Lectures in Nonlinear Analysis and Differential Equations

Doctoral School in Mathematics and Computer Science
Department of Mathematics and Computer Science, University of Calabria

April 8-12, 2019

General Information

The course, oganized within the Ph.D. programme in Mathematics and Computer Science, is intended for doctoral students and young researchers interested in Nonlinear Analysis and Differential Equations.
The classes will be given in Aula Pitagora (ex MT11), Department of Mathematics and Computer Science, University of Calabria, Cubo 30B, first floor.
During the same period our University will host some talks given by some participants, these are included here for completeness.

Course Lecturers

- **Alberto Cabada**, University of Santiago de Compostela, Spain, alberto.cabada@usc.es,
  Monotone Iterative Techniques for Ordinary Differential Equations.

  The Iterative methods are a fundamental tool in order to ensure the existence of solutions of Nonlinear Boundary Value Problems. This theory is valid both for Ordinary and Partial Differential Equations. Such theory is strongly related to the method of lower and upper solutions, which allows us to ensure the existence of a solution of the considered problem lying between a pair of ordered functions that satisfy some suitable inequalities. In particular, we have information not only about the existence of solutions, but also about the location of some of them. The combination of the lower and upper solutions method and monotone iterative techniques allow us to approximate the given solutions and, in some particular cases, obtain their exact expression. It is important to note that this recursive method has also a deep dependence on comparison results for suitable linear operators. Such comparison results are equivalent to the constant sign of the kernel of the related integral operator, the so-called Green's function. Is for this that, to apply such theory to a particular problem, we must study the sign of such function. Since to calculate the expression of these functions is, in general, very complicated, we have developed a Mathematica package, see http://library.wolfram.com/infocenter/MathSource/8825/, where the exact expression of the Green's function is obtained when the coefficients of the linear equation are constant. Even in such a case, the obtained expressions are very complicated to deal with. Is for this that, for a wider set of two-point boundary conditions and when non-constant coefficients are considered, by means of spectral theory, we can give the exact value of the parameters for which such Green's function has constant sign without calculate their exact expression.

- **Aleksander Ćwiszewski**, Nicolaus Copernicus University, Poland, aleks@mat.umk.pl,
  Topological Approach to Dynamics of PDEs.

  Lecture plan:

  1. Dynamical systems: semi-flows, stationary points, full orbits, Lyapunov functionals and attractors in finite dimensional spaces.
  2. Construction of semi-flows for parabolic and hyperbolic PDEs and their compactness properties.

3. Towards topological approach: invariant sets, isolating blocks, homotopy types, Conley and Rybakowski-Conley indices and how they capture dynamics.
4. Computation formulae for Conley index and its relations with fixed point index.
5. Stationary solutions, connecting orbits and bifurcations via topological tools.

- **Tibor Krisztin**, University of Szeged, Hungary, krisztin@math.u-szeged.hu,
  Delay Differential Equations.

  Delay differential equations (DDEs) form a particular class of infinite dimensional dynamical systems. The time delay arises naturally in phenomena where the rate of change of the system depends not only on the present but also on the past states of the system. There is a wide range of applications in physics, chemistry, biology, and social sciences. In particular systems with a feedback control involve time delays.

  In this introductory course we present the fundamental ideas emphasizing the similarities and differences between DDEs and ordinary differential equations and some partial differential equations. Recent global results will be explained concerning the geometric structure of global attractors for some equations with monotone and non-monotone feedback functions. We discuss open problems and possible future directions as well.

Guest Speakers

- **Stefano Biagi**, Università Politecnica delle Marche, s.biagi@dipmat.univpm.it,
  Heteroclinic solutions for a class of boundary value problems associated with singular equations.

  We obtain existence results for strongly nonlinear BVPs of type

$$\text{(P)} \qquad \begin{cases} \Big(\Phi\big(k(t)\,x'(t)\big)\Big)' = f(t,x(t),x'(t)) & \text{a.e. on } \Lambda := [0,\infty), \\ x(0)=\nu_1,\ x(\infty)=\nu_2 \end{cases}$$

  where $\Phi : \mathbb{R} \to \mathbb{R}$ is a strictly increasing homeomorphism such that $\Phi(0) = 0$ (the Φ-*Laplacian operator*), $k : \Lambda \to \mathbb{R}$ is a non-negative continuous map such that

$$\big|\{t \in \Lambda :\ k(t) = 0\}\big| = 0,$$

  f is a Carathéodory function on $\Lambda \times \mathbb{R}^2$ and ν_1, $\nu_2 \in \mathbb{R}$ are fixed.

  Under mild assumptions, including a weak form of a Nagumo-Winter growth condition, we prove the existence of heteroclinic solutions of problem (P) in the Sobolev space $W^{1,p}_{\text{loc}}(\Lambda)$ (for some $p > 1$). Our approach is based on fixed point techniques suitably combined to the method of upper and lower solutions.

- **Francesco Esposito**, Università della Calabria, esposito@mat.unical.it,
  On the moving plane method for singular solutions to some semilinear and quasilinear elliptic problems.

  In this talk we will consider positive singular solutions to semilinear or quasilinear elliptic problems. We will deduce symmetry and monotonicity properties of the solutions via the moving plane procedure. In particular, we will consider the problem

$$\begin{cases} -\Delta_p u = f(u) & \text{in} \quad \Omega \setminus \Gamma \\ u > 0 & \text{in} \quad \Omega \setminus \Gamma \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

where $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$, $1 < p < +\infty$, is the p-Laplace operator, Ω is a bounded smooth domain of $\mathbb{R}^n$ with $n \geq 2$, which is convex and it is symmetric with respect to the hyperplane $\{x_1 = 0\}$; $\Gamma \subset \{x_1 = 0\}$ is a closed set of vanishing p-capacity. Furthermore, the nonlinearity f will be assumed to be locally Lipschitz continuous from above far from the singular set in the case $p = 2$ and with some additional assumption when $p \neq 2$, that will be discussed. Our results will be obtained by means of the moving plane technique, that goes back to the celebrated papers of Alexandrov and Serrin.

- **Umberto Guarnotta**, Università degli Studi di Catania, Italy, umberto.guarnotta@gmail.com,
  Multiple nodal solutions to a Robin problem with sign-changing potential and locally defined reaction.

  In this talk we present some results about both existence and multiplicity of nodal C^1-solutions to the following Robin boundary-value problem:

$$\begin{cases} -\mathrm{div}(a(\nabla u)) + \alpha(x)|u|^{p-2}u = f(x,u) & \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial n_a} + \beta(x)|u|^{p-2}u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1}$$

Here, Ω denotes a bounded domain in $\mathbb{R}^N$, $N \geq 3$, with a smooth boundary $\partial\Omega$, the coefficient α is essentially bounded but sign-changing, β lies in $C^{0,\gamma}(\partial\Omega)$ and takes non-negative values, $1 < p < +\infty$, the reaction $f : \Omega \times [-\theta, \theta] \to \mathbb{R}$ satisfies Carathéodory's conditions. Moreover, $a : \mathbb{R}^N \to \mathbb{R}^N$ indicates a strictly monotone map having appropriate regularity and growth properties (stemming from Lieberman's regularity theory [2] and the Pucci-Serrin maximum principle [5]), while $\frac{\partial}{\partial n_a}$ stands for the co-normal derivative associated with a.
Problem (1) exhibits at least three interesting features:

i) We do not require that $\xi \mapsto a(\xi)$ be $(p-1)$-homogeneous. So, meaningful differential operators, as the (p,q)-Laplacian, are incorporated in (1).

ii) The potential term $u \mapsto \alpha(x)|u|^{p-2}u$ turns out indefinite, because α can change sign.

iii) $t \mapsto f(x,t)$ is only locally defined, whence its behavior near zero matters, and no conditions at infinity are imposed.

Via truncation-perturbation-comparison techniques, results from Morse theory, besides variational methods, a nodal solution $\hat{u} \in C^1(\overline{\Omega})$ and two regular, constant-sign solutions u_+, v_- of (1) are obtained. As far as we know, the existence of sign-changing solutions to Robin problems that exhibit difficulties i)–iii) did not receive much attention up to now. Topics i) and, somehow, iii) have been recently addressed in [4], while [3] investigates ii) but for $a(\xi) := |\xi|^{p-2}\xi$.
Multiplicity is also treated. If $f(x,\cdot)$ is odd, we can find a whole sequence $\{u_n\} \subseteq C^1(\overline{\Omega})$ of nodal solutions to (1) such that $u_n \to 0$ in $C^1(\overline{\Omega})$. We exploit an abstract theorem by Kajikiya [1]. The work [3] contains similar results concerning Dirichlet problems without indefinite potential.

References

1. R. Kajikiya, A critical point theorem related to the symmetric mountain pass lemma and its applications to elliptic equations, *J. Functional Anal.*, vol. 255, pp 352-370, (2005).

2. G. Lieberman, The natural generalization of the natural conditions of Ladyzhenskaya and Ural'tseva for elliptic equations, *Comm. Partial Differential Equations*, vol. 16, pp 311-361, (1991).

3. S. A. Marano, S. J. N. Mosconi, and N. S. Papageorgiou, On a (p,q)-Laplacian problem with parametric concave term and asymmetric perturbation, *Rend. Lincei Mat. Appl.*, vol. 29, pp 109-125, (2018).

4. N. S. Papageorgiou and P. Winkert, Nonlinear Robin problems with a reaction of arbitrary growth, *Ann. Mat. Pura Appl.*, vol. 195, pp 1207-1235, (2016).

5. P. Pucci and J. Serrin, The Maximum Principle, Birkhäuser, Basel, (2007).

- **Władysław Klinikowski**, Nicolaus Copernicus University, Poland, wklin@mat.umk.pl
  Minimal periods of evolution equation of second order with Lipschitz nonlinearity.

  First we present the paper of James Robinson and Alejandro Vidal-Lopez "Minimal periods of semilinear evolution equations with Lipschitz nonlinearity revisited" (Journal of Differential Equations 254 (2013) 4279–4289). They fix some $\alpha \in [0,1)$ and consider equation

$$u_t + Au = f(u), \tag{2}$$

  where $A : D(A) \subset X \to X$ is a self-adjoint sectorial operator with nonnegative spectrum, X is a Hilbert space and $f : D(A^\alpha) \to X$ satisfies following condition: there exists $L > 0$ such that for all u, $v \in D(A^\alpha)$

$$\|f(u) - f(v)\| \le L \|A^\alpha (u - v)\|.$$

  If u is a periodic orbit of (2) with period T, then

$$T \ge L^{-\frac{1}{1-\alpha}} K_\alpha,$$

  where K_α depends only of α. We explain the main idea of proof which is a proper partition of spectrum $\sigma(A)$.
  Next we consider an equation of second order

$$u_{tt} + (\alpha + \beta A)u_t + Au = f(u), \tag{3}$$

  where $\alpha \ge 0$, $\beta \ge 0$, operator $A : D(A) \subset X \to X$ satisfies the same conditions as above and there exists $L > 0$ such that for all u, $v \in D\left(A^{\frac{1}{2}}\right)$

$$\|f(u) - f(v)\| \le L \left\|A^{\frac{1}{2}}(u - v)\right\|.$$

  We rewrite (3) in such form

$$\begin{cases} u_t = v \\ v_t = -(\alpha + \beta A)v - Au + f(u) \end{cases}$$

  and

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \mathbb{A} \begin{pmatrix} u \\ v \end{pmatrix} + \mathbb{F} \begin{pmatrix} u \\ v \end{pmatrix},$$

  where $\mathbb{X} := D\left(A^{\frac{1}{2}}\right) \times X$, $D(\mathbb{A}) := \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{X} | u + \beta v \in D(A), v \in D\left(A^{\frac{1}{2}}\right) \right\}$, $\mathbb{A} : D(\mathbb{A}) \subset \mathbb{X} \to \mathbb{X}$, $\mathbb{A} \begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} v \\ -\alpha v - A(u + \beta v) \end{pmatrix}$, $\mathbb{F} \begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} 0 \\ f(u) \end{pmatrix}$.
  First we consider the case $\alpha \ge 0$, $\beta = 0$. We show that if $\alpha = 0$, then there is no a minimal period. The next case is $\alpha > 0$, $\beta > 0$, but we assume in addition that spectrum of A consists only of point spectrum and correspondent eigenvectors form an orthonormal basis of X. Our aim is to show that in this case there exists a proper partition of spectrum $\sigma(\mathbb{A})$ and there exists a minimal period of periodic orbits of (3).

Course Schedule

MONDAY 8 APRIL:

9:00-11:00	Alberto Cabada *Monotone Iterative Techniques for Ordinary Differential Equations (Part 1)*
11:00-13:00	Aleksander Ćwiszewski *Topological Approach to Dynamics of PDEs (Part 1)*

TUESDAY 9 APRIL:

9:00-11:00	Tibor Krisztin *Delay Differential Equations (Part 1)*
11:00-13:00	Alberto Cabada *Monotone Iterative Techniques for Ordinary Differential Equation (Part 2)*

WEDNESDAY 10 APRIL:

9:00-11:00	Aleksander Ćwiszewski *Topological Approach to Dynamics of PDEs (Part 2)*
11:00-13:00	Tibor Krisztin *Delay Differential Equations (Part 2)*

THURSDAY 11 APRIL:

9:00-11:00	Alberto Cabada *Monotone Iterative Techniques for Ordinary Differential Equations (Part 3)*
11:00-13:00	Aleksander Ćwiszewski *Topological Approach to Dynamics of PDEs (Part 3)*
15:30-16:00	Stefano Biagi *Heteroclinic solutions for a class of boundary value problems associated with singular equations*
16:00-16:30	Francesco Esposito *On the moving plane method for singular solutions to some semilinear and quasilinear elliptic problems*
16:30-17:00	Umberto Guarnotta *Multiple nodal solutions to a Robin problem with sign-changing potential and locally defined reaction*
17:00-17:30	Władysław Klinikowski *Minimal periods of evolution equation of second order with Lipschitz nonlinearity*

FRIDAY 12 APRIL:

9:00-11:00	Tibor Krisztin *Delay Differential Equations (Part 3)*
11:00-13:00	Course Lecturers *Tutorials*