# SPECTRAL THEORY

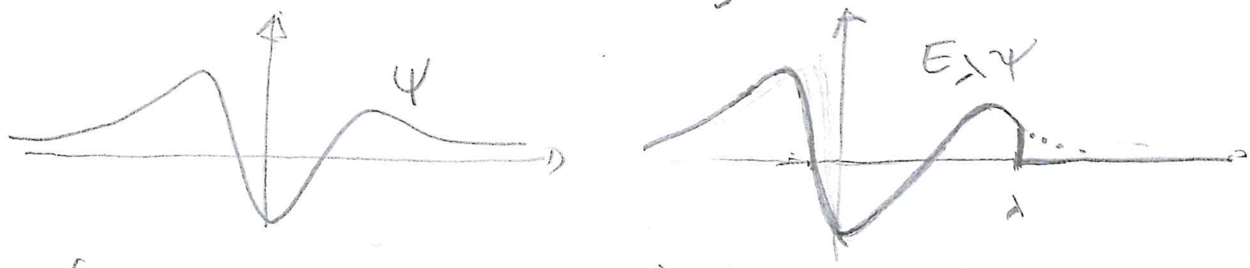
## 1. A FAMILY OF PROJECTIONS IN $L_2(\mathbb{R})$

For every $\lambda \in \mathbb{R}$ let us define the subspace $\mathcal{M}_\lambda \subseteq L_2(\mathbb{R})$, $\mathcal{M}_\lambda = \{ \psi \in L_2(\mathbb{R}) \mid \psi(x) = 0$ a.e. on $(\lambda, \infty)\}$. $\mathcal{M}_\lambda$ is a closed subspace, then there is a projection operator $E_\lambda \in \Pi(L_2(\mathbb{R}))$ such that $E_\lambda L_2(\mathbb{R}) = \mathcal{M}_\lambda$. Now, $E_\lambda \psi$ is the vector in $\mathcal{M}_\lambda$ with the smallest distance from $\psi$, that is to say

$$\| \psi - E_\lambda \psi \|^2 = \int |\psi(x) - (E_\lambda \psi)(x)|^2 dx = \int_{-\infty}^{\lambda} |\psi(x) - (E_\lambda \psi)(x)|^2 dx + \int_{\lambda}^{+\infty} |\psi(x)|^2 dx .$$

Then this value is minimum iff $(E_\lambda \psi)(x) = \psi(x)$ a.e. in $(-\infty, \lambda]$.

Thus $E_\lambda \psi(x) = \left\{ \begin{array}{ll} 0 & \text{if } x > \lambda \\ \psi(x) & \text{if } x \le \lambda \end{array} \right\} = \chi_{(-\infty,\lambda]}(x)\, \psi(x).$




($E_\lambda$ annihilates $\psi$ after $\lambda$).

This particular family $\{E_\lambda\}_{\lambda \in \mathbb{R}} \subseteq \Pi(L_2(\mathbb{R}))$ satisfy the following properties, as easily verified.

E.1) $\lambda \le \mu \Rightarrow E_\lambda \le E_\mu$

E.2) $\lim\limits_{\lambda \to -\infty} E_\lambda \psi = 0$, $\lim\limits_{\lambda \to \infty} E_\lambda \psi = \psi$, $\forall \psi$.

E.3) $\lim\limits_{\varepsilon \to 0} E_{\lambda+\varepsilon} \psi = E_\lambda \psi$, $\forall \psi$ (continuity on the right and on the left).

By making use of the correspondence $\lambda \to E_\lambda$, every interval $I = (a, b] \subseteq \mathbb{R}$ can be assigned a projection operator $E(I) = E_b - E_a$.

Properties (E.1)-(E.3) imply that the mapping

$$I \to E(I) = E_b - E_a$$

satisfies the following conditions.

(i) $E(\phi) = \phi$, $E(\mathbb{R}) = \mathbb{1}$

(ii) if $I_1 \cap I_2 = \phi$, then $E(I_1) \perp E(I_2)$,

indeed $E(I_1)E(I_2) = (E_{b_1} - E_{a_1})(E_{b_2} - E_{a_2}) =$

$= E_{b_1}E_{b_2} - E_{b_1}E_{a_2} - E_{a_1}E_{b_2} + E_{a_1}E_{a_2}$

$= E_{b_1} - E_{b_1} - E_{a_1} + E_{a_1} = \mathbb{0}$,

where $a_1 \le a_2, b_2$, $b_1 \le a_2, b_2$ since $(a_1, b_1] \cap (a_2, b_2] = \mathbb{0}$.

These properties allows us to extend the mapping $I \to E(I)$ to a mapping $E : \mathcal{B}(\mathbb{R}) \to \Pi(L_2(\mathbb{R}))$, where $\mathcal{B}(\mathbb{R})$ denote the family of all Borel sets. Indeed, each Borel set $\Delta$ is the countable set theoretic disjoint union of intervals: $\Delta = \cup_j I_j$, $I_j \cap I_k = \phi$ if $j \neq k$.

Then we define $E(\Delta)$ by

$$E(\Delta)\psi = \Sigma_j E(I_j)\psi = \chi_\Delta \psi.$$

Prop. 5.9 in TQ.1 ensures the convergence of the series to a projection operator $E(\Delta)$.

Hence, starting from $\{E_\lambda\}_\lambda$ we were able to construct a mapping $E : \mathcal{B}(\mathbb{R}) \to \Pi(L_2(\mathbb{R}))$ such that

(PV) $E(\phi) = \phi$, $E(\mathbb{R}) = \mathbb{1}$, $E(\cup \Delta_j)\psi = \Sigma_j E(\Delta_j)\psi$ if $\Delta_j \cap \Delta_k = \phi$.

Formally, this mapping $E$ satisfies the conditions for being a measure, the difference being that the values are projection operators rather than real numbers.

In fact, for any fixed $\psi \in \mathcal{H}$, with $\|\psi\| = 1$, by making use of $E : \mathcal{B}(\mathbb{R}) \to \Pi(L_2(\mathbb{R}))$, we obtain an ordinary measure

$$\mu_\psi : \mathcal{B}(\mathbb{R}) \to [0,1], \quad \Delta \to \mu_\psi(\Delta) = \langle \psi | E(\Delta)\psi \rangle;$$

indeed, $\mu_\psi(\emptyset) = 0$, $\mu_\psi(\mathbb{R}) = 1$, $\mu_\psi(\cup \Delta_i) = \sum_i \mu_\psi(\Delta_i)$ if $\Delta_j \cap \Delta_k = 0$ when $j \neq k$ are easily derived from (PV). To be precise, $\mu_\psi$ is a probability measure. It is obtained as a Stieltjes measure from the function $\alpha_\psi(\lambda) = \langle \psi | E_\lambda \psi \rangle$.

Hence, for every $\psi \in \mathcal{H}$, $\|\psi\| = 1$, we can introduce the (Stieltjes) integral with respect to $\mu_\psi$ : $\int f(\lambda) d_\lambda \alpha_\psi(\lambda)$, where $f$ is a complex function.

It is worth to explicitly specify the mathematical meaning of this integral. Let $\{\Pi_m\}$ a sequence of partitions. The partition $\Pi_m$ is identified by $N_m - 1$ intervals $(\lambda_1^{(m)}, \lambda_2^{(m)}], \dots, (\lambda_{N_m-1}^{(m)}, \lambda_{N_m}^{(m)}]$ with extremals $a^{(m)} = \lambda_1^{(m)}$ and $b^{(m)} = \lambda_{N_m}^{(m)}$. In order to define the integral on $\mathbb{R}$ we require that $\lim_{m\to\infty} \|\Pi_m\| = 0$, $\lim_m a^{(m)} = -\infty$, $\lim_m b^{(m)} = \infty$.

For each $\Pi_m$ we introduce the sum

$$S_m = \sum_{j=1}^{N_m-1} f(\tilde{\lambda}_j^{(m)})\, \mu_\psi\left((\lambda_j^{(m)}, \lambda_{j+1}^{(m)}]\right) = \sum_{j=1}^{N_m-1} f(\tilde{\lambda}_j^{(m)}) \langle \psi | \left(E_{\lambda_{j+1}^{(m)}} - E_{\lambda_j^{(m)}}\right)\psi \rangle$$

where $\tilde{\lambda}_j^{(m)} \in (\lambda_j^{(m)}, \lambda_{j+1}^{(m)}]$.

The function $f$ is said to be integrable with respect to $\mu_\psi$ if the limit

$$\lim_{m\to\infty} S_{N_m} = \lim_{m\to\infty} \sum_j f(\tilde{\lambda}_j^{(m)}) \langle \psi | (E_{\lambda_{j+1}^{(m)}} - E_{\lambda_j^{(m)}}) \psi \rangle$$

exists and it is the same for every sequence $\pi_m$ and for any choice of the $\tilde{\lambda}^{(m)}_j$. If this is the case, such a limit is denoted by

$$\int f(\lambda)\, d_\lambda \langle \psi | E_\lambda \psi \rangle,$$ and it is called the integral of $f$ with respect to $\mu_\psi$.

EXERCIZE: Show that $\int d_\lambda \langle \psi | E_\lambda \psi \rangle = \|\psi\|^2 = 1$

## 2. THE SPECTRAL REPRESENTATION OF F.

Now we show that the identical function $f(\lambda)$ is integrable and

$$\int \lambda \, d_\lambda \langle \psi | E_\lambda \psi \rangle = \langle \psi | F \psi \rangle.$$

Indeed

$$\left| \langle \psi | F \psi \rangle - \int \lambda \, d_\lambda \langle \psi | E_\lambda \psi \rangle \right| =$$

$$= \lim_m \left| \int x |\psi(x)|^2 dx - \sum_{j=1}^{N_m - 1} \tilde{\lambda}_j^{(m)} \langle \psi | E((\lambda_j^{(m)}, \lambda_{j+1}^{(m)}]) \psi \rangle \right|$$

$$= \lim_m \left| \sum_{j=1}^{N_m-1} \left( \int_{\lambda_j^{(m)}}^{\lambda_{j+1}^{(m)}} x |\psi(x)|^2 dx - \tilde{\lambda}_j^{(m)} \int_{\lambda_j^{(m)}}^{\lambda_{j+1}^{(m)}} |\psi(x)|^2 dx \right) \right|$$

$$= \lim_m \left| \sum_{j=1}^{N_m-1} \int_{\lambda_j^{(m)}}^{\lambda_{j+1}^{(m)}} (x - \tilde{\lambda}_j^{(m)}) |\psi(x)|^2 dx \right|$$

$$\leq \lim_n \left| \|\Pi_n\| \sum_{j=1}^{N_{n-1}} \int_{\lambda_j^{(n)}}^{\lambda_{j+1}^{(n)}} |\psi(x)|^2 dx \right| =$$

$$= \lim_n \|\Pi_n\| \int_{a^{(n)}}^{b^{(n)}} |\psi(x)|^2 dx = \lim_n \|\Pi_n\| \, \|\psi\|^2 = 0 .$$

Therefore, for all $\psi \in D_F$ we have

$$\langle \psi | F \psi \rangle = \int \lambda \, d_\lambda \langle \psi | E_\lambda \psi \rangle .$$

This relation is called the <u>spectral representation of F</u>. By exploiting the polar identity, we can slightly generalize it: for every $\psi, \varphi \in D_F$

$$\langle \psi | F \varphi \rangle \equiv \frac{1}{4} \{ \langle \psi+\varphi | F(\psi+\varphi) \rangle - \langle \psi-\varphi | F(\psi-\varphi) \rangle +$$

$$+ i \langle \psi - i\varphi | F(\psi - i\varphi) \rangle - i \langle \psi + i\varphi | F(\psi + i\varphi) \rangle \}$$

$$= \frac{1}{4} \left\{ \int \lambda \, d_\lambda \langle \psi+\varphi | E_\lambda (\psi+\varphi) \rangle + \cdots \right\}$$

$$= \int \lambda \, d_\lambda \langle \psi | E_\lambda \varphi \rangle$$

## 3. SPECTRAL THEOREM FOR SELF-ADJOINT OPERATORS.

We have seen that the multiplication operator F of $L_2(\mathbb{R})$ admits a family $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ of projection operators, whose properties (E.1)-(E.3) make possible to induce a Stieltjes integral such that the operator F itself can be represented as

$$\langle \psi | F\psi \rangle = \int \lambda \, d_\lambda \langle \psi | E_\lambda \psi \rangle, \quad \forall \psi \in D_F.$$

In fact, this possibility is not just a feature of the multiplication operator, but it is proved to hold for every selfadjoint operator.

**DEFINITION 3.1.** A family $\{E_\lambda\}_\lambda \subseteq \Pi(\mathcal{H})$ is a resolution of the identity if the following conditions hold

(R.1) $\lambda \le \mu$ implies $E_\lambda \le E_\mu$;

(R.2) $\lim_{\lambda\to-\infty} E_\lambda \psi = 0$, $\lim_{\lambda\to+\infty} E_\lambda \psi = \psi$, $\forall \psi \in \mathcal{H}$;

(R.3) $\lim_{\varepsilon\to 0^+} E_{\lambda+\varepsilon} \psi = E_\lambda \psi$, $\forall \psi \in \mathcal{H}$ (continuity on the right).

**THEOREM** (of the spectral representation for self-adjoint operators) Let A be a self-adjoint operator of a Hilbert space $\mathcal{H}$. Then there is a unique resolution of the identity $\{E_\lambda^A\}$ such that $\langle \psi | A\psi \rangle = \int \lambda \, d_\lambda \langle \psi | E_\lambda^A \psi \rangle$, $\forall \psi \in D_A$.

**COROLLARY.**

$$\langle \psi | A\varphi \rangle = \int \lambda \, d_\lambda \langle \psi | E_\lambda^A \varphi \rangle, \quad \forall \psi, \varphi \in D_A.$$

NOTICE that only continuity on the right is required

## 4. THE CASE OF FINITE DIMENSION OF $\mathcal{H}$.

The theorem just stated is the generalization of the ordinary spectral theorem for symmetric operators known in linear algebra of finite dimension, i.e. the theorem stating that any symmetric operator in a finite dimensional space has a basis of eigenvector. In this section this link is clarified.

**PROPOSITION 4.1.** If $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ is the resolution of the identity of a self-adjoint operator A in a finite dimensional Hilbert space $\mathcal{H}$, then $\forall\lambda_0\in\mathbb{R}$, $\lambda_1>\lambda_0$ exists such that $E_{\lambda_1}=E_{\lambda_0}$

**PROOF.** Let us suppose that $\lambda_0\in\mathbb{R}$ exists such that $E_{\lambda_1}\gneqq E_{\lambda_0}$ for all $\lambda_1>\lambda_0$. We can fix one such a value $\lambda_1$. Then $\lambda_2$ must exist, $\lambda_0<\lambda_2<\lambda_1$, such that $E_{\lambda_0}\lneqq E_{\lambda_2}\lneqq E_{\lambda_1}$: The first inequality is just the supposition that begins the proof; on the other hand, if $E_{\lambda_2}=E_{\lambda_1}$ for all $\lambda_2$, $\lambda_0<\lambda_2<\lambda_1$, we should have $E_{\lambda_0}=E_{\lambda_1}$ because of the right continuity of $E_\lambda$, against our supposition.
So, a sequence $\lambda_1,\lambda_2,\lambda_3,,\lambda_n$, exists such that $\lambda_0<\lambda_{n+1}<\lambda_n<\lambda_1$ satisfying $E_{\lambda_0}\lneqq E_{\lambda_{n+1}}\lneqq E_{\lambda_n}$; if we define $F_n=E_{\lambda_{n+1}}-E_{\lambda_n}\in\Pi(\mathcal{H})$, we have $F_n\perp F_m$ if $n\neq m$ because $(\lambda_n,\lambda_{n+1}]\cap(\lambda_m,\lambda_{m+1}]=\phi$.

Therefore for each $F_n$, $u_n \in F_n \cap \mathcal{H} \neq \{0\}$ exists such that $\|u_n\| = 1$. The countable family $\{u_n\}$ is a non-finite orthonormal system; it cannot exists if $\mathcal{H}$ is finite dimensional.

According to this proposition, $E_\lambda$ is constant on the right of every point $\lambda_0 \in \mathbb{R}$. Since $E_\lambda \psi \to 0$ if $\lambda \to -\infty$ and $E_\lambda \psi \to \psi$ if $\lambda \to +\infty$, we have to conclude that at least a $\lambda_0 \in \mathbb{R}$ must exist where $E_\lambda$ in not continuous on the left. Now, the set of such discontinuities must be finite, because each discontinuity point $\lambda_j$ is an eigenvalue of $A$, as we prove in the next proposition. Now, if we had a non-finite set of discontinuities, there must be a corresponding set of (non-vanishing) eigenvectors: $Au_j = \lambda_j u_j$. Two eigenvectors corresponding to different eigenvalues are orthogonal:

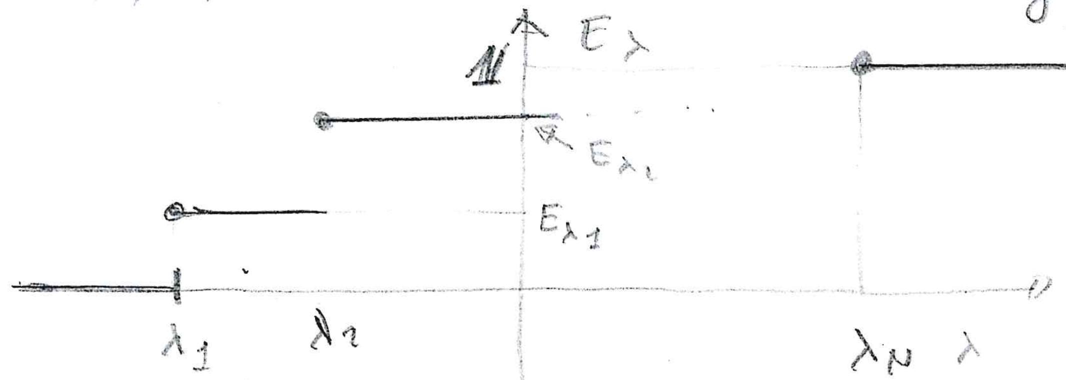
$$\langle Au_j | u_k \rangle = \lambda_j \langle u_j | u_k \rangle = \langle u_j | A u_k \rangle = \lambda_k \langle u_j | u_k \rangle$$

holds only if $\langle u_j | u_k \rangle = 0$ when $\lambda_j \neq \lambda_k$. Hence, we would have a non-finite set $\{u_j\}$ of orthogonal non-null vectors: This is not possible in a finite-dimensional Hilbert space.

**PROPOSITION 4.2.** Let $A$ be a self-adjoint operator. $\lambda_0 \in \mathbb{R}$ is an eigenvalue of $A$ iff $E_\lambda$ is not continuous (on the left) in $\lambda_0$.

**PROOF.** Let $\lambda_0$ be a point where $E_\lambda$ is not continuous. By Prop. 5.3 in TQ.1, this implies that $E_{\lambda_0-\frac{1}{n}}\psi \xrightarrow{n} E_{\lambda_0^-}\psi$, where $E_{\lambda_0^-} \in \Pi(\mathcal{H})$ but $E_{\lambda_0^-} < E_{\lambda_0}$; so $E_{\lambda_0} - E_{\lambda_0^-} \neq 0$, and $\exists\, \psi_0 \neq 0$ such that $(E_{\lambda_0} - E_{\lambda_0^-})\psi_0 = \psi_0$.
Therefore, $\langle \psi | A\psi_0 \rangle = \lim_{n} \sum_j \tilde{\lambda}_j \langle \psi | (E_{\lambda_{j+1}} - E_{\lambda_j})\psi_0 \rangle$.
Now, there is a unique interval $(\lambda_{j_0}, \lambda_{j_0+1}]$ that contains $\lambda_0$, and $E_{\lambda_0} - E_{\lambda_0^-} \leq E_{\lambda_{j_0+1}} - E_{\lambda_{j_0}}$, of course; so $(E_{\lambda_{j_0+1}} - E_{\lambda_{j_0}})\psi_0 = \psi_0$, $(E_{\lambda_{j+1}} - E_{\lambda_j})\psi_0 = 0$. Therefore $\langle \psi | A\psi_0 \rangle = \lim_{\|\pi\| \to 0} \tilde{\lambda}_{j_0} \langle \psi | \psi_0 \rangle = \langle \psi | \lambda_0 \psi_0 \rangle$, $\forall \psi$, i.e. $A\psi_0 = \lambda_0\psi_0$.

Hence, the resolution of the identity of a self-adjoint operator $A$ in a finite-dimensional Hilbert space $\mathcal{H}$ must be constant on $\mathbb{R}$, apart from a finite number of discontinuities $\{\lambda_1, \dots, \lambda_N\}$ on the left. Since $E_\lambda\psi \to \begin{cases} 0, & \lambda \to -\infty \\ \psi, & \lambda \to +\infty \end{cases}$, $E_\lambda = 0$ for $\lambda < \lambda_1$ and $E_\lambda = \mathbb{1}$ for $\lambda \geq \lambda_N$ ($\lambda_1, \dots, \lambda_N$ are ordered in increasing order).



PROPOSITION 9.3. Let $A$ be a self-adjoint operator of a finite dimensional Hilbert space $\mathcal{H}$, and let $\{\lambda_1, \ldots, \lambda_N\}$ the discontinuities of its resolution of the identity $E_\lambda$.
Then $A = \sum_{j=1}^{N} \lambda_j P_j$, where $P_j = E_{\lambda_j} - E_{\lambda_j^-}$,

with $E_{\lambda_j^-}\psi = \lim_{\lambda \to \lambda_j^-} E_\lambda \psi$.

PROOF. Proposition 5.9 in TQ.1 ensures that $E_{\lambda_j^-} \in \Pi(\mathcal{H})$ exists. Since $E_\lambda \leq E_{\lambda_j}$ if $\lambda < \lambda_1$, $E_{\lambda_j^-} \leq E_{\lambda_j}$, so that $P_j \in \Pi(\mathcal{H})$.

The spectral theorem implies
$$\langle \psi | A\varphi \rangle = \lim_{n\to\infty} \sum_k \tilde{\lambda}_k^{(n)} \langle \psi | (E_{\lambda^{(n)}_{k+1}} - E_{\lambda^{(n)}_k}) \varphi \rangle. \quad (1)$$

Now, when the norm $\|\Pi_n\|$ becomes smaller than $\max_j \{\lambda_{j+1} - \lambda_j\}$, in each interval $(\lambda^{(n)}_k, \lambda^{(n)}_{k+1}]$ there is at most one eigenvalue, i.e.
$$E_{\lambda^{(n)}_{k+1}} - E_{\lambda^{(n)}_k} = E_{\lambda_j} - E_{\lambda_j^-} \quad \text{if } \lambda_j \in (\lambda^{(n)}_k, \lambda^{(n)}_{k+1}],$$
$$E_{\lambda^{(n)}_{k+1}} - E_{\lambda^{(n)}_k} = \mathbb{O}, \quad \text{if } \lambda_j \notin (\lambda^{(n)}_k, \lambda^{(n)}_{k+1}],\ \forall j.$$
So (1) becomes
$$\langle \psi | A\varphi \rangle = \lim_{n\to\infty} \sum_j \tilde{\lambda}_j \langle \psi | (E_{\lambda_j} - E_{\lambda_j^-}) \varphi \rangle,$$
where $\tilde{\lambda}_j$ belong to the interval of the partition with $\lambda_j$.

If $n\to\infty$, $\|\Pi_n\| \to 0$ and thus, $\forall \psi, \varphi \in \mathcal{H}$,
$\langle \psi | A\varphi \rangle = \langle \psi | \sum_j \lambda_j P_j \varphi \rangle$, i.e. $A\varphi = \sum_j \lambda_j P_j$.

Since $\mathbb{1} = E_{\lambda_N + \varepsilon} - E_{\lambda_1 - \varepsilon}$, with $\varepsilon > 0$, we must have $\sum_j P_j = \mathbb{1}$; each subspace $\mathcal{M}_j = P_j \mathcal{H}$ is finite dimensional and hence has an orthonormal basis $\{u_1^{(j)}, \ldots, u_{N_j}^{(j)}\}$, so that

$$\psi = \sum_j P_j \psi = \sum_j \sum_{k=1}^{N_j} \langle u_k^{(j)} | \psi \rangle u_k^{(j)}.$$

Therefore, $\{u_k^{(1)}\} \cup \{u_k^{(2)}\} \cup \cdots \cup \{u_k^{(N)}\}$ is an ortonormal basis.

Furthermore

$$A\, u_k^{(j)} = \sum_m \lambda_m P_m u_k^{(j)} = \lambda_j u_k^{(j)}$$

because $P_m u_k^{(j)} = 0$ if $j \neq m$ ($\sum_m P_m = \mathbb{1} \in \Pi(\mathcal{H})$)

Thus $\{u_k^{(1)}\} \cup \{u_k^{(2)}\} \cup \cdots \cup \{u_k^{(N)}\}$ is an orthonormal basis of eigenvectors.

In other words, the theorem of the spectral representation reduces to the ordinary spectral theorem if $\dim \mathcal{H}$ is finite.

<u>EXERCIZE 1.</u> Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ be in $\mathcal{H} = \mathbb{C}^2$. Prove that

$$E_\lambda \psi = \begin{cases} 0 & \text{if } \lambda < -1, \\ \langle u_1 | \psi \rangle u_1, \text{ when } u_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}, & \text{if } -1 \leq \lambda < 1 \\ \mathbb{1}, & \text{if } \lambda \geq 1. \end{cases}$$

<u>EXERCIZE 2.</u> Prove that if $A \in \Pi(\mathcal{H})$, then

$$E_\lambda = \begin{cases} 0 & \text{if } \lambda < 0 \\ \mathbb{1} - A & \text{if } 0 \leq \lambda < 1 \\ \mathbb{1} & \text{if } \lambda \geq 1. \end{cases}$$

## 5. FUNCTIONAL CALCULUS OF SPECTRAL THEORY.

The aim of spectral functional calculus is to identify a spectral representation of an operator B that is function of a selfadjoint operator A whose spectral representatoon is determined by a resolution of the identity $E_\lambda$. We begin with the case where the function is $f(x)=x^2$, i.e. by searching for a spectral representatoon of the operator $f(A)=A^2$, were $\langle\psi|A\psi\rangle=\int\lambda\, d_\lambda\langle\psi|E_\lambda\psi\rangle$.

<u>LEMMA 5.1.</u> Let A be a self-adjoin operator with resolutoon of the identity $\{E_\lambda\}$. Then for any $\mu_0\in\mathbb{R}$ the following relation holds, provided that $\psi, \varphi, E_{\mu_0}\varphi \in D_A$.

$$\langle E_{\mu_0}\varphi|A\psi\rangle=\int^{\mu_0}\lambda\, d_\lambda\langle\varphi|E_\lambda\psi\rangle.$$

<u>Heuristic PROOF.</u> $\langle E_{\mu_0}\varphi|A\psi\rangle=\int_{-\infty}^{\mu_0}\lambda\, d_\lambda\langle\varphi|E_{\mu_0}E_\lambda\psi\rangle+\int_{\mu_0}\lambda\, d_\lambda\langle\varphi|E_{\mu_0}E_\lambda\psi\rangle$

In the first integral $E_{\mu_0}E_\lambda=E_\lambda$.

In the second integral $E_{\mu_0}E_\lambda=E_{\mu_0}$, therefore

$$\int_{\mu_0}^{\infty}\lambda\, d_\lambda\langle\varphi|E_{\mu_0}E_\lambda\psi\rangle=\lim_{n}\sum_j\tilde{\lambda}_j\langle\varphi|(E_{\mu_0}E_{\lambda_{j+1}}-E_{\mu}E_{\lambda_j})\psi\rangle=0.$$

Lemma 5.1. extends to prove that

$$\langle A\varphi|E_{\mu_0}\psi\rangle=\int^{\mu_0}\lambda\, d_\lambda\langle\varphi|E_\lambda\psi\rangle \text{ holds too.}$$

<u>COROLLARY.</u> If $\{E_\lambda\}$ is the resolutoon of the identity of $A=A^*$, then $[E_\mu, A]=0$.

PROPOSITION 5.1. If $\{E_\lambda\}$ is the resolution of the identity of $A = A^*$, then

$$\langle \psi | A^2 \psi \rangle = \int \lambda^2 \, d_\lambda \langle \psi | E_\lambda \psi \rangle ,$$

provided that $A\psi \in D_A$ and that the r.h.s. converges.

Heuristic PROOF. We have, by Lemma 5.1,

(i) $$\langle \psi | A^2 \psi \rangle = \langle A\psi | A\psi \rangle = \lim_m \sum_j \tilde{\lambda}_j \left( \langle A\psi | E_{\lambda_{j+1}} \psi \rangle - \langle A\psi | E_{\lambda_j} \psi \rangle \right)$$

$$= \sum_j \tilde{\lambda}_j \int_{\lambda_j}^{\lambda_{j+1}} \mu \, d_\mu \langle \psi | E_\mu \psi \rangle$$

(ii) $$\int \mu^2 d_\mu \langle \psi | E_\mu \psi \rangle = \lim_m \sum_j \int_{\lambda_j}^{\lambda_{j+1}} \mu^2 d_\mu \langle \psi | E_\mu \psi \rangle .$$

Then $$\left| \langle \psi | A^2 \psi \rangle - \int \mu^2 d_\mu \langle \psi | E_\mu \psi \rangle \right| =$$

$$= \lim_m \left| \sum_j \int_{\lambda_j}^{\lambda_{j+1}} (\tilde{\lambda}_j \mu - \mu^2) \, d_\mu \langle \psi | E_\mu \psi \rangle \right|$$

$$= \lim_m \left| \sum_j \int_{\lambda_j}^{\lambda_{j+1}} \{ (\tilde{\lambda}_j - \mu) \mu \} \, d_\mu \langle \psi | E_\mu \psi \rangle \right|$$

Now, we are free in choosing $\{\Pi_m\}$, provided that $\|\Pi_m\| \to 0$ and $a^{(m)} \to -\infty$, $b^{(m)} \to +\infty$. Let us choose $a^{(m)} = -b^{(m)}$ and $\|\Pi_m\| < \frac{1}{(b^{(m)})^2}$. Then

$$\left| \langle \psi | A^2 \psi \rangle - \int \mu^2 d_\mu \langle \psi | E_\mu \psi \rangle \right| \leq \lim_m \left| \sum_j \|\Pi_m\| b^{(m)} \int_{\lambda_j}^{\lambda_{j+1}} d_\mu \langle \psi | E_\mu \psi \rangle \right|$$

$$\leq \lim_m \|\Pi_m\| b^{(m)} \int_{-b^{(m)}}^{b^{(m)}} d_\mu \langle \psi | E_\mu \psi \rangle$$

$$\leq \lim_m \frac{1}{b^{(m)}} \|\psi\|^2 = 0$$

There is no difficulty to extending the proof to show that

$$\langle \psi | A^m \psi \rangle = \int \lambda^m d_\lambda \langle \psi | E_\lambda \psi \rangle,$$

provided that the r.h.s. converges and $A^{m-1}\psi \in D_A$.

This result can be effectively used to prove

<u>PROPOSITION 5.2.</u> If $\lambda_0$ is an eigenvalue of $A = A^*$, then it is a discontinuity of its resolution of the identity $\{E_\lambda\}$.

<u>Heuristic PROOF.</u> Let $\psi_0$ be an eigenvector of $\lambda_0$. Then, by Prop. 5.1, if $a < b < \lambda_0 < c < d$, we have

$$\langle (A-\lambda_0)\psi_0 | (A-\lambda_0)\psi_0 \rangle = 0 = \int (\lambda-\lambda_0)^2 d_\lambda \langle \psi_0 | E_\lambda \psi_0 \rangle =$$

$$= \int_{-\infty}^{a} (\lambda-\lambda_0)^2 d_\lambda \langle \psi_0 | E_\lambda \psi_0 \rangle + \int_a^b (\lambda-\lambda_0)^2 d_\lambda \langle \psi_0 | E_\lambda \psi_0 \rangle + \int_b^c (\lambda-\lambda_0)^2 d_\lambda \langle \psi | E_\lambda \psi_0 \rangle +$$

$$+ \int_c^d (\lambda-\lambda_0)^2 d_\lambda \langle \psi_0 | E_\lambda \psi_0 \rangle + \int_d^\lambda (\lambda-\lambda_0)^2 d_\lambda \langle \psi_0 | E_\lambda \psi_0 \rangle .$$

Being non-negative, all terms must be 0.

So, $0 = \int_a^b (\lambda-\lambda_0)^2 d_\lambda \langle \psi_0 | E_\lambda \psi_0 \rangle \geq (b-\lambda_0)^2 \int_a^b d_\lambda \langle \psi_0 | E_\lambda \psi_0 \rangle$.

Therefore $0 = \int_a^b d_\lambda \langle \psi_0 | E_\lambda \psi_0 \rangle = \langle \psi_0 | (E_b - E_a) \psi_0 \rangle$, i.e. $E_b\psi = E_a\psi$ for all $a, b$ with $a < b < \lambda_0$; but $E_a \psi_0 \to 0$ if $a \to -\infty$; hence $E_b \psi_0 = 0 \ \forall b < \lambda_0$.

Analogously we can prove that $E_c \psi_0 = \psi_0 \ \forall c > \lambda_0$.

Thus, $\lambda_0$ is a point of discontinuity for $E_\lambda$.

By iterating the procedure of prop. 5.1. it can be proved that for any polinomial $P_N(\lambda)=a_0+a_1\lambda+\dots+a_N\lambda^N$, the operator $P_N(A)=a_0+a_1A+\dots+a_NA^n$, where $A=A^*$, admits the spectral representation

$$\langle\varphi|P_N(A)\psi\rangle=\int P_N(\lambda)\,d_\lambda\langle\varphi|E_\lambda\psi\rangle,$$

$\{E_\lambda\}$ being the resolution of the identity of A, provided that $\varphi$ and $\psi$ are taken in a suitable domain. The fact that $P_N(A)$ can be determined by means the scalar function $P_N(\lambda)$ according the spectral representation, suggests to extend this representation to analytic functions

$$f(\lambda)=\sum_{n=0}^{\infty}a_n\lambda^n,$$

which are limits of polinomials. Then, given an analytic function $f(\lambda)=\sum_{n=0}^{\infty}a_n\lambda^n$, we can denote by $f(A)$ the operator determined by the equation

$$\langle\varphi|f(A)\psi\rangle=\int f(\lambda)\,d_\lambda\langle\varphi|E_\lambda\psi\rangle.$$

A first problem that occurs, obviously, is to establish the domain of the vectors $\psi$ for which such $f(A)$ exists. For the time being, however, we address our attention to establish the relation between $f(A)$ and the series $\sum_{n=0}^{\infty}a_nA^n$.

To investigate such a relation it is worth to make recourse to the notion of weak convergence.

<u>DEFINITION 5.1.</u> A sequence $\{A_n\}$ of linear operators of $\mathcal{H}$ weakly converges to the operator $A$ on a subspace $D \subseteq \mathcal{H}$ if $\lim_{n\to\infty} \langle\psi|A_n\psi\rangle = \langle\psi|A\psi\rangle,\ \forall\psi\in D$. In such a case we write $w\text{-}\lim_n A_n = A$ on $D$.

If $w\text{-}\lim_n A_n = A$, then $\lim_n \langle\varphi|A_n\psi\rangle = \langle\varphi|A\psi\rangle,\ \forall\varphi,\psi\in D$.

<u>PROPOSITION 5.3.</u> Given an analytic function $f(\lambda)=\sum_{n=0}^{\infty} a_n\lambda^n$, there is a subspace $D$ such that $w\text{-}\lim_m \sum_{j=0}^{m} a_j A^j = f(A)$.

<u>PROOF.</u> Let $D$ be the domain $D = S_p(\{\phi\in D_{A^m}\cap D_{f(A)},\ \phi=(E_{\mu_2}-E_{\mu_1})\psi\})$.

If $\phi=(E_{\mu_2}-E_{\mu_1})\psi\in D$, then

$$\lim_m \left|\langle\varphi|\sum_{j=0}^{m} a_j A^j\phi\rangle - \langle\varphi|f(A)\phi\rangle\right| =$$

$$= \lim_m \left|\int \sum_{j=0}^{m} a_j\lambda^j\, d_\lambda\langle\varphi|(E_{\mu_2}-E_{\mu_1})E_\lambda\psi\rangle - \int f(\lambda)\, d_\lambda\langle\varphi|(E_{\mu_2}-E_{\mu_1})E_\lambda\psi\rangle\right|$$

$$= \lim_m \left|\int_{\mu_1}^{\mu_2} \left(\sum_{j=0}^{m} a_j\lambda^j - f(\lambda)\right) d_\lambda\langle\varphi|E_\lambda\psi\rangle\right|$$

$= 0$ because $\sum_{j=0}^{\infty} a_j\lambda^j$ uniformly converges to $f(\lambda)$ on $(\mu_1,\mu_2]$. ∎

According to prop. 5.3, while the equalities

$$e^A = \sum_{n=0}^{\infty}\frac{A^n}{n!} \quad \text{or} \quad \sin A = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{A^{2n-1}}{(2n-1)!}$$

have problems of convergence if $A$ is not bounded, on the domain $D$ they hold in the weak sense.

With regard to the problem of identifying the domain of the operator $f(A)$, we will make use of the following result.

> If $f$ is a measurable function and $A$ is a self-adjoint operator, then the domain of definition of $f(A)$ is
> $$D_{f(A)} = \{ \psi \in \mathcal{H} \mid \int |f(\lambda)|^2 d_\lambda \langle \psi | E_\lambda \psi \rangle < +\infty .$$

Accordingly, it may happen that $D_{f(A)} \supseteq D_A$.

EXAMPLE. Let us consider the analytic function $f(\lambda) = e^{i\lambda} = \sum_m \frac{(i\lambda)^m}{m!}$. Now, for all $\psi \in \mathcal{H}$ we have

$$\int |e^{i\lambda}|^2 d_\lambda \langle \psi | E_\lambda \psi \rangle = \int d_\lambda \langle \psi | E_\lambda \psi \rangle =$$

$$= \lim_{\substack{a(n) \to -\infty \\ b(n) \to +\infty}} \left( \langle \psi | E_{b(n)} \psi \rangle - \langle \psi | E_{a(n)} \psi \rangle \right) = \|\psi\|^2 .$$

Therefore, $D_{e^{iA}} = \mathcal{H}$, independently of $D_A$.

In general, $\langle \psi | f^*(A) \psi \rangle = \int \overline{f(\lambda)} \, d_\lambda \langle \psi | E_\lambda \psi \rangle$.

So, $(f(A))^* = \bar{f}(A)$. Now, since $h(\lambda) = e^{-i\lambda} e^{i\lambda} = 1$

we find $\langle \psi | (e^{-iA} \cdot e^{iA}) \psi \rangle = \langle \psi | h(A) \psi \rangle = \int e^{-i\lambda} e^{i\lambda} d_\lambda \langle \psi | E_\lambda \psi \rangle$
$= 1, \forall \psi$.

Therefore $(e^{iA})^{-1} = e^{-iA} = (e^{iA})^*$. This implies
$\langle e^{iA} \psi | e^{iA} \psi \rangle = \langle (e^{iA})^* e^{iA} \psi | \psi \rangle = \langle e^{-iA} e^{iA} \psi | \psi \rangle = \|\psi\|^2$.

Thus, $e^{iA}$ is a unitary operator, for all $\psi$.

Notice that in this case $E_M \psi \in D, \forall \psi \in D_{e^{iA}} \equiv \mathcal{H}$, therefore $\bar{D} = \mathcal{H}$.

## PROBLEMS.

5.1. Let $\chi_{(\mu_0,\lambda_0]}$ the characteristic functional of $(\mu_0,\lambda_0]$, i.e. $\chi_{(\mu_0,\lambda_0]}(\lambda)=1$ if $\lambda\in(\mu_0,\lambda_0]$ and $\chi_{(\mu_0,\lambda_0]}(\lambda)=0$ if $\lambda\notin(\mu_0,\lambda_0]$. Prove that

$$\chi_{(\mu_0,\lambda_0]}(A)=E_{\lambda_0}-E_{\mu_0} \text{ and } \chi_{(-\infty,\lambda]}(A)=E_\lambda.$$