-p}\ln L}\right]$, where $L\geq 2$, there exist three constants $\alpha_1,\alpha_2,\tau>1$ such that, $\forall\beta>\frac{\pi}{16}\frac{\delta^{1+p}}{1+\tau}$,*

$$\mu^{\beta,\gamma}\left[\sigma_{\gamma}^{(\delta)}(\delta 0)\cdot\sigma_{\gamma}^{(\delta)}(\delta L(\delta,\gamma))\right]\leq\exp\left\{-\frac{\ln L}{2\alpha_2\beta}\left[1-\gamma^p\left(1+\frac{\ln 3}{\ln 2}\right)\right]+\frac{\gamma^{1+2p}}{4\alpha_1\beta}\right\}. \qquad (43)$$

We also mention that it is possible to prove by means of a polymer expansion that, if the temperature of the system is high enough, the decay of the truncated correlation function of two block-spins, when their mutual distance diverges, is at most exponential. For the proof of this result we refer the reader to [PS], [G1] and [G] Theorem 6.2.3.

2.2 Lower bound

To have a more precise characterization of the asymptotic behaviour of the correlation function of two block-spins, it is not enough to give an estimate from above, since this would imply only that the decay in space of such function is not faster than the decay of its upper bound.

To get a lower bound estimate of block-spin correlation functions we will follow a strategy which is in general valid for d-dimensional Kac rotators models with $d \geq 1$.

For any $\beta > \beta_{mf} = 1$ we will procede along these lines:

1) First we will confine the system to a subset $\Delta \subset\subset \mathbb{R}^d$ sufficiently large to which we will eventually impose free b.c.. Since the angular part of the interaction among the $\sigma_\gamma^{(\delta)}$'s is subject to a random ferromagnetic potential induced by their moduli, we will restrict ourseves to the event in $\mathcal{B}(S)$ such that all the moduli of the empirical magnetization defined inside Δ are larger than a given strictly positive value;

2) second, we will consider a given subset Λ of Δ. Inside Λ we will repalce the original Kac potential with a weaker one and we will replace the values assumed by the moduli of the empirical magnetization with the values given in the preceding point;

3) third, we will set to zero some of the elements of the interaction matrix, reducing our system to a system where the block-spins interact only with their nearest neighbours. The intensity of the new interaction in Λ will be set proportional to the minimum value assumed by the moduli of the empirical magnetizations given in 1) and will be eventually set to zero outside Λ (which means assuming outside Λ free b.c.). A this point the estimate from below of the correlation function of two block-spins is reduced to the estimate from below of the same quantity for a nearest neighbour model;

4) last, we will estimate the Gibbs measure of the event in $\mathcal{B}(S)$ considered at point 1).

2.3 The 2-dimesional case

Before entering in the details of the computations, we make some more remarks.

- In this case, large deviation estimates on the size of the modulus of the empirical magnetization, analogous to the ones valid for the one-dimensional case, are not available. Thus we will restrict ourselves to assume free b.c. outside Δ.

- As in the one-dimesional case, even in the two-dimensional one we can prove the correlation function of two empirical magnetizations for our model to be bounded from below by the two-point correlation function of a particular Villain model [Vi]. Therefore, the proof of our result is reduced to an application of the Fröhlich-Spencer argument for this latter model [FrS], [FrS1].

More precisely:

1. We will consider the system to be restricted to a large set Δ, such that $\lambda(\Delta) \leq Kl^2(\delta,\gamma)$, with $K > 1$, and we will assume outside Δ free b.c.. Since the Gibbs measure of
$$V_{\Delta}^{(\delta)\prime}(m_\beta;\zeta) := \bigcup_{\delta n\in\Delta} V_n^{(\delta)\prime}(m_\beta;\zeta) , \tag{44}$$
$$V_n^{(\delta)\prime}(m_\beta;\zeta) := \left\{m^{(\delta)} \in \mathcal{M} : \left|m^{(\delta)}(\delta n)\right| \leq m_\beta - \zeta\right\}, \tag{45}$$
is positive and exponentially small when $\gamma \downarrow 0$, it suffices to restrict our attention to the complementary event and so to give an estimate of
$$\mu_{\Delta}^{\beta,\gamma}\left[\sigma_\gamma^{(\delta)}(\delta 0)\cdot\sigma_\gamma^{(\delta)}(\delta l(\delta,\gamma))\,\mathbf{1}_{\left(V_\Delta^{(\delta)\prime}(m_\beta;\zeta)\right)^c}\right]. \tag{46}$$

2. At this point, we can replace the original interaction inside a given subset Λ of Δ with a weaker one and set the values of the moduli of the block-spins equal to $(m_\beta - \zeta)$.

3. Last, we will reduce the range of the interaction obtaining a nearest neighbour potential. Then, cutting out the interaction with the boundary of Λ, we will restrict ourselves to the case of free b.c.. Inside Λ, the resulting Hamiltonian will be
$$H_{\beta,\delta,\gamma}^{nn}(\theta_{\Lambda_\delta}) := -\delta^2(m_\beta-\zeta)^2\left(\frac{\delta}{\gamma}\right)^2 c_2(\gamma)\times \times\frac{1}{2}\sum_{\Lambda_\delta\ni n,k\,:\,|n-k|\leq 1}\cos(\theta_n-\theta_k) , \tag{47}$$
depending on β through $m_\beta-\zeta$. The lower bound estimate of the correlation function of two block-spins is now reduced to an estimate of the same quantity for the classical XY model defined by $H_{\beta,\delta,\gamma}^{nn}(\theta_{\Lambda_\delta})$. At this point we are free to use the argument of Fröhlich and Spencer [FrS], [FrS1], but we have to take care of the factor $\delta^2(m_\beta-\zeta)^2\left(\frac{\delta}{\gamma}\right)^2 c_2(\gamma)$ giving the intensity of the interaction for fixed values of β.

Theorem 5 *Let $\beta > \beta_{mf} = 1$, m_β be the solution of the mean field equation and $\zeta \in (0, m_\beta)$. $\forall\delta \in \left(\gamma, \frac{1}{2(\sqrt{2}+1)}\right]$ with $\gamma > 0$, there exists a constant β_0 such that, for any $\beta > \beta_0$, there exists a function $\beta_1(\beta) \geq \frac{1}{2\pi}$ for which $\lim_{\beta\uparrow\infty}\frac{\beta_1(\beta)}{\beta} = 1$ and*
$$\begin{aligned}&\mu^{\beta,\gamma}\left[\sigma_\gamma^{(\delta)}(\delta 0)\cdot\sigma_\gamma^{(\delta)}(\delta l(\delta,\gamma))\right]\\ &\geq K'(m_\beta-\zeta)^2\left(1-K(l(\delta,\gamma))^2\eta(\gamma)\,e^{-\frac{\beta}{\gamma^2}\frac{\delta^2\zeta^2}{c(\beta,J)}}\right)\times\\ &\times e^{-\frac{\ln l(\delta,\gamma)}{2\pi\beta_1(\beta)\delta^2\left(m_\beta-\zeta\right)^2\left(\frac{\delta}{\gamma}\right)^2 c_2(\gamma)}},\end{aligned} \tag{48}$$

where $K'>0$ and $K,\eta(\gamma)>1$, while $c(\beta,J)$ is a suitable positive constant.

Proof. Let $\beta>1$and $\delta\in\left(\gamma,\frac{1}{2\left(\sqrt{2}+1\right)}\right]$ with $\gamma>0$. We follow the strategy described above. We first consider our system to be confined in a large δ-measurable set $\Delta\subset\subset\mathbb{R}^2$such that $\lambda(\Delta)\leq Kl^2(\delta,\gamma)$, with $K>1$, and assume outside Δ free b.c.. From correlation inequalities it follows that

$$\mu^{\beta,\gamma}\left[\sigma_\gamma^{(\delta)}(\delta 0)\cdot\sigma_\gamma^{(\delta)}(\delta l_d(\delta,\gamma))\right]\geq\mu_\Delta^{\beta,\gamma}\left[\sigma_\gamma^{(\delta)}(\delta 0)\cdot\sigma_\gamma^{(\delta)}(\delta l_d(\delta,\gamma))\right]. \tag{49}$$

As in the one-dimensional case (see [G1] Theorem 7) we get

$$0\leq\mu_\Delta^{\beta,\gamma}\left[\sigma_\gamma^{(\delta)}(\delta 0)\cdot\sigma_\gamma^{(\delta)}(\delta l(\delta,\gamma))\mathbf{1}_{V_\Delta^{(\delta)\prime}(m_\beta;\zeta)}\right]\leq(m_\beta-\zeta)^2Kl^2(\delta,\gamma)\eta(\gamma)e^{-\frac{\beta}{\gamma^2}\frac{\delta^2\zeta^2}{c(\beta,J)}}. \tag{50}$$

Hence,

$$\mu^{\beta,\gamma}\left[\sigma_\gamma^{(\delta)}(\delta 0)\cdot\sigma_\gamma^{(\delta)}(\delta l_d(\delta,\gamma))\right]\geq\mu_\Delta^{\beta,\gamma}\left[\sigma_\gamma^{(\delta)}(\delta 0)\cdot\sigma_\gamma^{(\delta)}(\delta l(\delta,\gamma))\mathbf{1}_{\left(V_\Delta^{(\delta)\prime}(m_\beta;\zeta)\right)^c}\right] \tag{51}$$

and

$$\begin{aligned}&\mu_\Delta^{\beta,\gamma}\left[\sigma_\gamma^{(\delta)}(\delta 0)\cdot\sigma_\gamma^{(\delta)}(\delta l(\delta,\gamma))\mathbf{1}_{\left(V_\Delta^{(\delta)\prime}(m_\beta;\zeta)\right)^c}\right]\\&\geq(m_\beta-\zeta)^2\mu_\Delta^{\beta,\gamma}\left[\frac{\sigma_\gamma^{(\delta)}(\delta 0)}{\left|\sigma_\gamma^{(\delta)}(\delta 0)\right|}\cdot\frac{\sigma_\gamma^{(\delta)}(\delta l(\delta,\gamma))}{\left|\sigma_\gamma^{(\delta)}(\delta l(\delta,\gamma))\right|}\mathbf{1}_{\left(V_\Delta^{(\delta)\prime}(m_\beta;\zeta)\right)^c}\right].\end{aligned} \tag{52}$$

Let now Λ be a δ-measurable subset of Δ containing $\delta l(\delta,\gamma)$. By the consideration given at the second point of our scheme we have

$$\begin{aligned}&\mu_\Delta^{\beta,\gamma}\left[\frac{\sigma_\gamma^{(\delta)}(\delta 0)}{\left|\sigma_\gamma^{(\delta)}(\delta 0)\right|}\cdot\frac{\sigma_\gamma^{(\delta)}(\delta l(\delta,\gamma))}{\left|\sigma_\gamma^{(\delta)}(\delta l(\delta,\gamma))\right|}\mathbf{1}_{\left(V_\Delta^{(\delta)\prime}(m_\beta;\zeta)\right)^c}\right]\\&=\int\mu^\gamma\left(d(\sigma_\gamma)_{\Delta\setminus\Lambda}\right)\left[\frac{\mu_\Lambda^\gamma\left[e^{-\beta H_\gamma\left(\cdot|(\sigma_\gamma)_{\Lambda^c}\right)}\frac{\sigma_\gamma^{(\delta)}(\delta 0)}{\left|\sigma_\gamma^{(\delta)}(\delta 0)\right|}\cdot\frac{\sigma_\gamma^{(\delta)}(\delta l(\delta,\gamma))}{\left|\sigma_\gamma^{(\delta)}(\delta l(\delta,\gamma))\right|}\mathbf{1}_{\left(V_\Lambda^{(\delta)\prime}(m_\beta;\zeta)\right)^c}\right]}{\mu_\Lambda^\gamma\left[e^{-\beta H_\gamma\left(\cdot|(\sigma_\gamma)_{\Lambda^c}\right)}\mathbf{1}_{\left(V_\Lambda^{(\delta)\prime}(m_\beta;\zeta)\right)^c}\right]}\times\right.\\&\left.\times\mu_\Lambda^\gamma\left[e^{-\beta H_\gamma\left(\cdot|(\sigma_\gamma)_{\Lambda^c}\right)}\mathbf{1}_{V_{\delta,\Lambda}^{\prime(\delta),c}(m_\beta;\zeta)}\right]\mathbf{1}_{\left(V_{\Delta\setminus\Lambda}^{(\delta)\prime}(m_\beta;\zeta)\right)^c}\left((\sigma_\gamma)_{\Delta\setminus\Lambda}\right)\frac{e^{-\beta H_\gamma\left((\sigma_\gamma)_{\Delta\setminus\Lambda}\right)}}{\mu_\Delta^\gamma\left[e^{-\beta H_\gamma\left((\sigma_\gamma)_\Delta\right)}\right]}\right].\end{aligned} \tag{53}$$

Then, it is enough to estimate

$$\frac{\mu_{\Lambda}^{\gamma}\left[e^{-\beta H_{\gamma}\left(\cdot|(\sigma_{\gamma})_{\Lambda^c}\right)}\frac{\sigma_{\gamma}^{(\delta)}(\delta 0)}{\left|\sigma_{\gamma}^{(\delta)}(\delta 0)\right|}\cdot\frac{\sigma_{\gamma}^{(\delta)}(\delta l(\delta,\gamma))}{\left|\sigma_{\gamma}^{(\delta)}(\delta l(\delta,\gamma))\right|}\mathbf{1}_{\left(V_{\Lambda}^{(\delta)\prime}(m_{\beta};\zeta)\right)^{c}}\right]}{\mu_{\Lambda}^{\gamma}\left[e^{-\beta H_{\gamma}\left(\cdot|(\sigma_{\gamma})_{\Lambda^c}\right)}\mathbf{1}_{\left(V_{\Lambda}^{(\delta)\prime}(m_{\beta};\zeta)\right)^{c}}\right]} \tag{54}$$

$$:=\mu^{\beta,\gamma}_{\Lambda,\left(V_{\Lambda}^{(\delta)\prime}(m_{\beta};\zeta)\right)^{c}}\left[\frac{\sigma_{\gamma}^{(\delta)}(\delta 0)}{\left|\sigma_{\gamma}^{(\delta)}(\delta 0)\right|}\cdot\frac{\sigma_{\gamma}^{(\delta)}(\delta l(\delta,\gamma))}{\left|\sigma_{\gamma}^{(\delta)}(\delta l(\delta,\gamma))\right|}|\left(\sigma_{\gamma}^{(\delta)}\right)_{\Lambda^c}\right].$$

Therefore, since the moduli of the $\left(\sigma_{\gamma}^{(\delta)}\right)_{\Lambda^c}$'s are bound to assume values larger than $(m_{\beta}-\zeta)$, we get

$$\mu^{\beta,\gamma}_{\Lambda,\left(V_{\Lambda}^{(\delta)\prime}(m_{\beta};\zeta)\right)^{c}}\left[\frac{\sigma_{\gamma}^{(\delta)}(\delta 0)}{\left|\sigma_{\gamma}^{(\delta)}(\delta 0)\right|}\cdot\frac{\sigma_{\gamma}^{(\delta)}(\delta l(\delta,\gamma))}{\left|\sigma_{\gamma}^{(\delta)}(\delta l(\delta,\gamma))\right|}|\left(\sigma_{\gamma}^{(\delta)}\right)_{\Lambda^c}\right] \tag{55}$$

$$\geq \mu^{-,\beta}_{\Lambda,\left(V_{\Lambda}^{(\delta)\prime}(m_{\beta};\zeta)\right)^{c}}\left[\frac{\sigma_{\gamma}^{(\delta)}(\delta 0)}{\left|\sigma_{\gamma}^{(\delta)}(\delta 0)\right|}\cdot\frac{\sigma_{\gamma}^{(\delta)}(\delta l(\delta,\gamma))}{\left|\sigma_{\gamma}^{(\delta)}(\delta l(\delta,\gamma))\right|}|\left(\sigma_{\gamma}^{(\delta)}\right)_{\Lambda^c}\right]$$

$$\geq\left\{\int_{[0,1]^{|\Lambda_\delta|}}\prod_{n\in\Lambda_\delta}dm_n\prod_{n\in\Lambda_\delta}\frac{\nu^{\left(\frac{\delta}{\gamma}\right)^2}(dm_n)}{dm_n}\left(\prod_{n\in\Lambda_\delta}\mathbf{1}_{\left(m_\beta-\zeta,1\right]}(m_n)\right)\int_{[-\pi,\pi]^{|\Lambda_\delta|}}\prod_{n\in\Lambda_\delta}\frac{d\theta_n}{2\pi}\times\right.$$

$$\left.\times e^{\beta\left(\frac{\delta}{\gamma}\right)^2\delta^2\left(m_\beta-\zeta\right)^2\left[\sum_{n,k\in\Lambda_\delta}\frac{1}{2}+\sum_{n\in\Lambda_\delta}\sum_{k\in\Lambda_\delta^c}\right]\mathbf{1}\left(\delta|n-k|\leq\frac{1}{2}-\delta\right)\cos(\theta_n-\theta_k)}\right\}^{-1}\times$$

$$\times\left[\int_{[0,1]^{|\Lambda_\delta|}}\prod_{n\in\Lambda_\delta}dm_n\prod_{n\in\Lambda_\delta}\frac{\nu^{\left(\frac{\delta}{\gamma}\right)^2}(dm_n)}{dm_n}\left(\prod_{n\in\Lambda_\delta}\mathbf{1}_{\left(m_\beta-\zeta,1\right]}(m_n)\right)\int_{[-\pi,\pi]^{|\Lambda_\delta|}}\prod_{n\in\Lambda_\delta}\frac{d\theta_n}{2\pi}\times\right.$$

$$\left.\times e^{\beta\left(\frac{\delta}{\gamma}\right)^2\delta^2\left(m_\beta-\zeta\right)^2\left[\sum_{n,k\in\Lambda_\delta}\frac{1}{2}+\sum_{n\in\Lambda_\delta}\sum_{k\in\Lambda_\delta^c}\right]\mathbf{1}\left(\delta|n-k|\leq\frac{1}{2}-\delta\right)\cos(\theta_n-\theta_k)}\cos\left(\theta_0-\theta_{l(\delta,\gamma)}\right)\right]$$

$$:=\mu_{\Lambda}^{-,\beta,\left(m_\beta-\zeta\right)^2}\left[\cos\left(\theta_0-\theta_{l(\delta,\gamma)}\right)|\left\{\theta_n\right\}_{n\in\Lambda_\delta^c}\right].$$

We now proceed applying what stated in the third point of our scheme and reduce the interaction to the one given in (47). Let us denote the Gibbs measure associated to this potential by $\mu^{\beta,\delta,\gamma,nn}$. Cutting out the boundary interactions, we obtain

$$\mu_{\Lambda}^{-,\beta,\left(m_\beta-\zeta\right)^2}\left[\cos\left(\theta_0-\theta_{l(\delta,\gamma)}\right)|\left\{\theta_n\right\}_{n\in\Lambda_\delta^c}\right]\geq\mu_{\Lambda_\delta}^{\beta,\delta,\gamma,nn}\left[\cos\left(\theta_0-\theta_{l(\delta,\gamma)}\right)\right]. \tag{56}$$

Now, since

$$\int \mu^{\gamma}\left(d\left(\sigma_{\gamma}\right)_{\Delta\setminus\Lambda}\right)\left[\mu_{\Lambda}^{\gamma}\left[e^{-\beta H_{\gamma}\left(\cdot|(\sigma_{\gamma})_{\Lambda^{c}}\right)}\mathbf{1}_{V_{\delta,\Lambda}^{\prime(\delta),c}\left(m_{\beta};\zeta\right)}\right]\times\right.$$
$$\left.\times\frac{e^{-\beta H_{\gamma}\left((\sigma_{\gamma})_{\Delta\setminus\Lambda}\right)}\mathbf{1}_{\left(V_{\Delta\setminus\Lambda}^{(\delta)\prime}\left(m_{\beta};\zeta\right)\right)^{c}}\left(\left(\sigma_{\gamma}\right)_{\Delta\setminus\Lambda}\right)}{\mu_{\Delta}^{\gamma}\left[e^{-\beta H_{\gamma}\left((\sigma_{\gamma})_{\Delta}\right)}\right]}\right]$$
$$=\mu_{\Delta}^{\beta,\gamma}\left[\left(V_{\Delta}^{(\delta)\prime}\left(m_{\beta};\zeta\right)\right)^{c}\right]$$
$$\geq 1-Kl^{2}\left(\delta,\gamma\right)\eta\left(\gamma\right)e^{-\frac{\beta}{\gamma^{2}}\frac{\delta^{2}\zeta^{2}}{c(\beta,J)}}, \tag{57}$$

the problem is reduced to an anologous one for a nearest neighbour model described by (47). We can now use, with minor modifications due to the intensity of the coupling constant among the (block) spins, the technique Fröhlich and Spencer set up in the case of the planar rotator model with nearest neigbour interactions [FrS]. We remark that the last case rely on the analogous estimate for the Villain model which can be viewed as an approximation of the planar rotator model at very low temperatures (see for example [G] Appendix B). In our case this approximation turn out to be valid at any fixed temperature below the mean field critical one provided the intensity of the interaction (47) becomes sufficiently large when γ is chosen very small. More precisely the model with interaction (47) can be approximated at order $\left(\frac{\delta}{\gamma}\right)^{-2}$ by the Villain model with Gibbs factor

$$V_{\frac{\beta}{2}\delta^{2}\left(m_{\beta}-\zeta\right)^{2}\left(\frac{\delta}{\gamma}\right)^{2}}\left(\left(\theta-\theta'\right)\right):=\sum_{k\in\mathbb{Z}}e^{-\frac{\beta\delta^{2}\left(m_{\beta}-\zeta\right)^{2}\left(\frac{\delta}{\gamma}\right)^{2}c_{2}(\gamma)}{4}\left[(\theta-\theta')+2\pi k\right]^{2}}$$
$$=\frac{1}{\sqrt{\pi\beta\delta^{2}\left(m_{\beta}-\zeta\right)^{2}\left(\frac{\delta}{\gamma}\right)^{2}c_{2}\left(\gamma\right)}}\sum_{k\in\mathbb{Z}}e^{-\frac{k^{2}}{\beta\delta^{2}\left(m_{\beta}-\zeta\right)^{2}\left(\frac{\delta}{\gamma}\right)^{2}c_{2}(\gamma)}-ik(\theta-\theta')}, \tag{58}$$

which reduce the problem to a lower bound estimate for the two spins correlation function for this model at low temperatures, which follows the Fröhlich-Spencer bound.

To make the paper more readable we omit the details and refer the reader to [G] Appendix B and C for the computations. ■

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