# MATHEMATICAL METHODS OF QUANTUM THEORY

## 1. HILBERT SPACES

DEFINITION 1.1. A mapping $\langle\cdot|\cdot\rangle : \mathcal{S}\times\mathcal{S}\to\mathbb{K}$, where $\mathcal{S}$ is a linear space over the field $\mathbb{K}$ of real or complex numbers, such that the following conditions hold:

(i) $\langle x|y+z\rangle = \langle x|y\rangle + \langle x|z\rangle$, $\forall x,y,z\in\mathcal{S}$,

(ii) $\langle x|\lambda y\rangle = \lambda\langle x|y\rangle$, $\forall x,y\in\mathcal{S}$, $\forall\lambda\in\mathbb{K}$,

(iii) $\langle x|y\rangle = \overline{\langle y|x\rangle}$, $\forall x,y\in\mathcal{S}$ ( $\overline{\phantom{x}}$ is complex conjugation),

(iv) $\langle x|x\rangle \geq 0$, $\forall x\in\mathcal{S}$,

(v) $\langle x|x\rangle = 0$ implies $x=0$

is an inner product, and $\mathcal{S}$ is an inner product space.

The mapping $\|\cdot\| : \mathcal{S}\to\mathbb{R}_+$, $x\to\|x\| = \sqrt{\langle x|x\rangle}$ turns out to be a norm for $\mathcal{S}$, induced by $\langle\cdot|\cdot\rangle$.
The inner product is continuous and if $x=\sum_n y_n$ then $\langle x|z\rangle = \sum_n\langle y_n|z\rangle$.

DEFINITION 1.2. If $\mathcal{H}$ is an inner product space and it is complete with respect to the induced norm, then $\mathcal{H}$ is a Hilbert space.

<u>EXAMPLES.</u> $\mathcal{L}_p(\mathbb{R}^n)$ is the linear space of measurable functions $f : \mathbb{R}^n \to \mathbb{C}$ such that $\int |f(\underline{x})|^p d\underline{x}^n < +\infty$ with respect to Lebesgue integration.

$L_p(\mathbb{R}^n)$ denotes the quotient $\mathcal{L}_p(\mathbb{R}^n)/\sim$, where $f_1 \sim f_2$ if $f_1(\underline{x}) = f_2(\underline{x})$ almost everywhere on $\mathbb{R}^n$. $L_p$ is a linear space with respect to the operations

$$[f_1]_\sim + [f_2]_\sim = [f_1 + f_2]_\sim \text{ and } \lambda [f]_\sim = [\lambda f]_\sim .$$

The space $L_2(\mathbb{R}^n)$ is an inner product space with respet to $\langle [f_1]_\sim | [f_2]_\sim \rangle = \int \overline{\varphi_1(\underline{x})}\, \varphi_2(\underline{x})\, d\underline{x}^n$ where $\varphi_1 \in [f_1]_\sim$ and $\varphi_2 \in [f_2]_\sim$. Moreover, $L_2(\mathbb{R}^n)$ turns out to be complete with respect to the norm induced by this inner product; thus $L_2(\mathbb{R}^n)$ is a Hilbert space.

Another example of Hilbert space is $\ell_2 = \{ \underline{z} = (z_n)_{n \in \mathbb{N}},\ z_n \in \mathbb{C} \text{ and } \sum_n |z_n|^2 < +\infty$. $\ell_2$ is a linear space and $\langle \underline{z} | \underline{w} \rangle = \sum_n \overline{z_n} w_n$ turns out to be an inner product, with respect to which $\ell_2$ turns out to be complete.

## 2. <u>ADJOINT OF AN OPERATOR</u>

In a finite dimensional space, say $\mathbb{C}^n$, the adjoint of a linear operator $T : \mathbb{C}^n \to \mathbb{C}^n$, is the operator $T^* : \mathbb{C}^n \to \mathbb{C}^n$ such that

$$\langle T\underline{x} | \underline{y} \rangle = \langle \underline{x} | T^* \underline{y} \rangle, \quad \forall \underline{x}, \underline{y} \in \mathbb{C}^n. \quad (1)$$

Hence, if $T$ is identified by a $n \times n$ matrix $\hat{T} = (t_{jk})$ such that $T\underline{x} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \hat{T}\underline{x} \equiv \begin{bmatrix} t_{11} & \cdots & t_{1n} \\ \vdots & t_{jk} & \\ & \cdots & t_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, then $T^*$ is identified by the matrix $\hat{T}^* = \overline{(\hat{T}^t)} = \begin{bmatrix} \overline{t_{11}} & & \overline{t_{n1}} \\ & \overline{t_{kj}} & \\ \overline{t_{1n}} & & \ddots \end{bmatrix}$

(here $\overline{\phantom{x}}$ denotes complex conjugation).

In a Hilbert space $\mathcal{H}$ the dimension can be not finite, e.g. in $L_2$ and $l_2$, and this allows the domain $D_T$ of a linear operator to be different from $\mathcal{H}$.
In order to take into account this feature, the general definition of adjoint of an operator must be more articulate.

<u>DEFINITION 2.1.</u> Let $T : D_T \to \mathcal{H}$ be a linear operator with domain $D_T$ dense in $\mathcal{H}$, i.e. such that $\overline{D_T} = \mathcal{H}$. The domain

$$D_T^* = \{ y \in \mathcal{H} \mid \exists y^* \in \mathcal{H}, \langle Tx \mid y \rangle = \langle x \mid y^* \rangle, \forall x \in D_T \}$$

is a linear subspace. Then the adjoin $T^*$ of $T$ is $T^* : D_T^* \to \mathcal{H}$, $y \to T^* y = y^*$;
in such a way $\langle Tx | y \rangle = \langle x | T^* y \rangle$ $\forall x \in D_T, \forall y \in D_T^*$ holds, of course.

This definition is consistent, because for every $y \in D_T^*$, the vector $y^*$ such that $\langle Tx | y \rangle = \langle x | y^* \rangle$ is unique. Indeed, let $y_1^*$ and $y_2^*$ be such that
$\langle Tx | y \rangle = \langle x | y_1^* \rangle = \langle x | y_2^* \rangle$, i.e.
$\langle x | y_1^* - y_2^* \rangle = 0$, $\forall x \in D_T$. Since $\overline{D_T} = \mathcal{H}$, a sequence $(x_n)_n \subseteq D_T$ exists such that $\lim_n x_n = y_1^* - y_2^*$.
Hence $\| y_1^* - y_2^* \|^2 = \langle y_1^* - y_2^* | y_1^* - y_2^* \rangle = \lim_n \langle x_n | y_1^* - y_2^* \rangle = 0$,
i.e. $y_1^* = y_2^*$.

<u>DEFINITION 2.2.</u> An operator $T$ is symmetric if $\langle Tx | y \rangle = \langle x | Ty \rangle$ $\forall x, y \in D_T$.

If $T$ is symmetric, then $T^*$ is an extension of $T$, of course. In general, the following implication holds.

$$T_1 \subseteq T_2 \quad \text{implies} \quad T_2^* \subseteq T_1^*. \qquad (2)$$

When $T^* = T$, the operator $T$ is said self-adjoint. Implication (2) entails that, in general, a symmetric operator is not self-adjoint, because $T \subseteq T^*$ does not ensure $D_T = D_{T^*}$.

EXAMPLE: multiplication operator.

Let us consider the Hilbert space $L_2(\mathbb{R})$; to simplify the notation, we denote the equivalence class $[f]_\sim$ generated by any function belonging to $[f]_\sim$, such as $f$ itself: $[f]_\sim \equiv f$. The transformation of a function $\psi$ into the function $\varphi(x) = x\psi(x)$ can define a "multiplication" operator of $L_2(\mathbb{R})$ provided that $x\psi(x) \in \mathcal{L}_2(\mathbb{R})$:

$$\tilde{F} : L_2(\mathbb{R}) \to L_2(\mathbb{R}), \quad [\psi]_\sim \longrightarrow F[\psi]_\sim = [\varphi]_\sim, \quad \varphi(x) = x\psi(x).$$

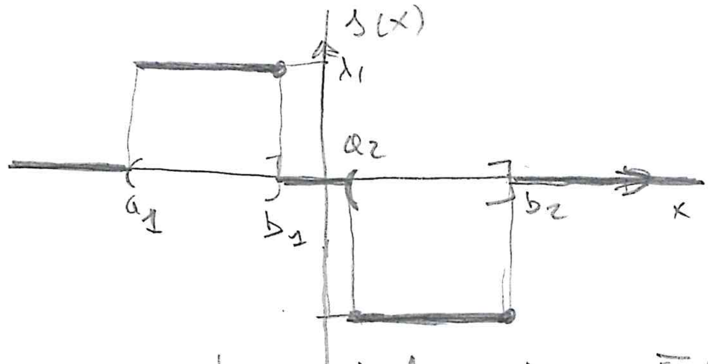
But there are $\psi \in \mathcal{L}_2(\mathbb{R})$ for which $x\psi(x) \notin \mathcal{L}_2(\mathbb{R})$, e.g. $\psi(x) = \frac{1}{1+|x|}$. Thus, the multiplication operator cannot be defined on the whole $L_2(\mathbb{R})$. There are different consistent choices for the domain of the multiplication operator:

(i) The subspace $\Sigma(\mathbb{R})$ of all step functions:

$$\Sigma(\mathbb{R}) = \{ s : \mathbb{R} \to \mathbb{C} \mid s(x) = \textstyle\sum_{n=1}^{N} \lambda_n \chi_{(a_n, b_n]}(x) \},$$

where $\chi_{(a_n, b_n]}$ is the characteristic functional of $(a_n, b_n]$

and $\lambda_n \in \mathbb{C}$.
(If $x \notin (a_n, b_n]\ \forall n$, $s(x) = 0$).




Then the multiplication operator defined on $\Sigma(\mathbb{R})$ is
$$F_1 : \Sigma(\mathbb{R}) \to L_2(\mathbb{R}) \ , \ s \to F_1 s \ , \ (F_1 s)(x) = x s(x)$$
is consistently defined because $x\, s(x) \in \mathcal{L}_2(\mathbb{R})$, $\forall s \in \Sigma(\mathbb{R})$.
(ii) The largest subspace on which the multiplication operator can be defined is
$$D_F = \{\psi \in L_2(\mathbb{R}) \mid x\psi(x) \in \mathcal{L}_2(\mathbb{R})\}.$$
Thus,
$$F : D_F \to L_2(\mathbb{R}) \ , \ \psi \to F\psi \ , \ (F\psi)(x) = x\psi(x),$$
is the most extended multiplication operator.

Both $F_1$ and $F$ are densely defined.

PROPOSITION 2.1. $F$ is self-adjoint.

PROOF. If $\varphi \in D_F$ then $\langle F\psi | \varphi\rangle = \int x \overline{\psi(x)} \varphi(x) dx =$
$= \int \overline{\psi(x)} (x\varphi(x)) dx = \langle \psi | F\varphi \rangle$, $\forall \psi \in D_F$. Thus $D_F \subseteq D_F^*$.
If $\varphi \in D_F^*$, then $\langle F\psi | \varphi\rangle = \int x\, \overline{\psi(x)}\, \varphi(x) dx = \int \overline{\psi(x)} (x \varphi(x)) dx =$
$= \langle \psi | \varphi^* \rangle = \int \overline{\psi(x)}\ \varphi^*(x)\, dx$, $\forall \psi \in D_F$, i.e.
$$\int \overline{\psi(x)}\, (x\varphi(x) - \varphi^*(x))\, dx = 0 \quad \forall \psi \in D_F,$$
in particular, $\forall \psi = \chi_{(a,b]}$; therefore
$\int \chi_{(a,b]}(x) (x\varphi(x) - \varphi^*(x))\, dx = \int_a^b (x\varphi(x) - \varphi^*(x)) dx = 0$, $\forall (a,b] \subseteq \mathbb{R}$;
i.e. $x\varphi(x) = \varphi^*(x) \in \mathcal{L}_2(\mathbb{R})$ almost everywhere. Thus $D_F^* \subseteq D_F$.

# 3. ORTHONORMAL SYSTEMS AND BASES.

## 3.1. Orthonormal Systems.

Given a Hilbert space $\mathcal{H}$, an orthonormal system is a family $\{u_\alpha\}_\alpha$ of vectors in $\mathcal{H}$ such that

$$\langle u_\alpha | u_\beta \rangle = \begin{Bmatrix} 0 & \text{if } \alpha \neq \beta \\ 1 & \text{if } \alpha = \beta \end{Bmatrix} = \delta_{\alpha\beta}. \qquad (3)$$

In a Hilbert space of finite dimension $n$, say $\mathbb{C}^n$, every orthonormal system has $n$ vectors at most.

In $l_2$ we can define a vector $\underline{e}_m = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \text{-}m$ for each $m \in \mathbb{N}$.

The family $\{\underline{e}_m\}_m$ satisfies $\langle \underline{e}_m | \underline{e}_n \rangle = \delta_{mn}$.

Therefore, $\{\underline{e}_m\}$ is an orthonormal system; it is not finite, in particular $\{\underline{e}_m\}$ is a countable set.

In general, an orthonormal system can be also not countable. This cannot occur if $\mathcal{H}$ is separable. We recall that a Hilbert space is separable if a countable set $\mathcal{K} \subseteq \mathcal{H}$ exists such that $\overline{\mathcal{K}} = \mathcal{H}$.

**PROPOSITION 3.1.** If $\mathcal{H}$ is separable then every orthonormal system $\{u_\alpha\}_\alpha$ is at most countable.

PROOF. For each $\alpha$, let us define the spherical neighborough $S_\alpha$ of radius $\frac{1}{2}$. Every $S_\alpha$ cannot contain another vector $u_\beta$ of the orthonormal system because $\|u_\alpha - u_\beta\| = \sqrt{2}$.

Furthermore, every $S_\alpha$ does not intersecate any other $S_\beta$.

Let $\mathcal{K}$ be a countable set such that $\overline{\mathcal{K}} = \mathcal{H}$.

In every $S_\alpha$ there are infinite vectors of $\mathcal{K}$.

Let us fix just one $x_\alpha \in K \cap S_\alpha$ for each $S_\alpha$; in so doing there is a bijective correspondence between $\{x_\alpha\}_\alpha$ and $\{u_\alpha\}$. But $\{x_\alpha\} \subseteq \mathcal{K}$, so that $\{x_\alpha\}$ is countable at most. Due to the bijectivity of the correspondence $\{x_\alpha\}_\alpha \leftrightarrow \{u_\alpha\}_\alpha$, also $\{u_\alpha\}_\alpha$ must be countable.

All Hilbert spaces $\mathbb{R}$, $\mathbb{C}$, $\mathbb{R}^n$, $\mathbb{C}^n$, $l_2$, $L_2(\mathbb{R}^n)$ are separable. Therefore, an orthonormal system in one of these space is countable at most. In particular, in $\mathbb{R}^n$, $\mathbb{C}^n$ must be finite.

DEFINITION 3.1. An orthonormal basis of a separable Hilbert space $\mathcal{H}$ is an orthonormal system $\{u_n\}_n$ such that for every vector $x \in \mathcal{H}$ a sequence of numbers $\{\lambda_n\}_n$ exists such that

$$x = \sum_n \lambda_n u_n. \qquad (4)$$

If $\{u_n\}$ is an orthonormal basis of $\mathcal{H}$, then for each $x \in \mathcal{H}$, $\lambda_n = \langle u_n | x \rangle$, indeed $\langle u_n | x \rangle = \langle u_n | \sum_m \lambda_m u_m \rangle = \sum_m \lambda_m \langle u_n | u_m \rangle = \lambda_n$ by (3).

PROPOSITION 3.2. If $\{u_n\}_n$ is an orthonormal system, then $\sum_n |\langle u_n | x \rangle|^2 \leq \|x\|^2$, $\forall x \in \mathcal{H}$, and the sign = holds iff $x = \sum_n \langle u_n | x \rangle u_n$.

PROOF. The thesis follows from $\|x - \sum_{n=1}^{m} \langle u_n | x \rangle u_n\| \geq 0$.

In a Hilbert space $\sum_n \lambda_n u_n$ converges iff $\sum_n |\lambda_n|^2$ converges.

PROPOSITION 3.3. If $\{u_m\}_m$ is an orthonormal system in a separable Hilbert space, then the following statements are equivalent.

(i) $\{u_m\}_m$ is a basis;

(ii) $\langle x|x\rangle = \sum_m \langle x|u_m\rangle\langle u_m|x\rangle$, $\forall x \in \mathcal{H}$ (first Parseval identity);

(iii) $\langle x|y\rangle = \sum_m \langle x|u_m\rangle\langle u_m|y\rangle$, $\forall x,y \in \mathcal{H}$ (second Parseval identity);

(iv) $\{u_m\}_m$ is a maximal orthonormal system of $\mathcal{H}$;

(v) $\mathcal{H} = \overline{Sp(\{u_m\}_m)}$.

## 4. MATRIX REPRESENTATION OF OPERATORS

If $\mathcal{H}$ is a Hilbert space of dimension $N$ (finite), then then fixed an orthonormal basis $\{u_m\}_{m=1}^N$, the correspondence

$$\mathcal{H} \to \mathbb{C}^N, \quad x \to \hat{x} = \begin{bmatrix} \langle u_1|x\rangle \\ \langle u_2|x\rangle \\ \vdots \\ \langle u_N|x\rangle \end{bmatrix} \tag{5}$$

is an isomorphism of linear space.

If $T$ is any linear operator of $\mathcal{H}$, then $T$ can be represented by a matrix $\hat{T} = (t_{jk})$, where $t_{jk} = \langle u_j|Tu_k\rangle$, i.e.

$$\hat{T} = \begin{bmatrix} \langle u_1|Tu_1\rangle & \langle u_1|Tu_2\rangle & \cdots & \langle u_1|Tu_N\rangle \\ \vdots & \vdots & \langle u_j|Tu_k\rangle & \vdots \\ \langle u_N|Tu_1\rangle & \cdots & \cdots & \langle u_N|Tu_N\rangle \end{bmatrix}.$$

Indeed, for every $x \in \mathcal{H}$, the vector $y = Tx$ is represented by $\hat{y} = \begin{bmatrix} \langle u_1|Tx\rangle \\ \vdots \\ \langle u_N|Tx\rangle \end{bmatrix}$ according to (5).

Since $x = \sum_m \langle u_m|x\rangle u_m$, we have $\hat{y} = \begin{bmatrix} \langle u_1|Tu_1\rangle & \cdots & \langle u_1|Tu_N\rangle \\ & & \\ \langle u_N|Tu_1\rangle & & \end{bmatrix} \begin{bmatrix} \langle u_1|x\rangle \\ \vdots \\ \langle u_N|x\rangle \end{bmatrix} = \hat{T}\hat{x}$.

If $\mathcal{H}$ has not finite dimension, any orthonormal basis $\{u_n\}_n$ is countable. In this case $\mathcal{H}$ is always isomorphic to the Hilbert space $l_2$.
Indeed, fixed an orthonormal basis $\{u_n\}_n \subseteq \mathcal{H}$, for any $x \in \mathcal{H}$ the sequence $\begin{bmatrix} \langle u_1|x\rangle \\ \langle u_2|x\rangle \\ \vdots \\ \langle u_n|x\rangle \\ \vdots \end{bmatrix} \equiv \hat{x}$

is a vector of $l_2$ for the Parseval identity (Prop. 3.3 ii). Then, the correspondence

$$\mathcal{H} \to l_2 , \quad x \to \hat{x} = \begin{bmatrix} \langle u_1|x\rangle \\ \vdots \\ \langle u_n|x\rangle \\ \vdots \end{bmatrix}$$

is an isomorphism.
The matrix representation $\hat{T}$ requires some care, because of convergence and domain problems. However, the matrix representation of an operator $T$ always exists under the condition that an orthonormal basis $\{u_n\}_n$ exists such that

$$\{u_n\}_n \subseteq D_T \cap D_{T^*} .$$

Indeed, in this case we have, $\forall\, x \in D_T$, define $y = Tx$. Then

$$\langle u_n|Tx\rangle = \langle T^* u_n|x\rangle = \sum_m \langle T^* u_n|u_m\rangle\langle u_m|x\rangle$$

$$= \sum_m \langle u_n|Tu_m\rangle \langle u_m|x\rangle = \sum_m t_{nm} \langle u_m|x\rangle ,$$

i.e. $\quad \hat{y} = \hat{T}\hat{x}$ , where $\hat{T} = \begin{bmatrix} \hat{t}_{11} \cdots & \hat{t}_{1m} \cdots \\ \hat{t}_{21} & \vdots \quad \hat{t}_{nm} \end{bmatrix}$

is a matrix with countable rows and columns, and $\hat{t}_{nm} = \langle u_n|Tu_m\rangle$;

## PROBLEMS.

4.1. The student is asked to show that if $T$ has a matrix representation $\hat{T}$, then $T^*$ has a matrix representation given by the transpose of the complex conjugate of $\hat{T}$, i.e. $\widehat{T^*} = \overline{\hat{T}}^t$.

4.2. The student is asked to show that for a symmetric operator $T$ the matrix representation $\hat{T}$ is hermitean, i.e. $\hat{t}_{jk} = \overline{\hat{t}}_{kj}$.

4.3. Prove that:

(i) $T \subseteq T^{**}$;

(ii) $T_1^* + T_2^* \subseteq (T_1 + T_2)^*$, $(\lambda T)^* = \bar{\lambda} T^*$

(iii) $T_1 \subseteq T_2 \Rightarrow T_2^* \subseteq T_1^*$

(iv) $T_2^* T_1^* \subseteq (T_1 T_2)^*$.

# 5. PROJECTION OPERATORS

Let $\mathcal{H}$ be a separable and complex Hilbert space. Let us consider a closed subspace $\mathcal{M} \subseteq \mathcal{H}$, $\mathcal{M} \neq \{0\}$. A non zero vector $u_1 \in \mathcal{M}$ exists, with $\|u_1\| = 1$, then a maximal orthonormal system $\{u_j\}_j \subseteq \mathcal{M}$ exists too, which is an orthonormal basis of $\mathcal{M}$ by PROP. 3.3. Given any $x \in \mathcal{H}$ $\sum_j |\langle u_j | x\rangle|^2$ converges ($\sum_{j=1}^N |\langle u_j | x\rangle|^2 \leq \|x\|^2$ by PROP. 3.2), so the series $\sum_j \langle u_j | x \rangle u_j$ must converge in $\mathcal{M}$ because $\mathcal{M}$ is an Hilbert space by itself. Hence, a mapping

$$P : \mathcal{H} \to \mathcal{M}, \quad x \to Px = \sum_j \langle u_j | x \rangle u_j$$

can be defined for each closed subspace $\mathcal{M}$, called the projection operator on the subspace $\mathcal{M}$.

In fact, $Px$ is the vector of $\mathcal{M}$ that realizes the least distance from $x$, and this feature motivates the name "projection". Indeed, any vector $y \in \mathcal{M}$ can be written as $y = \sum_j \lambda_j u_j$. We shall prove that $\|x - \sum_j \langle u_j|x\rangle u_j\|^2 \leq \|x - \sum_j \lambda_j u_j\|^2$:

$$\|x - \sum_j \lambda_j u_j\|^2 = \|(x - \sum_j \langle u_j|x\rangle u_j) + \sum_j (\langle u_j|x\rangle - \lambda_j) u_j\|^2.$$

By computing the hand rightside in terms of inner product, and exploiting $\langle u_j | u_k \rangle = \delta_{jk}$, we find $\|x - \sum_j \lambda_j u_j\|^2 =$

$$= \|x - \sum_j \langle u_j|x\rangle\|^2 + \sum_j |\langle u_j|x\rangle - \lambda_j|^2 + \sum_k \langle u_k|x\rangle\langle x|u_k\rangle +$$

$$+ \sum_{jk} \langle x|u_j\rangle\langle u_k|x\rangle\langle u_j|u_k\rangle - \sum_k \lambda_k \langle x|u_k\rangle +$$

$$+ \sum_{jk} \lambda_k \langle x|u_j\rangle\langle u_j|u_k\rangle + \sum_j \langle x|u_j\rangle\langle u_j|x\rangle - \sum_j \overline{\lambda_j}\langle u_j|x\rangle +$$

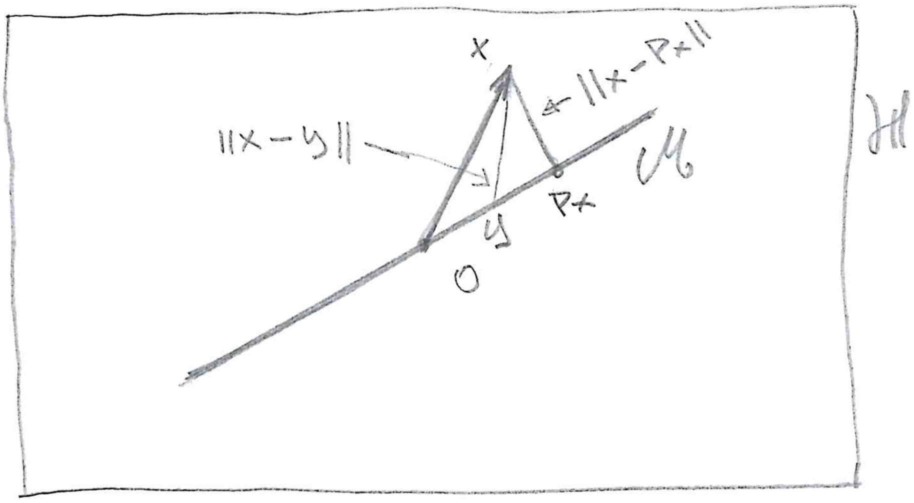
$$- \sum_{jk} \langle x|u_j\rangle\langle u_k|x\rangle\langle u_j|u_k\rangle + \sum_{jk} \overline{\lambda_j}\langle u_k|x\rangle\langle u_j|u_k\rangle =$$

$$= \|x - \sum_j \langle u_j|x\rangle u_j\|^2 + \sum_j |\langle u_j|x\rangle - \lambda_j|^2,$$

where Prop. 3.2. has been used. Thus

$$\|x - y\|^2 = \|x - Px\|^2 + \sum_j |\langle u_j|x\rangle - \lambda_j|^2 \geq \|x - Px\|^2. \quad (6)$$

From (6) we infer that $Px$ is the unique $y \in \mathcal{M}$ realizing least distance.




5.1. PROPERTIES OF PROJECTION OPERATORS

According to our definition, a projection operator $P$ is completely identified by its subspace $\mathcal{M}$ generated by its orthonormal basis $\{u_j\}_j$. Now we can consider the set $\mathcal{M}^\perp = \{y \in \mathcal{H} \mid \langle y|x\rangle = 0 \ \ \forall x \in \mathcal{M}\}$ of all vectors orthogonal to $\mathcal{M}$.

<u>PROBLEM 5.1.</u> Prove that $\mathcal{M}^\perp$ is a closed subspace of $\mathcal{H}$.

Since $\mathcal{M}^\perp$ is a closed subspace, it has an orthonormal basis $\{v_k\}_k$ ($v_k \perp u_j \ \ \forall j, k$, of course), and there is a projection operator $P^\perp$ on $\mathcal{M}^\perp$:

$$P^\perp x = \sum_k \langle v_k|x\rangle v_k \ .$$

Now, the orthonormal system $\{w_m\}_m = \{u_j\}_j \cup \{v_k\}_k$ is an orthonormal basis of the whole $\mathcal{H}$. Indeed, $\{w_m\}_m$ is a maximal orthonormal system of $\mathcal{H}$: if $x_0 \in \mathcal{H}$ and $\langle x_0|w_m\rangle = 0$, $\forall m$, then $x_0 \in \mathcal{M}^\perp$ because for every $z \in \mathcal{M}$ we have $\langle x_0|z\rangle = \sum_j \langle x_0|u_j\rangle\langle u_j|z\rangle = 0$.
But, similarly, $\langle x_0|y\rangle = 0 \ \ \forall y \in \mathcal{M}^\perp$; then $x_0 \in \mathcal{M}^\perp \cap (\mathcal{M}^\perp)^\perp$; now, $\mathcal{M}^\perp \cap (\mathcal{M}^\perp)^\perp = \{0\}$; so $x_0 = 0$.

Since $\{w_m\}_m = \{u_j\}_j \cup \{v_k\}_k$ is maximal in $\mathcal{H}$, according to Prop. 3.2. it is an orthonormal basis of $\mathcal{H}$. Therefore,

$$\forall x \in \mathcal{H}, \quad x = \Sigma_m \langle w_m | x \rangle w_m = \Sigma_j \langle u_j | x \rangle u_j + \Sigma_k \langle v_k | x \rangle v_k$$
$$= Px + P^\perp x .$$

This equality implies $P^\perp = \mathbb{1} - P$.
By making use of this last relation we obtain

$$P^{\perp\perp} = (P^\perp)^\perp = \mathbb{1} - P^\perp = \mathbb{1} - (\mathbb{1} - P) = P,$$

i.e., $P^{\perp\perp} = P$, equivalent to $\mathcal{M}^{\perp\perp} = \mathcal{M}$.

PROPOSITION 5.1. Every projection operator is linear

PROOF. $P(\lambda x + \mu y) = \Sigma_j \langle u_j | \lambda x + \mu y \rangle u_j =$
$= \lambda \Sigma_j \langle u_j | x \rangle u_j + \mu \Sigma_j \langle u_j | y \rangle u_j$
$= \lambda Px + \mu Py$.

PROPOSITION 5.2. A projection operator is bounded, self-adjoint and idempotent ($P^2 = P$).

PROOF. Bounded: $\|Px\|^2 = \langle \Sigma_j \langle u_j | x \rangle u_j | \Sigma_k \langle u_k | x \rangle u_k \rangle =$
$= \Sigma_j |\langle u_j | x \rangle|^2 \leq \|x\|^2$;

Furthermore, if $x \in \mathcal{M}$, $Px = x$; then, $\|P\| = 1$.

Self-adjoint: $\langle Px | y \rangle = \langle \Sigma_j \langle u_j | x \rangle u_j | \Sigma_i \langle u_i | y \rangle u_i + \Sigma_k \langle v_k | y \rangle v_k \rangle =$
$= \Sigma_j \overline{\langle u_j | x \rangle} \langle u_j | y \rangle, \ \forall x, y \in \mathcal{H}$;

Analogously we find $\langle x | Py \rangle = \Sigma_i \overline{\langle u_i | x \rangle} \langle u_i | y \rangle = \langle Px | y \rangle$.

Idempotent: $P^2 x = P(Px) = P \Sigma_j \langle u_j | x \rangle u_j = \Sigma_j \langle u_j | x \rangle P u_j$;
but $P u_j = u_j$; thus $P^2 x = \Sigma_j \langle u_j | x \rangle u_j = Px$.

Boundness, self-adjointness and idempotence completely characterize a projection operator, according to Prop. 5.3

PROPOSITION 5.3. If $T : \mathcal{H} \to \mathcal{H}$ is a bounded, self-adjoint and idempotent operator, then it is a projection operator.

PROOF. The image $T\mathcal{H}$ is a closed subspace:
Indeed, (i) $x \in T\mathcal{H} \Rightarrow x = Tz = T^2 z = Tx$, i.e. $x = Tx$;
(ii) if $\{x_n\} \subseteq T\mathcal{H}$ and $\lim_n x_n = x$, then $\lim_n Tx_n = Tx \in T\mathcal{H}$, by boundness;
but $Tx_n = x_n$; therefore $Tx = x \in T\mathcal{H}$.
Now, if $y \in (T\mathcal{H})^\perp$ then $Ty = 0$.
Indeed, $\langle y | x \rangle = 0$, $\forall x \in T\mathcal{H}$, i.e. $\langle y | Tz \rangle = 0$ $\forall z \in \mathcal{H}$;
but $\langle y | Tz \rangle = \langle Ty | z \rangle = 0$ $\forall z \in \mathcal{H} \Rightarrow Ty = 0$.
Let $\{w_n\}_n = \{u_j\}_j \cup \{v_k\}_k$ the orthonormal basis of $\mathcal{H}$ formed by an orthonormal basis $\{u_j\}_j$ of $(T\mathcal{H})$ and an orthonormal basis $\{v_k\}_k$ of $(T\mathcal{H})^\perp$;
then

$$Tx = T\left(\Sigma_j \langle u_j | x \rangle u_j + \Sigma_k \langle v_k | x \rangle v_k\right) =$$

$$= \Sigma_j \langle u_j | x \rangle T u_j + \Sigma_k \langle v_k | x \rangle T v_k$$

$$= \Sigma_j \langle u_j | x \rangle u_j .$$

Thus $T$ coincides with the projection operator of the subspace $\mathcal{M} = T\mathcal{H}$.

## 5.2. ALGEBRAIC PROPERTIES

Let us denote by $\Pi(\mathcal{H})$ the set of all projection operators of the complex and separable Hilbert space $\mathcal{H}$.

PROPOSITION 5.4. Let $P_1, P_2$ projection operators of $\mathcal{H}$. The product $P_1 P_2$ is a projection iff $P_1 P_2 = P_2 P_1$.

PROOF. $P_1 P_2$ is always bounded. Therefore, according to Prop. 5.3 it is sufficient to prove that $P_1 P_2$ is self-adjoint and idempotent. Suppose that $P_1 P_2 = P_2 P_1$. Now, since the domain of $P_1 P_2$ is the whole $\mathcal{H}$, we have

$$(P_1 P_2)^* = P_2^* P_1^* = P_2 P_1 = P_1 P_2 \quad \text{if } P_1 P_2 = P_2 P_1.$$

On the other hand, $(P_1 P_2)^2 = P_1 P_2 P_1 P_2 = P_1^2 P_2^2 = P_1 P_2$.

The converse is an exercise.

In general, if two operators A and B satisfy $AB = BA$, we shall say that they commute; hence, by introducing the form $[A, B] = AB - BA$, we can say that A and B commute whenever $[A, B] = 0$. Accordingly, Prop. 5.4. states that two projection operators $P_1, P_2$ commute iff $P_1 P_2 \in \Pi(\mathcal{H})$.

PROPOSITION 5.5. Let $P_1, P_2$ be projection operators with $\mathcal{M}_1 = P_1 \mathcal{H}$ and $\mathcal{M}_2 = P_2 \mathcal{H}$. Then $P_1 \perp P_2$, i.e. $\mathcal{M}_1 \perp \mathcal{M}_2$, iff $P_1 P_2 = 0$.

PROOF. If $P_1 P_2 = 0$, then $\langle P_1 x | P_2 y \rangle = \langle x | P_1 P_2 y \rangle = 0$, $\forall x, y \in \mathcal{H}$. Conversely, $\langle P_1 x | P_2 y \rangle = 0 \ \forall x, y \in \mathcal{H} \Rightarrow P_1 P_2 = 0$.

Set theoretic inclusion is a partial order relation in the class of all closed subspaces of $\mathcal{H}$, which can be used to define a (isomorphic) partial order relation on $\Pi(\mathcal{H})$ by:

$P_1 \leq P_2$ iff $\mathcal{M}_1 \subseteq \mathcal{M}_2$, where $\mathcal{M}_1 = P_1\mathcal{H}$, $\mathcal{M}_2 = P_2\mathcal{H}$.

PROPOSITION 5.6. Given $P_1, P_2 \in \Pi(\mathcal{H})$, the following statements are equivalent.
(i) $P_1 \leq P_2$ ; (ii) $P_1P_2 = P_1$ ; $\langle x|P_1x\rangle \leq \langle x|P_2x\rangle$, $\forall x \in \mathcal{H}$.

PROOF. (i) implies that an orthonormal basis $\{u_j\}$ of $\mathcal{M}_1$ and an orthonormal system $\{v_k\}$ exist such that $\{u_j\}\cup\{v_k\}$ is an orthonormal basis of $\mathcal{M}_2$.
Then

$$P_1P_2x = P_1\left(\sum_j \langle u_j|x\rangle u_j + \sum_k \langle v_k|x\rangle v_k\right)$$
$$= \sum_m \langle u_m|\textstyle\sum_j \langle u_j|x\rangle u_j + \sum_k \langle v_k|x\rangle v_k\rangle u_m$$
$$= \sum_{m,j} \langle u_m|u_j\rangle\langle u_j|x\rangle u_m + \sum_{mk}\langle u_m|v_k\rangle\langle v_k|x\rangle u_m$$
$$= \sum_j \langle u_j|x\rangle u_j = P_1x .$$

So (i) $\Rightarrow$ (ii). Conversely if (ii) holds, $\forall x \in \mathcal{M}_1$
$P_2x = P_2P_1x = P_1P_2x = P_1x = x$, i.e. $x \in \mathcal{M}_2$
($[P_1,P_2] = 0$ by Prop. 5.4.).
If (i) holds, then $\langle x|P_2x\rangle = \langle x|\sum_j \langle u_j|x\rangle u_j + \sum_k \langle v_k|x\rangle v_k\rangle$
$= \langle x|P_1x\rangle + \sum_k |\langle v_k|x\rangle|^2 \geq \langle x|P_1x\rangle$. So, (i) $\Rightarrow$ (3).
If (iii) holds, $0 \leq \langle x|x\rangle - \langle x|P_2x\rangle \leq \langle x|x\rangle - \langle x|P_1x\rangle$;
if $x \in \mathcal{M}_1$, $P_1x = x$, then $0 = \langle x|x\rangle - \langle x|P_1x\rangle \geq \langle x|x - P_2x\rangle \geq 0$;
therefore $0 = \langle x|(\mathbb{1} - P_2)x\rangle = \langle(\mathbb{1}-P_2)x|(\mathbb{1}-P_2)x\rangle$, i.e. $P_2x = x$.

PROP. 5.7. If $P_1, P_2 \in \Pi(\mathcal{H})$, then $(P_1+P_2) \in \Pi(\mathcal{H})$ iff $P_1 \perp P_2$.

PROOF. If $P_1 \perp P_2$, $P_1 P_2 = 0$, then $(P_1+P_2)^2 = P_1 + P_2$, i.e. $P_1 + P_2$ is idempotent, while self-adjointness and boundness are trivial.
Conversely, if $P_1 + P_2 \in \Pi(\mathcal{H})$, by Prop. 5.6. we have $P_1 \leq P_1 + P_2$, then $P_1 = P_1(P_1+P_2) = P_1^2 + P_1 P_2$, i.e. $P_1 P_2 = 0$.

LEMMA. 5.8. If $P_1 P_2 \in \Pi(\mathcal{H})$, then $P_1 P_2 \mathcal{H} = \mathcal{M}_1 \cap \mathcal{M}_2$, where $\mathcal{M}_1 = P_1 \mathcal{H}$ and $\mathcal{M}_2 = P_2 \mathcal{H}$.

PROOF. If $x \in P_1 P_2 \mathcal{H}$, then $P_1 P_2 x = x$, therefore $x = P_1 P_2 x = P_1^2 P_2 x = P_1 x$, i.e. $x \in \mathcal{M}_1$; analogously $x \in \mathcal{M}_2$ is proved.
Conversely, if $x \in \mathcal{M}_1 \cap \mathcal{M}_2$ then $P_1 P_2 x = P_1 x = x$, i.e. $x \in P_1 P_2 \mathcal{H}$.

PROPOSITION 5.8. Given $P_1, P_2 \in \Pi(\mathcal{H})$, $[P_1, P_2] = 0$ iff $Q_0, Q_1, Q_2 \in \Pi(\mathcal{H})$ exist, with $Q_0 \perp Q_1 \perp Q_2 \perp Q_0$ such that $P_1 = Q_1 + Q_0$ and $P_2 = Q_2 + Q_0$.

PROOF. If $P_1 = Q_1 + Q_0$ and $P_2 = Q_2 + Q_0$ with $Q_1 Q_0 = Q_2 Q_0 = Q_1 Q_2 = 0$, then $[P_1, P_2] = Q_1 Q_2 - Q_2 Q_1 + Q_0 Q_2 - Q_2 Q_0 + Q_0 Q_1 - Q_1 Q_0 + Q_0 Q_0 - Q_0 Q_0 = 0$
Conversely, if $[P_1, P_2] = 0$, define $Q_0 = P_1 P_2$; we have $Q_0 \mathcal{H} = \mathcal{M}_1 \cap \mathcal{M}_2$ by lemma 5.8. Let $\{u_j\}, \{v_k^{(1)}\}, \{v_k^{(2)}\}$ be orthonormal systems such that $\{u_j\}, \{u_j\} \cup \{v_k^{(1)}\}, \{u_j\} \cup \{v_k^{(2)}\}$ are orthonormal bases of $\mathcal{M}_1 \cap \mathcal{M}_2$, $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. Once defined $Q_m x = \sum_k \langle v_k^{(m)} | x \rangle v_k^{(m)}$, $m = 1, 2$, we have $P_1 = Q_0 + Q_1$, $P_2 = Q_0 + Q_2$, with $Q_0 \perp Q_1$ and $Q_0 \perp Q_2$.
Since $P_1 P_2 \in \Pi(\mathcal{H})$, $Q_0 = P_1 P_2 = Q_0^2 + Q_0 Q_2 + Q_1 Q_0 + Q_1 Q_2 = Q_0^2 + Q_1 Q_2$.
Therefore, $Q_1 Q_2 = 0$, i.e. $Q_1 \perp Q_2$.

## 5.3. CONVERGENCES IN $\Pi(\mathcal{H})$

Let us consider a monotone sequence $\{P_n\}$ of projection operators, e.g. a non-decreasing sequence: $P_n \le P_{n+1}$, $\forall n$. Since the smallest projection $\mathbb{0}$ and the greatest projection $\mathbb{1}$ bound every projection operator, i.e. $\mathbb{0} \le P \le \mathbb{1}$, $\forall P \in \Pi(\mathcal{H})$, $\{P_n\}$ is a non-decreasing sequence such that $P_n \le \mathbb{1}$, $\forall n$. Then the question arises whether has a limit for $n \to \infty$, as it happens for bounded monotone sequences of real numbers in ordinary analysis. In fact, unless $\{P_n\}$ is definitively constant, i.e. unless $n_0$ exists such that $P_n = P_m$, $\forall n, m > n_0$, such a sequence $\{P_n\}$ cannot converge with respect to the norm of bounded operators. Indeed, if $P_n \ne P_m$, a non-zero vector $x \in \mathcal{M}_m = P_m\mathcal{H}$ exists such that $P_n x = 0$. Then $\|(P_m - P_n)x\| = \|x\|$, i.e. $\|P_m - P_n\| = 1$ holds, and this implies that $\{P_n\}$ is not a Cauchy sequence with respect to the norm of bounded operators.

However, a weaker form of convergence holds.

PROPOSITION 5.9. Let $\{P_n\} \subseteq \Pi(\mathcal{H})$ a non-decreasing sequence. Then (i) for every $x \in \mathcal{H}$ the $\lim_{n\to\infty} P_n x = y$ exists;

(ii) The mapping

$$P : \mathcal{H} \to \mathcal{H}$$
$$x \to Px = \lim_n P_n x$$

is a projection operator for which $P_n \le P$, $\forall n$.

PROOF. (i) Let us define $Q_1 = P_1$, $Q_2 = P_2 - P_1, \dots, Q_n = P_n - P_{n-1}$ for every $n > 1$. Since $Q_n^2 = P_n^2 - 2P_{n-1}P_n + P_{n-1}^2 = P_n - P_{n-1} = Q_n$, we have $\{Q_n\} \subseteq \Pi(\mathcal{H})$ (self-adjointness and boundness trivial). Moreover, if $n \neq m$, say $n < m$, then

$$Q_n Q_m = P_n P_m - P_{n-1} P_m - P_n P_{m-1} + P_{n-1} P_{m-1} = \emptyset, \quad \text{and}$$

$$\sum_{n=1}^{N} Q_n = P_N \quad \text{hold.}$$

These relations imply that

$\forall x \in \mathcal{H}$, $\{P_N x\}_N$ is a Cauchy sequence of vectors. Indeed,

$$\|(P_M - P_N)x\|^2 = \|\sum_{j=N+1}^{M} Q_j x\|^2 = \sum_{j=N+1}^{M} \|Q_j x\|^2 = |S_M - S_N|, \quad (*)$$

where $S_N = \sum_{j=1}^{N} \|Q_j x\|^2 = \|P_N x\|^2 \leq \|x\|^2$,

so that $\{S_N\}_N$ converges in $\mathbb{R}$, being a non decreasing and bounded sequence. Therefore, $\{S_N\}_N$ is Cauchy and $(*)$ implies that $\{P_N x\}_N$ is Cauchy too:

$$\forall x \in \mathcal{H}, \ \exists \lim_n P_n x = Px.$$

(ii) The operator $P : \mathcal{H} \to \mathcal{H}$, $Px = \lim_n P_n x$ is linear and bounded: $\|P_n x\| \leq \|x\|$ implies $\|Px\| \leq \|x\|$.

Now, $P$ is self-adjoint. Indeed

$$\langle Px | y \rangle = \lim_n \langle P_n x | y \rangle = \lim_n \langle x | P_n y \rangle = \langle x | Py \rangle, \ \forall x, y \in \mathcal{H}.$$

Moreover, $P$ is idempotent. Indeed

$$\langle x | P^2 y \rangle = \langle Px | Py \rangle = \lim_n \langle P_n x | P_n y \rangle = \lim_n \langle x | P_n^2 y \rangle =$$
$$= \lim_n \langle x | P_n y \rangle = \langle x | Py \rangle, \ \forall x, y \in \mathcal{H}.$$

Exercise. Exploit $Px = \sum_{n=1}^{\infty} Q_n x$ to prove

$$P_N \leq P, \quad \forall N.$$

<u>PROBLEM.</u> The student is asked to prove that if $\{P_m\}_m$ is a non-increasing sequence of projection operators, then for every $x \in \mathcal{H}$ the $\lim_{m\to\infty} P_m x$ exists and $Px = \lim_{m\to\infty} P_m x$ defines a projection operator $P \geq P_m$, $\forall m$.