SEMANTICAL STRUCTURES FOR FUZZY LOGICS: AN INTRODUCTORY APPROACH

Gianpiero Cattaneo and Giuseppe Nisticò
Department of Mathematics, University of Calabria, Cosenza, Italy.

Abstract. Fuzzy propositional languages are introduced as sets L of formulas closed with respect to two binary operations, the connectives "and" and "or", and two unary operations, the "diametrical" negation and the "intuitionistic" negation. A classical semantical structure is given by a set of states or worlds S and a logical value function f: L x S ---→ [0,1]. In this way, to each well formed formula from L we can associate an ordered pair of subsets of S, the certainly-true and the certainly-false domains. The language is so represented into the propositional logic based on the preclusivity space (S,≠).

1 Introduction

In this work we are dealing with infinite-valued logics based on a propositional language consisting of a set L of well formed formulas closed with respect to two binary connectives, the conjunction "and" and the disjunction "or", and two unary connectives, the diametrical negation ("not") and the intuitionistic negation (to be "false"). (By means of these two denials we can further on construct a third kind of denial: "not true", according to the remark that "the history of logic provides ample precedent for distinguishing "false" from "not true"." [1])

The approach to infinite-valued logics considered here is essentially semantic in the sense that it is introduced a family of semantical states (or worlds) various formulas are applicable to and a truth value function, whose behaviour is of fuzzy kind, i.e. it assigns to any formula in any semantical state a truth value which is a real number between 0 and 1. In infinite-valued logics the more questioned assumption is the claim to assign a degree of indeterminacy or fuzziness to the statements which are neither true (truth value 1) nor false (truth value 0); this degree of indeterminacy being quantified by a well defined real number between 0 and 1. However, in this work we shall consider the study of infinite-valued logics as a point of departure to obtain informations on more general situations in which one deals with statements which, besides to the possibility of being true or false, require to take into account the possibility, what is more, that there could be some semantical situations in which we can state neither that the statement is true nor that it is false. In a certain sense, it is as if all truth values different from 1 or 0 are conceived as being collapsed into a unique truth value, "indeterminacy", and so the study of infinite-valued logic is led back to the study of a three-valued one.

For this reason, we shall represent any formula α, in a more correct way, by an ordered pair of subsets: the family of all semantical states (worlds) $S_T(\alpha)$ in which the formula α is a true sentence, e.g. the certainly-true domain of α, and the family $S_F(\alpha)$ of all semantical states in which the formula α is a false sentence, e.g. the certainly-false domain. Such an ordered pair of subsets of semantical states is a classical fuzzy proposition. Of course, all other semantical states constitute the indeterminacy-domain of the formula α.
This representation has been made in such a way that the main features of the original infinite-valued fuzzy logic are reproduced by a three-valued "fuzzy" logic, removing in this way the criticisms on the possibility of determining the exact truth values which must be assigned to statements which are neither true nor false.

The aim of this work is not to prove some astonishing logical theorem neither to state the basic thoerems of formal logic such as completeness, soundness and so on, but, rather, to explore the fundamental semantical structure of classical fuzzy logics in such a way to furnish the elements for a natural generalization to other non classical semantical situations such as preclusivity propositional logics;these lasts including as particular cases quantum logics. In a certain sense, it is as if we would seize the opportunity of studying the structure of classical fuzzy logics to mimic analogous structures of quantum logics.

2 Semantical structures for classical fuzzy logics.

In semantical logic we shall consider a set $L = \{\alpha, \beta, \gamma, \ldots\}$ of "formulas" and a set $S = \{x, y, z, \ldots\}$ of "semantical situations" or "states" or "worlds" the formulas are applicable to, according to the following

Definition 1. A semantical structure for classical fuzzy logic is a triplet (S,L,f), where S is a non-empty set of semantical states or worlds, L the non-empty set of all formulas of a suitable propositional language and $f: L \times S \dashrightarrow [0,1]$ is the fuzzy logical value function, or truth-value function, which associates to any ordered pair formed by a formula α and a semantical state x the logical value $f(\alpha,x) \in [0,1]$.

We shall say that in the semantical state x the formula α is a "true" sentence iff $f(\alpha,x) = 1$ and is a "false" one iff $f(\alpha,x)=0$; in all other cases the formula is neither true nor false, with a grade of "indeterminateness" or "fuzziness" expressed by $f(\alpha,x)$.

In a semantical structure for classical fuzzy logic (S,L,f), the set of formulas L is an ordered system $< L, 0, I, \wedge, \vee, -, \sim >$ of formulas of some propositional language where:

(a) L is an infinite set in which there exist two particular distinguished elements 0 and I, the absurd or contradictory and the certain or tautologous formula respectively, for which the following relations hold:

$$f(0,x) = 0 \ , \forall x \in S \qquad f(I,x) = 1, \forall x \in S$$

(b) $\wedge : L \times L \dashrightarrow L$ and $\vee : L\times L \rightarrow L$ are two binary operations which associate to any ordered pair of formulas α and β a new formula, "α and β" and "α or β" respectively,

(c) $- : L \dashrightarrow L$ and $\sim : L \dashrightarrow L$ are two unary operations which associate to any formula α a new formula $-\alpha$ and $\sim\alpha$, the diametrical negation of α, i.e. "non α", and the intuitionistic negation of α, i.e. "α is false", respectively, in such a way that the following axiom is satisfied

Axiom 1. The logical value function for the basic connectives $\wedge$, $\vee$, $-$, $\sim$ is given by the following rules :

(1) $f(\alpha \wedge \beta, x) := \text{g.l.b.}\ \{f(\alpha,x), f(\beta,x)\}$

(2) $f(\alpha \vee \beta, x) := \text{l.u.b.}\ \{f(\alpha,x), f(\beta,x)\}$

(3) $f(-\alpha,x) := 1 - f(\alpha,x)$

(4) $f(\sim\alpha,x) := \begin{cases} 1 & \text{iff } f(\alpha,x) = 0 \\ 0 & \text{iff } f(\alpha,x) \neq 0 \end{cases}$

Using these two negation connectives, one can construct the formula $\#\alpha := -\sim-\alpha$ which give rise to a new unary operation associating with any formula α the "exclusion negation" $\#\alpha$. By means of the previous axiom 1, we obtain that the logical value function relative to the exclusion negation is given by

$$f(\#\alpha,x) := \begin{cases} 0 & \text{iff} \quad f(\alpha,x)=1 \\ 1 & \text{iff} \quad f(\alpha,x)\neq 1 \end{cases}$$

Owing to this behaviour, the formula $\#\alpha$ should be read "α is not true".

With the help of the two denials, the diametrical and the intuitionistic ones, it is possible to introduce the modal operators of possibility and necessity respectively as

$$\Diamond\alpha := -\sim\alpha \qquad\qquad \Box\alpha := \sim-\alpha$$

which, according to axiom 1, are subject to the truth values

$$f(\Diamond\alpha,x) = \begin{cases} 1 & \text{iff} \quad f(\alpha,x) \neq 0 \\ 0 & \text{iff} \quad f(\alpha,x) = 0 \end{cases} \qquad f(\Box\alpha,x) = \begin{cases} 1 & \text{iff } f(\alpha,x)=1 \\ 0 & \text{iff } f(\alpha,x)\neq 1 \end{cases}$$

Notice that these modal unary operators of possiblity and necessity have just the truth values functions of the ones introduced by Łukasiewicz (see for instance [2]) in his infinite-valued system.

Definition 2. The semantical equivalence of two formulas from L is defined as follows:

if $f(\alpha,x) = f(\beta,x)$ for all $x \in S$ then we shall say that the formulas α and β are S-equivalent, simply equivalent, denoted by $\alpha \cong_S \beta$ or, if no confusion is likely, by $\alpha \cong \beta$.

Remark 1. It is now easy to see that the following formulas are semantically equivalent:

$$(1)\ (\alpha \vee \beta) \tilde{=} -(-\alpha \wedge -\beta) \quad (2)\ (\alpha \wedge \beta) \tilde{=} -(-\alpha \vee -\beta)$$

and so a fuzzy logic can be, more economically, introduced either as a system $\langle L,0,1,\wedge,-,\sim\rangle$ in which the disjunction is defined by the (1) or as a system $\langle L,0,1,\vee,-,\sim\rangle$ in which the conjunction is defined by the (2).

Remark 2. Other formulas which are mutually semantically equivalent are the following two groups:

$$\# \alpha \tilde{=} -\Box\alpha \tilde{=} \sim \Box \alpha \tilde{=} \# \Box \alpha \qquad \text{(non-necessity)}$$

$$\sim\alpha \tilde{=} -\Diamond\alpha \tilde{=} \sim \Diamond \alpha \tilde{=} \# \Diamond \alpha \qquad \text{(impossibility)}$$

For these reasons, $\#\alpha$ is also read as "α is not necessary" and $\sim\alpha$ as "α is impossible".

Finally, as regards to the modal operators we have also

$$\Box \alpha \tilde{=} - \Diamond -\alpha \quad \text{and} \quad \Diamond\alpha \tilde{=} -\Box -\alpha$$

2.1 The classical Chrysippian part.

Once given a semantical structure for classical fuzzylogic (S,L,f) we introduce the following

Definition 1. A formula a is said to be a classical Chrysippian formula iff it can assume only the two truth values, either 0 or 1, otherwise is a very "fuzzy formula".

The set of all classical Chrysippian formulas from L will be denoted by L_c. The restriction of the fuzzy logical value function f to $S \times L_c$ becomes a mapping $f_c : L_c \times S \longrightarrow \{0,1\}$, which is called the classical two-valued function. Therefore, we can take into account the "classical Chrisyppian" substructure (S,L_c,f_c) where S is the original set of semantical states, L_c the set of all classical Chrysippian formulas and f_c is the restriction to $L_c \times S$ of the original fuzzy logical value function, whose range contains the two values 0 and 1 only, false and true.

This classical Chrysippian substructure just coincides with the Watanabe approach to classical two-valued logic as expressed in [3] and [4] and so semantical structures for fuzzy logics can be considered as generalizations of the Watanabe approach to classical two-valued logic.

Notice that for any formula α of L

(1) there exists at least the classical Chrysippian formula $\Box\alpha$ of L_c such that $f(\alpha,x)=1$ iff $f(\Box\alpha\ ,x)=1$; moreover, if $\alpha\varepsilon L_c$ is another classical Chrysippian formula which satisfies the previous condition then $a\cong\Box\alpha$.

(2) there exists at least the classical Chrysippian formula $\Diamond\alpha$ of L_c such that $f(\alpha,x) = 0$ iff $f(\alpha,x) = 0$;,moreover,if $b\varepsilon L_c$ is another classical Chrysippian formula which satisfies the previous condition then $b\cong\alpha$.

A truth-functional treatment of the modalities was not possible in two-valued logic since $a\cong\Diamond a\cong\Box a$ so that modal distinctions collapse.

3 Fuzzy interpretations of formalized languages of zero order.

In this section we shall consider the structure (see [5])

$$B:=\langle[0,1]\ ,\ 0,1,\cap,\cup,-,\tilde{}\ \rangle$$

which is called the real unit interval Brouwer-Zadeh algebra where

$$r_1 \cap r_2 := \text{g.l.b. } \{r_1,r_2\} \tag{1}$$

$$r_1 \cup r_2 := \text{l.u.b. } \{r_1,r_2\} \tag{2}$$

$$-r: = 1 - r \tag{3}$$

$$\sim r: = \begin{cases} 1 & r=0 \\ 0 & r\neq 0 \end{cases} \tag{4}$$

A particular substructure of the real unit interval Brouwer-Zadeh algebra B is $B_c := \langle\{0,1\}\ ,0,1,\cap,\cup,\ -\rangle$ which consists of the two elements 0,1 only and is equipped with the restriction to the set of these two elements of the operations defined on B, once noticed that the two unary operations - and ~ coincide on {0,1}. Of course, B_c is the boolean algebra consisting of two elements only,i.e. the "two elements boolean algebra".

Definition 1. A B-interpretation or fuzzy interpretation (or valuation or realization) mapping for the propositional language L is a mapping $\nu : L \dashrightarrow [0,1]$ which assigns to every formula an element of $[0,1]$ in such a way that the following statements hold:

(1) $\nu(\alpha \wedge \beta) = \nu(\alpha) \cap \nu(\beta)$

(2) $\nu(\alpha \vee \beta) = \nu(\alpha) \cup \nu(\beta)$

(3) $\nu(-\alpha) = -\nu(\alpha)$

(4) $\nu(\sim\alpha) = {}^{\sim}\nu(\alpha)$

(5) $\nu(0) = 0, \nu(I) = 1.$

A fuzzy interpretation or fuzzy realization of the propositional language L is any pair $<B, \nu>$ where B is the real unit interval Brouwer-Zadeh albegra and is any fuzzy intepretation mapping.

In a semantical structure for fuzzy logic (S,L,f) for any fixed state x the mapping

$$f_x : L \dashrightarrow [0,1]$$

defined by $f_x(\alpha) := f(\alpha,x)$ is a B-interpretation mapping generated by the state x and so for every fixed semantical state $x \in S$ the pair $<B,f_x>$ is called the S fuzzy interpretation or the S fuzzy realization of the propositional language L generated by the state $x \in S$.

Definition 2. A formula α of the propositional language L is a true sentence in a S-realization $<B,f_x>$, written $\models_x \alpha$, iff $f_x(\alpha)=1$. In this case we shall say that the formula α is true in the world x.

A formula α is a tautology, to be more precise a S-tautology, written $\models_S \alpha$ or, simply, $\models \alpha$, with respect to the universe of the discurse S iff α is a true sentence for every S-interpretation f_x of L, that is in any possible world $x \in S$ the language L is applicable to .

The semantical equivalence relation between sentences from L is now expressed by the set $\{f_x : x \in S\}$ of all fuzzy interpretation mappings generated by states, as follows:

$$\alpha \equiv \beta \text{ iff } f_x(\alpha) = f_x(\beta), \forall\ x \in S. \qquad \text{(eq)}$$

That is two formulas of the language L are two equivalent sentences iff they show the same truth values whatever be the world x in S.

4. Propositional representation of fuzzy logics

We shall define as classical fuzzy proposition based on the reference space S any ordered pair (A_1,A_0) of subsets of S such that $A_1 \cap A_0 = \emptyset$. In particular we have the two trivial propositions $O=(\emptyset, S)$, the absurd one, and $I=(S,\emptyset)$, the certain one. If $x \varepsilon A_1$ then the classical fuzzy proposition (A_1,A_0) is confirmed by or is "true" in the state x, while if $x \varepsilon A_0$ then it is refuted or is "false" in this state. Of course, a classical fuzzy proposition can never be simultaneously confirmed and refuted by any state $x \varepsilon S$. If neither $x \varepsilon A_1$ nor $x \varepsilon A_0$, then the classical fuzzy proposition (A_1,A_0) is neither confirmed nor refuted, that is its truth value is "indeterminate". At any rate we stress that we have no need to assign a grade of indeterminacy or fuzziness to the propositions under examination.

In the following the set of all classical fuzzy propositions based on S will be denoted by $L_f(S)$; this last can be regarded as a distributive BZ-algebra [(8)] $<L_f(S), O, I, \cap, \cup, \sim, ->$, called the classical fuzzy propositional logic, where we define

$$(A_1,A_0) \cap (B_1,B_0) = (A_1 \cap B_1, A_0 \cup B_0) \qquad \text{(conjunction)}$$

$$(A_1,A_0) \cup (B_1,B_0) = (A_1 \cup B_1, A_0 \cap B_0) \qquad \text{(disjunction)}$$

$$\sim(A_1,A_0) = (A_0, S/A_0) \qquad \text{(intuitionistic not)}$$

$$-(A_1,A_0) = (A_0, A_1) \qquad \text{(diametrical not)}$$

Therefore, in classical fuzzy propositional logic the conjunction $p \wedge q$ of two propositions is understood to be a proposition that is true if and only if each of the propositions p,q are true and is false if and only if at least one of the propositions p,q is false; in all other cases the conjunction is indeterminate. Analogously, if either component in a disjunction $p \vee q$ is true the disjunction is true, if both components p,q are false then the disjunction pVq is false and in all other cases the disjunction is indeterminate.

The intuitionistic denial of a proposition is true whenever the original proposition is false and is false whenever the original proposition is not true for wathever reason. On the contrary, the diametrical denial of a true proposition is false, of a false one is true and of an indeterminate one indeterminate.

We can also give rise to the proposition

$$\#(A_1,A_0)=-\sim-(A_1,A_0) = (S/A_1,A_1) \qquad \text{(exclusion not)}$$

which is true iff the starting proposition is not true and is false iff the starting proposition is true.
With the help of the two non-standard orthocomplementations, the diametrical one and the intuitionistic one, it is possible to introduce the two modal-like operators of "possibility" and "necessity" according to the following definitions:

$$\Diamond(A_1,A_0)=-\sim(A_1,A_0) = (S/A_0,A_0) \quad \text{(possibility)}$$

$$\Box(A_1,A_0) =\sim-(A_1,A_0)= (A_1,S/A_1) \quad \text{(necessity)}$$

Notice that the partial ordering induced from this BZ-algebra is

$$(A_1,A_0)\sqsubseteq(B_1,B_0) \text{ iff } A_1\sqsubseteq B_1 \text{ and } B_0 \sqsubseteq A_0 \quad \text{(ordering)}$$

and the exact or closed part, i.e. the collection of all classical fuzzy propositions which coincide with their intuitionistic bi-denial, is the set

$$L_c(S)= \{(A,S/A) : A\sqsubseteq S\}$$

Let now (S,L,f) be a semantical structure for classical fuzzy logic, in analogy with [6] we can associate to any formula α the following three subsets of the set of all semantical states:

The certainly-true domain of α consisting of all semantical states with respect to which the formula α becomes a true sentence:

$$S_T(\alpha): = \{x\varepsilon S: f(\alpha, x)=1\} \qquad (1)$$

The certainly-false domain of α consisting of all semantical states to which the formula α becomes a false sentence:

$$S_F(\alpha) := \{x \in S : f(\alpha, x) = 0\} \qquad (2)$$

The indeterminacy-domain of α consisting of all other semantical states to which we can affirme that the formula α is neither true nor false:

$$S_I(\alpha) := \{x \in S : 0 < f(\alpha,x) < 1\} \qquad (3)$$

This being stated, we can associate with any formula $\alpha \epsilon L$ the fuzzy proposition $ext(\alpha)=(S_T(\alpha),S_F(\alpha))$ from $L_f(S)$, also called the extension of α. In this way we have introduced a mapping

$$ext: L \longrightarrow L_f(S)$$

which satisfies the conditions:

(1) $ext\ (\alpha \wedge \beta) = ext(\alpha) \cap ext\ (\beta)$

(2) $ext\ (\alpha \vee \beta) = ext(\alpha) \cup ext\ (\beta)$

(3) $ext\ (-\alpha) = -\ ext\ (\alpha)$

(4) $ext\ (\sim\alpha) = \sim ext(\alpha)$

(5) $ext\ (O) = (\emptyset,S)$, $ext(I) = (S,\emptyset)$

Therefore, we can say that the pair $(L_f(S), ext)$ is a classical propositional or $L_f(S)$ representation or interpretation of the set L of all formulas of our fuzzy logic.

The classical Chrysippian formulas from L_c are mapped by ext into the set $L_c(S)$ of all classical closed propositions

$$ext(\alpha) \in L_c(S), \quad \forall \alpha \epsilon L_c$$

In general, the mapping ext is not one-to-one: indeed there could be different formulas α and β whose certainly-true and certainly-false domains coincide but which are profoundly different on the indeterminacy domain.

5 Semantical relations of quasi-order on L

In the context of a semantical structure for classical fuzzy logic, one can introduce three interesting binary relations of quasi-ordering on the set of all formulas L. First of all, we have the (ld) Relation of logical deduction:

$$\alpha \precsim \beta \quad \text{iff} \quad f(\alpha, x) \leq f(\beta, x), \forall x \in S. \qquad \text{(ld)}$$

Therefore, β is logically deduced from α iff in any S-relation $\langle B, f_x \rangle$ we have that $f_x(\alpha) \leq f_x(\beta)$. It is straightforward to prove that

$$\sim\alpha \precsim -\alpha \lesssim \#\alpha, \ \forall\alpha. \qquad \text{(ne)}$$

$$\Box\alpha \precsim \alpha \precsim \Diamond\alpha, \ \forall\alpha \qquad \text{(mo)}$$

Besides the semantical relation of logical deduction we can further on introduce two other semantical binary relations:

(si) Relation of semantical inference:

$$\alpha \models \beta \quad \text{iff} \quad S_T(\alpha) \subseteq S_T(\beta) \quad \text{and} \quad S_F(\beta) \subseteq S_F(\alpha) \qquad \text{(si)}$$

which, owing to the quasi-ordering relation of section 4, can be also restated in the following way:

$$\text{iff } ext(\alpha) \sqsubseteq ext(\beta)$$

(se) Relation of semantical entailment :

$$\alpha \vdash \beta \quad \text{iff} \quad S_T(\alpha) \subseteq S_T(\beta). \qquad \text{(se)}$$

Therefore, to say that a sentence α semantically entails another sentence β is to say that whenever α is true than so is β. On the other hand, to say that a sentence β is semantically inferred by a sentence α is to say that whenever α is true so is β and whenever β is false so is α.

Trivially, we have that

$$\alpha \precsim \beta \quad \text{implies} \quad \alpha \models \beta \quad \text{implies} \quad \alpha \vdash \beta \qquad \text{(im)}$$

but in general the converse is not true. In particular, from the previous definitions it follows that a formula α is a S-tautology

$$\models \alpha \quad \text{iff} \quad I \precsim \alpha \quad \text{iff} \quad S_T(\alpha) = S$$

Definition 1. A formula α is said to be self-contradictory, or, simply, a contradiction, iff $\alpha \models -\alpha$, i.e. iff $S_T(\alpha) = \emptyset$.

The previously introduced relations of quasi-ordering induce the following corresponding relations of equivalence on L:

$\alpha \tilde{=} \beta$ iff $\alpha \lesssim \beta$ and $\beta \lesssim \alpha$ (eq-ld)

iff $f(\alpha,x)=f(\beta,x)$, $\forall x \varepsilon S$

$\alpha \bar{=} \beta$ iff $\alpha \models \beta$ and $\beta \models \alpha$ (eq-si)

iff $S_T(\alpha)= S_T(\beta)$ and $S_F(\alpha)= S_F(\beta)$

$\alpha \simeq \beta$ iff $\alpha \vdash \beta$ and $\beta \vdash \alpha$ (eq-se)

iff $S_T(\alpha) = S_T(\beta)$

While the first of these equivalence relations is just the relation of semantical equivalence introduced in section 2, definition 2, the second of these equivalence relations can be named the extensional equivalence since $\alpha \equiv \beta$ iff $ext(\alpha) = ext(\beta)$.

The extensional equivalence class generated by a formula α will be denoted by $\|\alpha\|$ and its extension Ext ($\|\alpha\|$) is defined as the extension of any, and therefore all, of the formulas from $\|\alpha\|$. Therefore, to any extensional equivalence class of formulas $\|\alpha\|$ there corresponds a unique fuzzy proposition Ext($\|\alpha\|$)$=(S_T(\alpha), S_F(\alpha))$ and this correspondence Ext from the quotient set $L/\equiv$ onto $L_f(S)$ is one-to-one; in this way the algebraic structure of distributive BZ-poset of the set of all fuzzy propositions $L_f(S)$ can be translated onto the quotient set of all extensional equivalence classes $L/\equiv$. The formulas $\alpha_j \varepsilon \|\alpha\|$ are said to be formulas which realize the fuzzy proposition Ext($\|\alpha\|$). In this way we have that, in general, several different formulas realize a unique classical fuzzy proposition.

Example. Let $ext(\alpha) = (A_1,A_0)$ then

$ext(\sim\alpha \vee \alpha)= (A_1 \cup A_0,\emptyset)$ $\quad ext(-\alpha \vee \alpha) = (A_1 \cup A_0,\emptyset)$

$ext(\sim\alpha \wedge \alpha)= (\emptyset , S)$ $\quad ext(-\alpha \wedge \alpha) = (\emptyset, A_1 \cup A_0)$.

From which we get $({\sim}\alpha V\alpha)\equiv(-\alpha V\alpha)$. Notice that in general ${\sim}\alpha V\alpha \not\equiv -\alpha V\alpha$; for instance, if $f(\alpha,x)=1/4$ then $f(-\alpha V\alpha,x)= 3/4$ and $f({\sim}\alpha V\alpha,x)=1/4$.

Finally, we shall denote by $|\alpha|$ the equivalence class generated by α using the third equivalence relation; any such equivalence class is called property. All the formulas of a property $P:=|\alpha|$ are characterized by the same certainly-true domain which can be defined as the certainly-true domain of the whole property and will be denoted by $S_T(|\alpha|)$.

Of course, in general, the certainly-false domains of all formulas belonging to a certain property P are different among them and so it is not possible to individuate a unique certainly-false domain of P starting from the certainly false domains of the formulas $\alpha_j \varepsilon P$ which represent the property P.

Notice that the condition $\alpha \simeq \beta$ in general does not imply neither $(-\alpha)\simeq(-\beta)$ nor $({\sim}\alpha)\simeq({\sim}\beta)$ and so we cannot introduce the denials -P or ~P of the property $P=|\alpha|$ as the equivalence classes $|-\alpha|$ or $|{\sim}\alpha|$, respectively, generated by any of the formulas representing the property P. On the contrary, we have that $\alpha \simeq \beta$ implies $\#\alpha\simeq\#\beta$ and so the "denial" of property $P= |\alpha|$ can be defined as the property $\#P =|\#\alpha|$.
We write $P\sqsubseteq Q$ iff $\alpha\varepsilon P$, $\beta\varepsilon Q$ and α-β, i.e. iff $S_T(\alpha)\sqsubseteq S_T(\beta)$.

Since it is obviously $P= Q$ iff $S_T(P)=S_T(Q)$ the previous relation is a partial ordering on the set of all properties, denoted by pr(L), which turns out to be a Boolean algebra with respect to the operations, whatever be $P=|\alpha|$ and $Q=|\beta|$:

$$PVQ = |\alpha V\beta| \qquad P\Lambda Q= |\alpha\Lambda\beta| \qquad \# P=|\#\alpha|$$

In particular we have that

$$S_T(PVQ) = S_T(P)\cup S_T(Q) \qquad S_T(P\Lambda Q) = S_T(P)\cap S_T(Q)$$

$$S_T(\# P) = S/S_T(P)$$

6 Rules of inference

The fact that some formulas are logical consequence of others can be formalized generalizing the relations of quasiordering of section 5, by the following definitions:

Definition 1. In a semantical structure for classical fuzzy logic,a strong or standard rule of inference which associates with a finite number of premises $\{\alpha_1,\alpha_2,\ldots,\alpha_n\}$ from L a logical consequence or conclusion $\beta\epsilon L$ is defined iff

$$\alpha_2 \wedge \alpha_2 \wedge \ldots \wedge \alpha_n \precsim \beta$$

In our opinion, contrary to [7], [8], what seems minimally required to a binary connective κ on L to be an implication connective is that the following condition is satisfied:

$\alpha\wedge(\alpha \ \kappa \ \beta) \precsim \beta$ (modus ponens)

from which, in particular, it follows that

$$\models (\alpha\kappa\beta) \text{ implies } \alpha \precsim \beta$$

i.e. every tautology of the form $\alpha\kappa\beta$ provides a strong inference rule of the form "β is a logical consequence of α".

Examples. In this connection we can introduce two implication connectives, also called i-material conditionals, in terms of the basic logical connectives conjunction, disjunction and intuitionistic negation

$$\alpha \Rightarrow \beta : = {\sim}\alpha \vee \beta$$

whose truth value function is

$$f(\alpha \Rightarrow \beta\ ) = \begin{cases} 1 & \text{iff } f(\alpha,x)=0 \\ f(\beta,x) & \text{iff } f(\alpha,x)\neq 0 \end{cases}$$

and the Sasaki material conditional defined as $\alpha \Rightarrow\!\!> \beta : = {\sim}\alpha \vee (\alpha\wedge\beta)$

with associated truth valued function

$$f(\alpha \Rightarrow\!\!> \beta\ ,x) = \begin{cases} 1 & \text{iff } f(\alpha,x) = 0 \\ \max\{ f(\alpha,x),f(\beta,x)\} & \text{iff } f(\alpha,x)\neq 0 \end{cases}$$

In this way, $(\alpha{=}{>}{>}\beta) \underset{\sim}{<} (\alpha{=}{>}\beta)$. Moreover, these two implication connectives satisfy also the following strong rule of inference:

$\sim\beta\wedge(\alpha\kappa\beta) \underset{\sim}{<} \sim\beta$ (i-modus tollens)

Notice that, once denoted these two implication connectives by κ, we have that

$$\models_x (\alpha\kappa\alpha),\ \forall x\in S/S_I(\alpha)$$

Analogously, the d-material contidionals are defined in terms of conjuction, disjunction and diametrical negation as

$$\alpha\to\beta := -\alpha\vee\beta \text{ and } \alpha{-}{>}{>}\beta := -\alpha \vee (\alpha\wedge\beta)$$

which satisfy the inference rule

$-\beta \wedge (\alpha\ \kappa\ \beta) \underset{\sim}{<} -\beta$ (d-modus tollens)

On the other hand, if one take into account the fact that any formula can be represented as a fuzzy proposition by the extension mapping, there is a more appropriate definition of rule of inference according to the definition

Definition 2. A rule of inference which associates with any family of premises $\{\alpha_j : j\in J\}$ from L a logical consequence or conclusion β is defined iff the following conditions hold:

(1-a) $\bigcap S_T(\alpha_j) \sqsubseteq S_T(\beta)$

(1-b) $S_F(\beta) \sqsubseteq \bigcup S_F(\alpha_j)$

i.e. whenever all the formulas α_j are true then also the formula β is true and if the formula β is false then at least one of the formulas α_j is false. This fact is noted as $\{\alpha_j : j\in J\} \models \beta$.

Proposition 1. The formula α satisfies the "contradictory" inference rule for the diametrical negation: $\alpha \models (-\alpha)$ iff $S_T(\alpha)=\emptyset$.

For this reason a formula $\alpha\neq 0$ such that $S_T(\alpha) = \emptyset$ holds has been called a self-contradictory formula of the language L; a self-contradiction is thus characterized by the fact that there is no semantical state in which the formula can be interpreted as a true sentence. Notice that there is no classical Chrysippian non trivial formula which satisfies the contradiction inference rule.

In conclusion, the diametrical negation violates the law of self-contradiction which requires that a formula that implies its negation is limited to the constant absurdity. Watanabe asserts that "this law is so fundamental in human thinking that any violation of this law by a proposed logic would deny its qualification as a logic "[9]. This may be linked to the believe that the fact that for classical Chrysippian formulas if the self-contradiction law is violated by α then every other formula β is deduced from α. In the fuzzy approach to inference rules this is not the case since, in general, $S_F(\beta) \not\sqsubseteq S_F(\alpha)$.

Proposition 2. If α is a self-contradictory formula, then $\alpha \models \beta$ and $a \models -\beta$ iff $S_T(\beta) \cup S_F(\beta) \sqsubseteq S_F(\alpha)$.

Besides the previous rules of inference we can consider the weak rules of inference according to the following

Definition 3. In a semantical structure for classical fuzzy logic a weak rule of inference which associates with a set of premises $\{\alpha_j : j \in J\}$ a logical consequence $\beta \in L$ is defined iff the following condition holds:

$$\bigcap S_T(\alpha_j) \sqsubseteq S_T(\beta)$$

A weak rule of inference will be noted by $\{\alpha_j : j \in J\} \vdash \beta$. Of course, if α is a self-contradiction then $\forall \beta$, $(\alpha \vdash \beta)$.

6.1 The classical Chrysippian logic.

Once given a semantical structure for classical fuzzy logic, as we have seen in section 2.1, it is possible to single out the system $\langle L_c, O, I, \wedge, - \rangle$ of all classical Chryssipian formulas endowed with two connectives, $a \wedge b$ and $-a$, and defining two other connectives $a \vee b := -(-a \wedge -b)$ and $a \to b := -a \vee b$. Of course, if one take into account the larger system of all fuzzy formulas $\langle L, O, I, \wedge, \vee, -, \sim \rangle$ it is easy to check the following semantical equivalences involving classical Chrysippian formulas only: $-a \stackrel{\sim}{\equiv} \sim a \stackrel{\sim}{=} \# a$ and $a \to b \stackrel{\sim}{=} a \twoheadrightarrow b \stackrel{\sim}{=} a \Rightarrow b \stackrel{\sim}{\equiv} a \Rightarrow\!\!> b$. In the classical Chrisippian system also the three kinds of rules of inference for any finite number of premises coalesce, in particular we have

$$a \precsim b \text{ iff } a \models b \text{ iff } a \vdash b$$

Moreover, we have the following equivalent statements:

$$\models (a \to b) \text{ iff } a \leq b \qquad \text{(ti)}$$

And so, in the classical Chrysippian system every tautology of the form $\models (a \rightarrow b)$ is equivalent to an inference rule of the form "b is a logical consequence of a".

These considerations, with the aim to avoid some misunderstandings in extension to quantum logics, lead to a clarification of the Jauch-Piron affirmation that "the role of $a \lesssim b$ (in clasical logic) corresponds obviously to the conditional, and in logic the conditional is considered a proposition just as all the other. It is the proposition which affirms 'if a then b'. (....) [On the contrary, the object $\lesssim$] is not a new element of the language, but rather a relation between certain elements of the language, and it is therefore something entirely different from the other elements of the language " [10]. Of course, it is a binary relation between formulas, to be more precise the logical deduction relation, which, as a sign, pertains to the metalanguage and is semantically specified by the set of all semantical states the language is applicable to.

Quoting [7]: "In view of the ordering properties, it is tempting, indeed it is often done, to let the partial [quasi-] order $\lesssim$ to play the role of implication in the logic. However, it has been objected to this, since implication is treated as a binary connective. This implication should be on the same linguistic level as conjunction and disjunction. HOwever, $\lesssim$ is a relation on L rather than a binary operation on L. To insist on using $\lesssim$ to play the role of implication amount to a violation of the sacrosanct distinction between object language and metalanguage. The relation $\lesssim$ should be viewed as a statement about "deducibility". One might read '$a \lesssim b$' as 'b is deducible from a'."

Therefore, $\lesssim$ is a sign of the metalanguage which differs from the sign $\rightarrow$ of the language, the very conditional connective. The relation $a \lesssim b$ does not correspond to the conditional connective $a \rightarrow b$, but, rather, is equivalent to state that $a \rightarrow b$ is a tautology, that is a quite different thing. Precisely, it is equivalent to the fact that 'from a we can infer b' or that 'b is a logical consequence of a'.

References

1. B.C. Van Fraassen, The labyrinth of quantum logics, in "Boston Studies in the Philosophy of Science", vol. XIII, (D. Reidel Pub. Co., Dordrecht-Holland, 1974).

2. N. Rescher, Many-valued logic (Mc Graw-Hill, New York, 1969).

3. S. Watanabe, Theory of propensity, in: R.S. Cohen and M.W. Wartofsky Eds., Language, Logic, and Method (D. Reidel Pub. Co., Dordrecht-Holland, 1983).

4. S. Watanabe, Knowing and Guessing (J. Wiley, New York, 1969).

5. G. Cattaneo and G. Marino, Brouwer-Zadeh posets and fuzzy set theory in Proceedings of 1st Napoli Meeting on Fuzzy Systems, Napoli, (1985) 59-70.

6. B. Mielnik, Quantum logic: is it necessarily orthocomplemented?, in: M. Flato et al. Eds., Quantum Mechanics, Determinism, Causality and Particles, (D. Reidel Publ. Co., Dordrecht-Holland, 1976).

7. L. Hermann, E.L. Mardsen and R. Piziak, Implication connectives in orthomodular lattices, Notre Dame J. Formal Logic XVI (1975) 305.

8. G. M. Hardegree, The conditional in quantum logic, Synthese 29(1974) 63.

9. S. Watanabe, A generalized fuzzy set theory, IEEE Trans. Syst., Man Cybern., SMC-8(1978)756.

10. J.M. Jauch and C. Piron, What is Quantum Logic?, in P.G.O. Freund, C.J. Goebel and J. Nambu Eds., Quanta (University of Chicago Press, 1970).