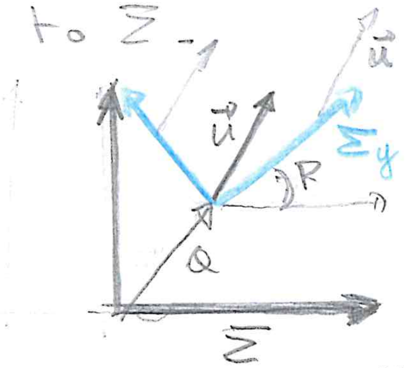
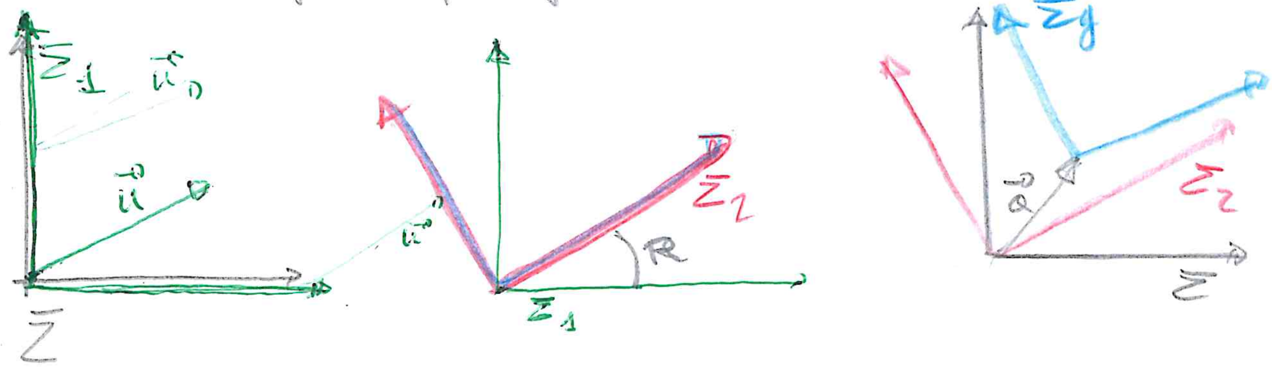
TQ.4 QUANTUM THEORY OF ISOLATED SYSTEM

1. GROUP THEORETICAL INTERLUDE

The group of Galilei $\mathcal{G}$ is the group of transformations generated by spatial translations, spatial rotations an galileian boosts. It can be identified as the group of transformations g that transform a reference frame Σ into another reference frame Σ_g which moves with constant velocity $\vec{u}$ with respect to Σ.



This interpretation allow to establish that g can be realized first by transforming Σ into a reference frame Σ_1 identical to Σ but moving with velocity $\vec{u}$ with respect to Σ; then transforming Σ_1 into a frame Σ_2 rotated as Σ_g is rotatated with respect to Σ_g; finally, by tranforming Σ_2 by the translation that relates the origin of Σ to the origin of Σ_g at the considered instant.



So, every $g \in \mathcal{G}$ is the product of a translation Υ, a rotation R and a boost B:

$$g = \Upsilon R B .$$

The triplet Υ, R, B is unique if $g = \Upsilon R B$.
Υ is completely identified by the translation vector $\vec{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$, and transforms a spatial vector $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ into $\vec{x} + \vec{a} = \begin{bmatrix} x + a_x \\ y + a_y \\ y + a_z \end{bmatrix}$; therefore $\Upsilon = h_1(a_x) h_2(a_y) h_3(a_z)$, where $h_1(a_x), h_2(a_y), h_3(a_z)$ are translations along x, y, z of a value a_x, a_y, a_z respectively.
Analogously $B = h_7(u_x) h_8(u_y) h_9(u_z)$, where $h_7(u_x), h_8(u_y), h_9(u_z)$ are boosts along x, y, z with relative velocities u_x, u_y, u_z respectively.
It is evident that the triplet u_x, u_y, u_z to realize B, as well as the triplet a_x, a_y, a_z to realize Υ, is unique.

While the rotation R to realize g as $g = \Upsilon R B$ is unique, the triplet of rotations $h_4(\alpha), h_5(\beta), h_6(\gamma)$ of rotations around x, y, z of angles α, β, γ to realize R as $R = h_4(\alpha) h_5(\beta) h_6(\gamma)$, is not unique in general; however, if we restrict to small rotations, i.e. to rotations that transform Σ_1 into Σ_2 such that $|\widehat{x_1 x_2}| + |\widehat{y_1 y_2}| + |\widehat{z_1 z_2}| < \frac{\pi}{2}$, then the triplet α, β, γ is unique with the natural condition $0 \leq \alpha < 2\pi$, $0 \leq \beta < 2\pi$, $0 \leq \gamma < 2\pi$.

PROBLEM. (a) Find a rotation R for which there is not a unique triplet α, β, γ ($0 \le \alpha < 2\pi, 0 \le \beta < 2\pi, 0 \le \gamma < 2\pi$) such that $R = h_4(\alpha) h_5(\beta) h_6(\gamma)$.
(b) Show that for small rotations such a triplet is unique.

The argument above shows that for every $g \in \mathcal{G}$ there is a set $\underline{x} = (x_1, x_2, \dots, x_9)$ of nine real values such that

$$g = h_1(x_1) h_2(x_2) \cdots h_9(x_9).$$

The vector $\underline{x} \in \mathbb{R}^9$ of the parameters identifying g is unique in a neighborough of the identity transformation $e \in \mathcal{G}$, i.e. the neutral element of $\mathcal{G}$. If we denote the subgroups of translations along x, y, z by H_1, H_2, H_3, the subgroups of rotations around x, y, z by H_4, H_5, H_6, and the subgroups of boosts along x, y, z by H_7, H_8, H_9, then $\mathcal{G}$ can be defined as the group generated by $H_1, H_2, \dots, H_9$: $\mathcal{G} = G(H_1, H_2, \dots, H_9)$.

Each H_j is an abelian subgroup; in particular $h_j(0) = e, \forall j$ and $h_j(x+y) = h_j(x) h_j(y)$.

While the sub-groups $\mathcal{T} = G(H_1, H_2, H_3)$ and $\mathcal{B} = G(H_7, H_8, H_9)$ of translations and boosts are abelian, the subgroup of rotations $\mathcal{R} = G(H_4, H_5, H_6)$ is not-abelian.

2. MATRIX REALIZATION OF $\mathcal{G}$..

From a mathematical point of view, a boost identified by a relative velocity $\vec{u}$, i.e. the galilean transformation $g = h_x(u_x)\,h_y(u_y)\,h_z(u_z)$, is a transformation that transforms a spatial vector $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$ into $\begin{bmatrix} x+u_x t \\ y+u_y t \\ z+u_z t \end{bmatrix}$ in correspondence with the value t of the "time" parameter. Such a transformation can be realized through the familiar matrix product by identifying the spatial vector at "time" t with the vector $\begin{bmatrix} t \\ x \\ y \\ z \\ 1 \end{bmatrix} \in \mathbb{R}^5$, and the transformations $h_x(u_x), h_y(u_y), h_z(u_z)$ by the matrices

$$\hat{h}_x(u_x) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ u_x & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \hat{h}_y(u_y) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ u_y & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \hat{h}_z(u_z) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ u_z & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

which are matrices of $GL(\mathbb{R},5)$. Indeed, we see that

$$\begin{bmatrix} t \\ x+u_x t \\ y+u_y t \\ z+u_z t \\ 1 \end{bmatrix} = \hat{g} \begin{bmatrix} t \\ x \\ y \\ z \\ 1 \end{bmatrix} = \hat{h}_x(u_x)\,\hat{h}_y(u_y)\,\hat{h}_z(u_z) \begin{bmatrix} t \\ x \\ y \\ z \\ 1 \end{bmatrix}.$$

According to the same procedure, also rotations and translations can be realized as matrices

$$\hat{h}_1(x_1) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & x_1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \hat{h}_2(x_2) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & x_2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \hat{h}_3(x_3) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (2.1)$$

$$\hat{h}_4(x_4) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & cx_4 & -sx_4 & 0 \\ 0 & 0 & sx_4 & cx_4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \hat{h}_5(x_5) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & cx_5 & 0 & sx_5 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -sx_5 & 0 & cx_5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \hat{h}_6(x_6) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & cx_6 & -sx_6 & 0 & 0 \\ 0 & sx_6 & cx_6 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where $cx \equiv \cos x$, $sx \equiv \sin x$.

According to these correspondences, every element $g \in G$, identified by the nine parameters $(x_1, x_2, \ldots, x_9) \equiv \underline{x}$, can be realized as the matrix

$$\hat{g} \equiv \hat{g}(\underline{x}) = \hat{h}_1(x_1)\,\hat{h}_2(x_2) \cdots \hat{h}_9(x_9) \in GL(\mathbb{R}, 5)$$

If we denote the set of all these matrices by $\hat{G}$, we see that $\hat{G}$ is a subgroup of $GL(\mathbb{R},5)$ and that

$$\hat{\ } : G \to \hat{G}, \quad g = g(\underline{x}) \longrightarrow \hat{g}(\underline{x}) = \hat{h}_1(x_1)\hat{h}_2(x_2)\cdots\hat{h}_9(x_9)$$

is a group isomorphism such that $\hat{g}(\underline{0}) = \mathbb{1}_5 = \begin{bmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{bmatrix}$.

Therefore, G can be identified with the matrix group $\hat{G}$. One advantage in adopting this identification is that given $\hat{g}_1, \hat{g}_2 \in \hat{G}$, $\widehat{g_1 g_2} = \hat{g}_1 \hat{g}_2$ can be directly computed as matrix product.

2.1. Lie algebra of $\hat{G}$. According to the above introduced matrix realization of G, each subgroup H_j is realized by a one-parameter subgroup $\hat{H}_j$ of matrices: $\hat{H}_j = \{\hat{h}_j(s),\ s \in \mathbb{R}\}$. For every H_j, the derivative $\hat{a}_j = \frac{d}{ds}\hat{h}_j(s)\big|_{s=0}$ is: (where no entry is written, 0 is implied)

$$\hat{a}_1 = \begin{bmatrix} &&&& \\ &&&&1 \\ &&&&0 \\ &&&&0 \\ &&&& \end{bmatrix}, \quad \hat{a}_2 = \begin{bmatrix} &&&& \\ &&&&0 \\ &&&&1 \\ &&&&0 \\ &&&& \end{bmatrix}, \quad \hat{a}_3 = \begin{bmatrix} &&&& \\ &&&&0 \\ &&&&0 \\ &&&&1 \\ &&&& \end{bmatrix}, \quad \hat{a}_4 = \begin{bmatrix} &&&& \\ &0&0&0& \\ &0&0&-1& \\ &0&1&0& \\ &&&& \end{bmatrix}, \quad \hat{a}_5 = \begin{bmatrix} &&&& \\ &0&0&1& \\ &0&0&0& \\ &-1&0&0& \\ &&&& \end{bmatrix}, \quad \hat{a}_6 = \begin{bmatrix} &&&& \\ &0&-1&0& \\ &1&0&0& \\ &0&0&0& \\ &&&& \end{bmatrix}$$

$$\hat{a}_7 = \begin{bmatrix} &&&& \\ 1&&&& \\ 0&&&& \\ 0&&&& \\ &&&& \end{bmatrix}, \quad \hat{a}_8 = \begin{bmatrix} &&&& \\ 0&&&& \\ 1&&&& \\ 0&&&& \\ &&&& \end{bmatrix}, \quad \hat{a}_9 = \begin{bmatrix} &&&& \\ 0&&&& \\ 0&&&& \\ 1&&&& \\ &&&& \end{bmatrix}$$

These nine matrices are not invertible.
<u>We see that they are linearly independent.</u>
The linear space of matrices generated by them is called the Lie algebra of $\check{G}$.

2.2 Analicity of the group operations

Any element $\check{h}_j(x_j)$ defined by eq. (2.1) is an analytic function of x_j; hence, being $\check{g}(\underline{x})$ the product $\hat{h}_1(x_1) \cdots \hat{h}_9(x_9)$ a product of these matrices, its entries are sums of products of entries of the $\check{h}_j(x_j)$'s, and therefore they are analytic functions of $\underline{x}$: $\check{g}(\underline{x})$ is an analytic function of $\underline{x}$.
Also the converse holds: each parameter x_j is an an analytic function of the (entries of) $\check{g}(\underline{x}) = \check{g}$. Indeed, let us denote the entries of $\check{g}$ by d_{jk}. Then we have

$$\check{g}(\underline{x}) \equiv \hat{g} = \begin{bmatrix} d_{11} & \cdots & d_{15} \\ \vdots & \ddots \; d_{jk} & \vdots \\ \vdots & & \vdots \end{bmatrix} = \hat{h}_1(x_1)\,\hat{h}_2(x_2) \cdot \cdots \hat{h}_9(x_9) = \tag{2.2}$$

$$= \left[\begin{array}{c|c|c} 1 & 0\;0\;0 & 0 \\ \hline R\begin{bmatrix} x_7 \\ x_8 \\ x_9 \end{bmatrix} & R & \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} \\ \hline & & 1 \end{array}\right] \quad \text{where } R = R^x_{x_4} R^y_{x_5} R^z_{x_6} \in SO(3).$$

Eq. (2.2) shows that $x_1 = d_{25}$, $x_2 = d_{35}$, $x_3 = d_{45}$.
Therefore, the parameters of spatial translations are always analytic functions of (the entries of) $\check{g}$.

Now we consider the parameters determining the central block of the formal matrix in (2.2), i.e. of $R = R^x_{x_4} R^y_{x_5} R^z_{x_6}$. A straightforward computation yields

$$R = \begin{bmatrix} d_{22} & d_{23} & d_{24} \\ \vdots & \ddots & \vdots \\ \cdot & & d_{44} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cx_4 & -sx_4 \\ 0 & sx_4 & cx_4 \end{bmatrix} \begin{bmatrix} cx_5 & 0 & sx_5 \\ 0 & 1 & 0 \\ -sx_5 & 0 & cx_5 \end{bmatrix} \begin{bmatrix} cx_6 & -sx_6 & 0 \\ sx_6 & cx_6 & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} cx_5\, cx_6 & -cx_5\, sx_6 & sx_5 \\ sx_4\, sx_5\, cx_6 + cx_4\, sx_6 & -sx_4\, sx_5\, sx_6 + cx_4\, cx_6 & -sx_4\, cx_5 \\ -cx_4\, sx_5\, cx_6 + sx_4\, sx_6 & cx_4\, sx_5\, sx_6 + sx_4\, cx_6 & cx_4\, cx_5 \end{bmatrix}. \tag{2.3}$$

Eq. (2.3) shows that $d_{24} = sx_5$, i.e. $x_5 = \arcsin d_{24}$: x_5 is an analytic function of d_{24}, and hence of $\hat{g}$, in a neighborough of $\hat{e} = \mathbb{1}$.

We see that $d_{23} = -cx_5\, sx_6$. Therefore $x_6 = \arcsin \dfrac{d_{23}}{\cos(\arcsin d_{24})}$: x_6 is an analytic function of $\hat{g}$ in a neighborough of $\hat{e} = \mathbb{1}$. Since $d_{34} = -sx_4\, cx_5$, $x_4 = \arcsin \dfrac{d_{34}}{cx_5}$:

x_4 is an analytic function of $\hat{g}$ in a neighborough of $\hat{e} = \mathbb{1}$. Thus, the following Proposition holds.

PROP. 2.1. The parameters x_i are analytic functions of $\hat{g}$ in a neighborough of $\hat{e} = \mathbb{1}$.

For our future purposes, the following proposition will be useful.

PROPOSITION 2.2. Let f_j be real function such that

$$f_j(s) = b_j s + c_j s^2 + o_j(s^2) \qquad (2.4)$$

b_j, c_j being real constants and o_j infinitesimal of order greater than 2.

If $\quad \check{g}(s) = \hat{h}_1(f_1(s))\hat{h}_2(f_2(s)) \cdots \hat{h}_g(f_g(s)) = \mathbb{1} + d_{j_0}\hat{a}_{j_0}s^2 + o(s^2)$, (2.5)

then $b_j = 0$, $\forall j$, and $c_j = 0$ $\forall j \neq j_0$ and $c_{j_0} = d_{j_0}$.

PROOF. By making use of (2.4), since $\hat{h}_j(t) = \mathbb{1} + \hat{a}_j t + \hat{b}_j t^2 + \hat{o}_j(t^2)$,

$$\begin{aligned}\check{g}(s) = &\{\mathbb{1} + \hat{a}_1(b_1 s + c_1 s^2) + \hat{b}_1(b_1 s + c_1 s^2)^2\} \times \\ &\times \{\mathbb{1} + \hat{a}_2(b_2 s + c_2 s^2) + \hat{b}_2(b_2 s + c_2 s^2)\} \times \\ &\times \cdots \\ &\times \{\mathbb{1} + \hat{a}_{j_0}(b_{j_0} s + c_{j_0} s^2) + \hat{b}_{j_0}(b_{j_0} s + c_{j_0} s^2)\} \times \\ &\times \cdots \\ &\times \{\mathbb{1} + \hat{a}_g(b_g s + c_g s^2) + \hat{b}_g(b_g s + c_g s^2)^2\} \\ = &\ \mathbb{1} + \left(\Sigma_j b_j \hat{a}_j\right) s + o(s) \qquad (2.6)\end{aligned}$$

By (2.5) we find that the following equation hold $\forall s$.

$$\mathbb{1} + \left(\Sigma_j b_j \hat{a}_j\right) s + o(s) = \mathbb{1} + \hat{a}_{j_0} d_{j_0} s^2 \qquad (2.7).$$

Therefore $\Sigma_j b_j \hat{a}_j = 0$, that implies $b_j = 0$ $\forall j$ since the $\hat{a}_j$ are linearly independent.

As a consequence (2.5) becomes

$\hat{g}(s) = \mathbb{1} + \left(\sum_j \hat{a}_j c_j\right) s^2 + o(s^2)$ and (2.5) implies

$$\mathbb{1} + \left(\sum_j \hat{a}_j c_j\right) s^2 + o(s^2) = \mathbb{1} + \hat{a}_{j_0} d_{j_0} s^2 .$$

Once again, the linear independence of the $\hat{a}_j$'s implies

$$c_{j_0} = d_{j_0} \text{ and } c_j = 0 \text{ if } j \neq j_0 .$$

3. ISOLATED SYSTEM

An isolated phisical system is a system that does not interact with the rest of the universe. This implies that physical phenomena the system is involved in are indistinguishable from the physical phenomena that happen once everything is moved according to any transformation g of the Galilei group $\mathcal{G}$.

In a quantum theory the basic physical concepts are observable A and expectation value Ev, that is to say the procedure for measuring A and the procedure to select specimens of the physical system in such a way realize the expectation value Ev, procedures that include the concrete apparatuses to realize them. Therefore, to "move everything according to $g \in \mathcal{G}$" entails, in a quantum theory, the existence of two mappings

$$S_g^{(1)}: \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H}), \quad S_g^{(2)}: \mathcal{R}(\mathcal{H}) \to \mathcal{R}(\mathcal{H})$$

such that $S_g^{(1)}[\rho] = \rho'$ is the density operator that represents the expectation value one obtain with the procedure represented by ρ moved according to g, and $S_g^{(2)}[A] = A'$ is the self-adjoint operator that represents the measuring procedure obtained moving the procedure of A by g.

Coherently with the concept of isolated system formulated above, $S_g^{(1)}$ and $S_g^{(2)}$ must satisfy the following conditions.

(S1) $S_g^{(1)}$ and $S_g^{(2)}$ are bijective for every $g \in \mathcal{G}$.

(S2) $Tr(\rho A) = Tr(S_g^{(1)}[\rho] S_g^{(2)}[A])$ for all g, for all $A \in \mathcal{R}(\mathcal{H})$ and for all $\rho \in \mathcal{S}(\mathcal{H})$ such that $Ev(A)$ exists.
This condition, quite simply establi-shes the wanted indistinguishability with respect to transformations of $\mathcal{G}$, since $Tr(\rho A)$ is just the expectation value of A.

(S3) For every measurable real function f such that $f(A) \in \mathcal{R}(\mathcal{H})$,
$$S_g^{(2)}[f(A)] = f\left(S_g^{(2)}[A]\right).$$
This condition establishes that adding to the apparatuses that measure A and A' (after transformation) the same mathematical function f that transforms the outcomes of the measurem-ents of A and A' does not affect indistinguishability.

The pair $(S_g^{(1)}, S_g^{(2)})$ is called <u>quantum symmetry transformation</u>

3.1. IMPLICATIONS OF WIGNER THEOREM.

Eugene Wigner proved an important theorem about "Wigner transformations", which will be very effective in the development of the quantum theory of an isolated system.

DEF. 3.1. A mapping
$$S : \Pi_1(\mathcal{H}) \to \Pi_1(\mathcal{H})$$
is a Wigner transformation if it is bijective and $Tr(P_1 P_2) = Tr(S[P_1]S[P_2])$, $\forall P_1, P_2 \in \Pi_1(\mathcal{H})$.

THEOREM 3.1. (WIGNER). If $S : \Pi_1(\mathcal{H}) \to \Pi_1(\mathcal{H})$ is a Wigner transformation, then either a unitary operator or an anti-unitary operator U exists such that
$$S[P] = UPU^{-1}, \quad \forall P \in \Pi_1(\mathcal{H}).$$
Moreover, if V is a unitary or antiunitary operator that realizes the same Wigner transformation, i.e. such that
$$S[P] = VPV^{-1}, \quad \forall P \in \Pi_1(\mathcal{H}),$$
then $V = e^{i\alpha} U$, where α is a real constant.

PROOF. See appendix

Wigner theorem is not about quantum symmetry transformations, so it cannot be immediately applied to them. However, every quantum symmetry transformation uniquely identifies a Wigner transformation that completely determines the quantum symmetry transformation according to the following proposition.

PROPOSITION 3.1. Let $(S^{(1)}, S^{(2)})$ be a quantum symmetry transformation. Then the following statements hold.

(i) $S^{(1)}$ is a convex isomorphism of $\mathfrak{S}(\mathcal{H})$; in particular, therefore, $S^{(1)}[\rho] \in \Pi_1(\mathcal{H})$ iff $\rho \in \Pi_1(\mathcal{H})$.

(ii) $S^{(2)}$ is linear.

(iii) $S^{(2)}[E] \in \Pi(\mathcal{H})$ iff $E \in \Pi(\mathcal{H})$

(iv) $E \leq F$ iff $S^{(2)}[E] \leq S^{(2)}[F]$

(v) $P \leq E$ iff $S_1[P] \leq S_2[E]$, $\forall P \in \Pi_1(\mathcal{H})$, $\forall E \in \Pi(E)$

(vi) $E \perp F$ iff $S_2[E] \perp S_2[F]$, $\forall E, F \in \Pi(\mathcal{H})$

(vii) $P \perp Q$ iff $S_1[P] \perp S_1[Q]$, $\forall P, Q \in \Pi_1(\mathcal{H})$

(viii) $S_1[P] = S_2[P]$, $\forall P \in \Pi_1(\mathcal{H})$

(ix) $S_2[E] = \sum_j S_1[P_j]$, where $P_j \in \Pi_1(\mathcal{H})$, $E \in \Pi(\mathcal{H})$ and $E = \sum_j P_j$.

(x) $S : \Pi_1(\mathcal{H}) \to \Pi_1(\mathcal{H})$, $S[P] = S^{(1)}[P] = S^{(2)}[P]$ is a Wigner transformation.

PROOF. (i) If $\rho = \lambda\rho_1 + (1-\lambda)\rho_2$, with $\rho_1, \rho_2 \in \mathcal{S}(\mathcal{H})$, then

$Tr\left(S_1[\lambda\rho_1 + (1-\lambda)\rho_2]\, S_2[A]\right) = Tr\left((\lambda\rho_1 + (1-\lambda)\rho_2)A\right)$ by (S2)

$= \lambda Tr(\rho_1 A) + (1-\lambda) Tr(\rho_2 A) = \lambda Tr\left(S_1[\rho_1] S_2[A]\right) + (1-\lambda) Tr\left(S_1(\rho_2) S_2[A]\right)$

$= Tr\left(\left(\lambda S^{(1)}[\rho_1] + (1-\lambda) S^{(1)}[\rho_2]\right) S_2(A)\right)$,

$\forall\, S_2[A] \in \Omega(\mathcal{H})$, in particular $\forall\, |\varphi\rangle\langle\varphi| = P \in \Pi_1(\mathcal{H})$. Therefore,

$\langle\varphi|\left(\lambda S^{(1)}[\rho_1] + (1-\lambda) S^{(1)}[\rho_2]\right)\varphi\rangle = \langle\varphi|S^{(1)}[\lambda\rho_1 + (1-\lambda)\rho_2]\varphi\rangle$,

$\forall\, \varphi \in \mathcal{H}$, $\|\varphi\| = 1$, i.e. $S^{(1)}[\lambda\rho_1 + (1-\lambda)\rho_2] = \lambda S^{(1)}[\rho_1] + (1-\lambda) S^{(1)}[\rho_2]$.

The isomorphism property follows from the fact that $(S^{(1)-1}, S^{(2)-1})$ is a quantum symmetry transformation.

(ii) $\langle\varphi|S^{(2)}[\alpha A + \beta B]\varphi\rangle = Tr\left(S^{(1)}[P]\, S^{(2)}[\alpha A + \beta B]\right)$

where, $S^{(1)}[P] = |\varphi\rangle\langle\varphi|$, $P = |\psi\rangle\langle\psi|$, so that

$\langle\varphi|S^{(2)}[\alpha A + \beta B]\varphi\rangle = Tr\left(P(\alpha A + \beta B)\right) =$

$= Tr\left(P(\alpha A)\right) + Tr\left(P(\beta B)\right) =$

$= \alpha Tr\left(S^{(1)}[P] S^{(2)}[A]\right) + \beta Tr\left(S^{(1)}[P] S^{(2)}[B]\right) =$

$= \alpha\langle\varphi|S^{(2)}[A]\varphi\rangle + \beta\langle\varphi|S^{(2)}[B]\varphi\rangle =$

$= \langle\varphi|\left(\alpha S^{(2)}[A] + \beta S^{(2)}[B]\right)\varphi\rangle$.

(iii) If $E \in \Pi(\mathcal{H})$, then $f(E) = E$ where $f(\lambda) = \lambda^2$. Therefore $(S^{(2)}[E])^2 = f(S^{(2)}[E]) = S^{(2)}[E^2] \equiv S^{(2)}[E]$, by (S3).

(iv) If $F = E + G \in \Pi(\mathcal{H})$, with $E, G \in \Pi(\mathcal{H})$, i.e. $E \leq F$, then $\Pi(\mathcal{H}) \ni S^{(2)}[F] = S^{(2)}[E+G] = S^{(2)}[E] + S^{(2)}[G]$, by (ii). Since $S^{(2)}[E], S^{(2)}[G] \in \Pi(\mathcal{H})$, $S^{(2)}[E] \perp S^{(2)}[G]$, and $S^{(2)}[E] \leq S^{(2)}[F]$, of course.

(v) $P \leq E$ implies $Tr(PE) = \langle \varphi | E \varphi \rangle = 1$, where $P = |\varphi\rangle\langle\varphi|$

$S^{(1)}[P] = |\varphi\rangle\langle\varphi|$ by (i) and

$1 = Tr(S^{(1)}[P] S^{(2)}[E]) = \langle \varphi | S^{(2)}[E] \varphi \rangle$, i.e.

$S^{(1)}[P] \leq S^{(2)}[E]$.

(vi) is proved in (iv).

(vii) $S^{(1)}[P] \leq S^{(2)}[P]$, $S^{(1)}[Q] \leq S^{(2)}[Q]$ by (v);

$P \perp Q$ implies $S^{(2)}[P] \perp S^{(2)}[Q]$ by (vi).

Then $S^{(1)}[P] \perp S^{(1)}[Q]$.

(viii) If $S^{(1)}[P] \lneqq S^{(2)}[P]$, then

$\exists\, Q_0 \in \Pi_1(\mathcal{H})$, $Q_0 = S^{(1)}[P_0]$, $Q_0 \perp S^{(1)}[P]$

such that $S^{(1)}[P] + Q_0 \leq S^{(2)}[P]$.

Therefore $Q_0 \leq S^{(2)}[P]$, so that

$P_0 \leq P$, but $P_0 \perp P$, since $S^{(1)}[P_0] \perp S^{(1)}[P]$.

(ix) If $E = \sum_j P_j$, $P_j \in \Pi_1(\mathcal{H})$, $E \in \Pi(\mathcal{H})$, then

$S^{(2)}[E] = w\text{-}\sum_j S^{(2)}[P_j] = w\text{-}\sum_j S^{(1)}[P_j]$

$= \sum_j S^{(1)}[P_j]$.

(x) $Tr(S^{(1)}[P] S^{(1)}[Q]) = Tr(S^{(1)}[P] S^{(2)}[Q])$

$= Tr(PQ)$.

According to Prop. 3.1, the mappings $S^{(1)}$ and $S^{(2)}$ of a quantum symmetry transformation are the extensions of the same Wigner transformation that results from restricting both to $\Pi_1(\mathcal{H})$. Hence, according to Wigner theorem a unitary or antiunitary operator U exists such that

$$S^{(1)}[P] = S^{(2)}[P] = UPU^{-1}, \quad \forall P \in \Pi_1(\mathcal{H}).$$

<u>Exercize 3.1.</u> By exploiting the convexity of $S^{(1)}$ and the linearity of $S^{(2)}$ show that $S^{(1)}[\rho] = U\rho U^{-1}$ and $S^{(2)}[A] = UAU^{-1}$, $\forall \rho \in \mathfrak{S}(\mathcal{H}), \forall A \in \mathfrak{R}(\mathcal{H})$.
(hint: show before that w-$\lim_n B_n = B$ implies w-$\lim_n S^{(k)}[B_n] = S^{(k)}[B]$).

Of course, in order to realize $S^{(1)}[\rho]$ and $S^{(2)}[A]$, any operator $V = e^{i\alpha} U$ used: indeed

$$VBV^{-1} = e^{i\alpha}(UBU^{-1})e^{-i\alpha} = UBU^{-1}.$$

Therefore, for every $g \in G$, we can freely choose an operator V_g in the set $\{e^{i\alpha} U_g\} = \mathbf{U}_g$, called the <u>ray operator of U_g</u>, to realize $S^{(1)}$ and $S^{(2)}$ according to $S^{(1)}_g[\rho] = V_g \rho V_g^{-1}$, $S^{(2)}_g[A] = V_g A V_g^{-1}$.
We stipulate to choose $V_e = \mathbb{1}$. This is a consistent choice because $S^{(1)}_e[\rho] = \rho = \mathbb{1}\rho\mathbb{1}^{-1}$ and $S^{(2)}_e[A] = A = \mathbb{1}A\mathbb{1}^{-1}$.
Let us suppose that a fixed operator V_g is chosen $\forall g$.
This means that $\forall g_1, g_2$, $V_{g_1 g_2}$, V_{g_1} and V_{g_2} are fixed.
Now, to "move everything" according to

$g_1 g_2$ can be realized by a unique transformation $g = g_1 g$ or, equivalently by first moving according to g_2 and then moving the result according to g_1. Since all these transformations are symmetry transformations, the group property

$$(S4) \qquad S^{(m)}_{g_1 g_2} = S^{(m)}_{g_1} S^{(m)}_{g_2} \quad , \quad m = 1, 2$$

must hold. Hence,

$$S^{(m)}_{g_1 g_2}[B] = V_{g_1 g_2} B V_{g_1 g_2}^{-1} = S^{(m)}_{g_1}\left(S^{(m)}_{g_2}[B]\right) = \left(V_{g_1} V_{g_2}\right) B \left(V_{g_1} V_{g_2}\right)^{-1}.$$

Therefore, according to Wigner theorem

$$V_{g_1} V_{g_2} = \omega(g_1, g_2) V_{g_1 g_2} \; , \quad \omega(g_1, g_2) \in \mathbb{C} \, , \; |\omega(g_1, g_2)| = 1 .$$

Thus, in the quantum theory of an isolated system a correspondence

$V : \mathcal{G} \to \mathcal{U}(\mathcal{H})$ must exist, $\left(\mathcal{U}(\mathcal{H}) = \{V \text{ unitary or anti-unitary}\}\right)$

such that $\quad S^{(m)}_g [B] = V_g B V_g^{-1}$, and

$$(PR) \quad V_e = \mathbb{1} \quad , \quad V_{g_1} V_{g_2} = \omega(g_1, g_2) V_{g_1 g_2} \quad , \quad |\omega(g_1, g_2)| = 1 .$$

A correspondence $V : \mathcal{G} \to \mathcal{U}(\mathcal{H})$ that satisfies conditions (PR) is called <u>projective representation</u>.

In a projective representation we have

$$\omega(e, e) = 1 .$$

Indeed, $V_e V_e = \mathbb{1} = V_{ee} = \omega(e,e) V_{ee}$.

<u>Exercise 3.2.</u> Prove that $\omega(g, e) = \omega(e, g) = 1$.

If $V : \mathcal{G} \to \mathcal{U}(\mathcal{H})$ is a projective representation of Galilei group $\mathcal{G}$, then V_g is unitary. Indeed, according to section 1, 1.7,

$$\begin{aligned} g = g(x) &= h_1(x_1)\, h_2(x_2) \cdots h_g(x_g) \\ &= h_1\left(\tfrac{x_1}{2}+\tfrac{x_1}{2}\right) h_2\left(\tfrac{x_2}{2}+\tfrac{x_2}{2}\right) \cdots h_g\left(\tfrac{x_g}{2}+\tfrac{x_g}{2}\right) \\ &= h_1\left(\tfrac{x_1}{2}\right) h_1\left(\tfrac{x_1}{2}\right) h_2\left(\tfrac{x_2}{2}\right) h_2\left(\tfrac{x_2}{2}\right) \cdots h_g\left(\tfrac{x_g}{2}\right) h_g\left(\tfrac{x_g}{2}\right). \end{aligned}$$

Then $V_g = \omega(x_1, x_2, .., x_g)\, V^2_{h_1(\frac{x_1}{2})}\, V^2_{h_2(\frac{x_2}{2})} \cdots V^2_{h_g(\frac{x_g}{2})}$.

Now, each $V_{h_j(\frac{x_j}{2})}$ can be unitary or anti-unitary; in any case, however, its square $V^2_{h_j(\frac{x_j}{2})}$ must be unitary. Therefore V_g is unitary.

Thus, the quantum symmetry transformations associated to Galilei group in the quantum theory of an isolated system are realized through a <u>unitary</u> projective representation.

3.2 STONE THEOREM AND ITS IMPLICATIONS

If $A = A^*$ and $\langle\psi|A\psi\rangle = \int \lambda \, d_\lambda \langle\psi|E_\lambda\psi\rangle$ (E_λ resolution of the identity of A), the operator e^{iAt} is the unitary operator defined by $\langle\psi|e^{iAt}\psi\rangle = \int e^{i\lambda t} d_\lambda \langle\psi|E_\lambda\psi\rangle$ for all $\psi \in \mathcal{H}$. The operator e^{-iAt} is unitary too, of course and it is the inverse of e^{iAt}, $\forall t$:

$$\langle\psi|(e^{iAt})^*\psi\rangle = \overline{\langle\psi|e^{iAt}\psi\rangle} = \overline{\int e^{i\lambda t} d_\lambda \langle\psi|E_\lambda\psi\rangle}$$
$$= \int e^{-i\lambda t} d_\lambda \langle\psi|E_\lambda\psi\rangle = \langle\psi|e^{-iAt}\rangle.$$

Moreover, if $h(\lambda) = f(\lambda)g(\lambda)$, where $f(\lambda) = e^{i\lambda t_1}$, $g(\lambda) = e^{i\lambda t_2}$, then $\langle\psi|h(A)\psi\rangle = \langle\psi|e^{iA(t_1+t_2)}\psi\rangle$
$= \int f(\lambda)g(\lambda) d_\lambda \langle\psi|E_\lambda\psi\rangle = \langle\psi|f(A)g(A)\psi\rangle$
$= \langle\psi|e^{iAt_1}e^{iAt_2}\psi\rangle$. Therefore

$$e^{iA(t_1+t_2)} = e^{iAt_1}e^{iAt_2}$$

Hence, the mapping $t \to e^{iAt}$ satisfies

$$e^{iA0} = \mathbb{1}, \quad e^{iA(t_1+t_2)} = e^{iAt_1}e^{iAt_2}.$$

If we compute the derivative $\frac{d}{dt}(e^{iAt}\psi)$ we obtain $iAe^{iAt}\psi$. Indeed

$$\langle\psi|\frac{d}{dt}e^{iAt}\psi\rangle = \lim_{\delta\to 0} \frac{\langle\psi|(e^{iAt+\delta} - e^{iAt})\psi\rangle}{\delta} =$$
$$= \lim_{\delta\to 0} \frac{\langle\psi|e^{iAt}(e^{iA\delta} - \mathbb{1})\psi\rangle}{\delta} =$$
$$= \langle\psi|e^{iAt}\lim_{\delta\to 0}(e^{iA\delta} - \mathbb{1})\psi\rangle =$$
$$= \lim_{\delta\to 0} \int \frac{(e^{i\delta\lambda} - 1)}{\delta} d_\lambda \langle e^{-iAt}\psi|E_\lambda\psi\rangle =$$
$$= \int i\lambda \, d_\lambda \langle e^{-iAt}\psi|E_\lambda\psi\rangle = \langle\psi|iAe^{iAt}\psi\rangle.$$

So, the mapping $t \to e^{iAt}$ satisfies
$0 \longrightarrow \mathbb{1}$, $t_1+t_2 \longrightarrow e^{iAt_1} e^{iAt_2}$, $\frac{d\, e^{iAt}}{dt} = iA e^{iAt}$.

STONE THEOREM. Let $V: \mathbb{R} \to \mathcal{U}(\mathcal{H})$ be a unitary representation of the group of real numbers, i.e. it satisfies
$$V_0 = \mathbb{1}, \quad V_{t_1+t_2} = V_{t_1} V_{t_2}$$
Then a self-adjoint operator A exists such that
$$V_t = e^{iAt}.$$

PROOF. First of all $B = \frac{d\, V_t}{dt}\Big|_{t=0} = +iA$, where $A = A^*$.

Indeed $V_t = \mathbb{1} + Bt + o(t)$ and $(V_t)^{-1} = V_{-t}$.
therefore $\mathbb{1} = V_t V_{-t} = V_t V_t^* = (\mathbb{1} + Bt + o_1(t))(\mathbb{1} + B^* t + o_2^*(t))$
$$= \mathbb{1} + (B+B^*)t + o(t), \quad \forall t,$$
i.e. $B = -B^*$; if $A := -iB$, $A = A^*$ and $B = +iA$.

Now, $\frac{d\, V_t}{dt} = iA V_t$ holds; indeed,
$$\frac{d\, V_t}{dt} = \lim_{s \to 0} \frac{V_{t+s} - V_t}{s} = \lim_{s\to 0} \frac{V_s - \mathbb{1}}{s} V_t$$
$$= B V_t = iA V_t.$$
Finally, $\frac{d}{dt}\left(e^{-iAt} V_t\right) = -iA e^{-iAt} V_t + e^{-iAt} iA V_t$
$$= 0;$$
So, $e^{-iAt} V_t = \text{constant} = e^{-iA0} V_0 = \mathbb{1}$;
thus $V_t = (e^{-iAt})^{-1} = e^{iAt}$.

Stone theorem has important implications on projective representations of $\mathcal{G}$. Any element $g \in \mathcal{G}$ can be factorized as

$g = h_1(x_1) h_2(x_2) \cdots h_g(x_g)$.

Each subgroup H_j satisfies

$$h_j(x+y) = h_j(x) h_j(y).$$

If $V: \mathcal{G} \to \mathcal{U}(\mathcal{H})$ is a projective representation, then $V_g = D(g) V_{h_1(x_1)} V_{h_2(x_2)} \cdots V_{h_g(x_g)}$, $|D(g)| = 1$.

Let us consider any $V_{h_j(x)} \equiv V_x^{(j)}$.

$V^{(j)}: \mathbb{R} \to \mathcal{U}(\mathcal{H})$ is a unitary projective representation of $\mathbb{R}$; in particular

$$V_0^{(j)} = \mathbb{1}, \qquad V_{x+y}^{(j)} = D(x;y) V_x^{(j)} V_y^{(j)}, \ |D(x;y)| = 1$$

holds. If V is the projective representation of the quantum theory of an isolated system, we can use $\phi^{(j)}(x) V_x^{(j)}$ instead $V_x^{(j)}$ to realize the quantum symmetry transformations, where $\phi^{(j)}(x) \in \mathbb{C}$, $\phi^{(j)}(0) = 1$, $|\phi(x)| = 1$.

PROPOSITION 3.2. A function $\phi^{(j)}$ exists such that the mapping $U^{(j)}: \mathbb{R} \to \mathcal{U}(\mathcal{H})$, $U_x^{(j)} = \phi^{(j)}(x) V_x^{(j)}$ satisfies $U_0^{(j)} = \mathbb{1}$ and $\frac{d U_x^{(j)}}{dx} = i A_j U_x^{(j)}$, where $i A_j = \left.\frac{d V_x^{(j)}}{dx}\right|_{x=0}$.

PROOF. The operator $B_j = i A_j = \frac{d}{dx} V_x^{(j)}\Big|_{x=0}$ is anti-hermitean, so that A_j is self-adjoint (see proof of Stone th.).

Now, $V_x^{(j)} V_y^{(j)} = \omega_j(x,y) V_{x+y}^{(j)}$ holds, with $|\omega_j(x,y)|=1$, $\omega_j(0,x)=1$.

By deriving with respect to y and taking $y=0$ we obtain

$$i A_j V_x^{(j)} = \frac{\partial \omega_j}{\partial y}(x,0) V_x^{(j)} + \omega_j(x,0) \frac{d V_x^{(j)}}{dx};$$

now, $\omega_j(x,0)=1$; since $\omega_j(x,y) = e^{i\xi(x,y)}$, $\frac{\partial \omega_j}{\partial x}(x,0) = i g(x)$, where $g(x)\in\mathbb{R}$. Therefore our equation becomes

$$i A_j V_x^{(j)} = i g(x) V_x^{(j)} + \frac{d V_x^{(j)}}{dx}.$$

Let us introduce $\phi_j(x) = e^{i\int_0^x g(t)\,dt}$, of course $\phi^{(j)}(0)=1$, $|\phi^{(j)}(x)|=1$.

Once defined $U_x^{(j)} = \phi^{(j)}(x) V_x^{(j)}$, by making use of the last equation we obtain

$$\frac{d U_x^{(j)}}{dx} = i g(x) \phi^{(j)}(x) V_x^{(j)} + \phi^{(j)}(x) \frac{d V_x^{(j)}}{dx}$$

$$= i g(x) U_x^{(j)} + \phi^{(j)}(x) \left(i A_j V_x^{(j)} - i g(x) V_x^{(j)} \right)$$

$$= i A_j U_x^{(j)}.$$

Since $U_0^{(j)} = \mathbb{1}$, the proof is complete.

In the quantum theory of an isolated system we can use $U_x^{(j)}$ instead of $V_x^{(j)}$ to realize the quantum transformations. Thus, the projective representation U such that $S_g[D] = U_g D U_g^{-1}$ can be always put in the form

$$\boxed{U_g = e^{iA_1 x_1} e^{iA_2 x_2} \cdots e^{iA_9 x_9}}, \quad \forall g = h_1(x_1) h_2(x_2) \dots h_9(x_9).$$

The nine operators $A_1, A_2, \dots, A_9$ are the self-adjoint generators of U.

3.3. IDENTIFICATION OF THE PROJECTIVE REPRESENTATIONS.

As important implication of Stone theorem we have obtained that in order to identify the projective representation of $\mathcal{G}$ that realizes the quantum symmetry transformations determined by Galilei group, it is sufficient to identify the nine self-adjoint generators $A_1, A_2, \dots, A_9$ such that $U_g = e^{iA_1x_1}e^{iA_2x_2}\dots e^{iA_9x_9}$.

In pursuing such an identification, an effective rôle is played by the constraints imposed by the structural properties of Galilei group, together with the application of the following proposition

PROPOSITION 3.3. Let $U:\mathcal{G}\to\mathcal{U}(\mathcal{H})$, $U_g = e^{iA_1x_1}\dots e^{iA_9x_9}$ be a projective representation of Galilei group. Then $\hat{h}_n(s)\hat{h}_m(s)\hat{h}_n(-s)\hat{h}_m(-s) = 1\!\!1 + d_{j_0}\hat{a}_{j_0}s^2 + o(s^2)$ implies

$$[A_m, A_n] = i(\alpha + d_{j_0}A_{j_0})$$

where α is a real constant.

PROOF. According to section 2.2, if $\hat{h}_n(s)\hat{h}_m(s)\hat{h}_n(-s)\hat{h}_m(-s) = 1\!\!1 + d_{j_0}\hat{a}_{j_0}s^2 + o(s^2)$ then

$$\hat{h}_n(s)\hat{h}_m(s)\hat{h}_n(-s)\hat{h}_m(-s) = \hat{h}_1(o_1(s^2))\hat{h}_2(o_2(s^2))\cdots\hat{h}_{j_0}(d_{j_0}s^2+o_{j_0}(s^2))\cdots\hat{h}_9(o_9(s^2)) \quad (1)$$

Since U is a projective representation, (1) implies

$$e^{iA_ns}e^{iA_ms}e^{-iA_ns}e^{-iA_ms} = e^{i\alpha(s)}e^{iA_1o_1(s^2)}e^{iA_2o_2(s^2)}\cdots e^{iA_{j_0}d_{j_0}s^2}e^{iA_{j_0}o_{j_0}(s^2)}\cdots e^{iA_9o_9(s^2)}.$$

Here $\alpha(s)$ is a real function; since the $O_j(s^2)$ are analytic and

$$e^{i\alpha(s)} = e^{iA_n s} e^{iA_m s} e^{-iA_n s} e^{-iA_m s} e^{-iA_g O_g(s^2)} \cdots$$

$\alpha(s)$ is analytic because the right hand side of this equation is analytic: $\alpha(s) = \alpha_1 s + \alpha_2 s^2 + o(s^2)$. By expending eq.(1) with respect to s we find

$$\left(\mathbb{1} + iA_n s - \frac{A_n^2}{2} s^2\right)\left(\mathbb{1} + iA_m s - \frac{A_m^2}{2} s^2\right)\left(\mathbb{1} - iA_n s - \frac{A_n^2}{2} s^2\right)\left(\mathbb{1} - iA_m s - \frac{A_m^2}{2} s^2\right) =$$

$$= \left(\mathbb{1} + i\alpha_1 s + i\alpha_2 s^2 - \frac{(\alpha_1 s + \alpha_2 s^2)^2}{2}\right)\left(\mathbb{1} + iO_1(s^2)A_1\right) \cdot \cdots \left(\mathbb{1} + iO_{j_0-1}(s^2)A_{j_0-1}\right)$$

$$\left(\mathbb{1} + i d_{j_0} s^2 A_{j_0}\right)\left(\mathbb{1} + iO_{j_0+1}(s^2)A_{j_0+1}\right) \cdots \left(\mathbb{1} + iO_g(s^2)A_g\right) + \\ + \tilde{o}(s^2)$$

After computation we find

$$\mathbb{1} + [A_m, A_m] s^2 = \mathbb{1} + i\alpha_1 s + i\alpha_2 s^2 + \tilde{O}(s^2).$$

so $\alpha_1 = 0$, that implies

$$\mathbb{1} + i[A_m, A_m] s^2 = \mathbb{1} + i\alpha_2 s^2 + i d_{j_0} A_{j_0} s^2 + \tilde{O}(s^2).$$

Thus $[A_m, A_m] = i(\alpha_2 + d_{j_0} A_{j_0})$.

COROLLARY. If $h_m(x) h_m(y) = h_m(y) h_m(x)$, then $[A_m, A_m] = i\alpha$.

By making use of prop. 3.3 we can explicity find how the generators of a projective representation of $\mathcal{G}$ must be related with each other.

Let us begin taking into account the product $\hat{h}_4(s)\hat{h}_5(s)\hat{h}_4(-s)\hat{h}_5(-s)$.

Here
$$\hat{h}_4(s) = \begin{bmatrix} \boxed{1} & & \\ & \begin{matrix} 1 & 0 & 0 \\ 0 & cs & -ss \\ 0 & ss & cs \end{matrix} & \\ & & \boxed{1} \end{bmatrix} = \begin{bmatrix} \boxed{1} & & \\ & \begin{matrix} 1 & 0 & 0 \\ 0 & 1-\frac{s^2}{2} & -s \\ 0 & s & 1-\frac{s^2}{2} \end{matrix} & \\ & & \boxed{1} \end{bmatrix} + \hat{o}_1(s^2),$$

$$\hat{h}_5(s) = \begin{bmatrix} \boxed{1} & & \\ & \begin{matrix} cs & -ss & 0 \\ ss & cs & 0 \\ 0 & 0 & 1 \end{matrix} & \\ & & \boxed{1} \end{bmatrix} = \begin{bmatrix} \boxed{1} & & \\ & \begin{matrix} 1-\frac{s^2}{2} & -s & 0 \\ s & 1-\frac{s^2}{2} & 0 \\ 0 & 0 & 1 \end{matrix} & \\ & & \boxed{1} \end{bmatrix} + \hat{o}(s^2)$$

and so on. Now the computatoon of the product can be carried out by collecting all term of order greater than 2 in a unique term infinitesimal of order greater than 2 $\hat{o}(s^2)$. In so doing we find

$$\hat{h}_4(s)\hat{h}_5(s)\hat{h}_4(-s)\hat{h}_5(-s) = \mathbb{1} + \hat{a}_6 s^2 + \hat{o}(s^2).$$

Therefore, according to prop. 3.3., we derive

$$[A_5, A_4] = i(\alpha_6 + A_6). \qquad (2)$$

Analogous computatoons yield

$$[A_6, A_5] = i(\alpha_4 + A_4), \qquad (2)$$
$$[A_4, A_6] = i(\alpha_5 + A_5).$$

Now we introduce the self-adjoint operators

$$J_x = -\alpha_4 - A_4, \quad J_y = -\alpha_5 - A_5, \quad J_z = -\alpha_6 - A_6. \qquad (3)$$

The operators $-J_x, -J_y, -J_z$ are the generators of the unitary one parameter groups

$$e^{-iJ_x\alpha}, \quad e^{-iJ_y\beta}, \quad e^{-iJ_z\gamma}. \qquad (4)$$

Now $e^{-iJ_x x_4} = e^{-i\alpha_4 x_4} e^{iA_4 x_4}$. Therefore it can replace $e^{iA_4\theta}$ in the projective representation

$$U_g = e^{iA_1x_1} \ldots e^{iA_4x_4} \ldots e^{iA_9x_9}$$

of a quantum theory of an isolated system, because it realizes the same quantum transformation realized by $e^{iA_4x_4}$. Thus, we establish $-J_x, -J_y, -J_z$ as the self-adjoint generators corresponding to rotations. For the new equivalent generators $-J_x, -J_y, -J_z$ the relations (2) become

$$[J_j, J_k] = i\hat{\epsilon}_{jkl} J_l, \qquad j,k,l \in \{x,y,z\}$$

where $\hat{\epsilon}_{jkl}$ is the Levi-Civita tensor restricted by the condition $j \neq l \neq k$, defined by

$$\hat{\epsilon}_{jkl} = \begin{cases} 0 & \text{if } j = k \\ 1 & \text{if } j,k \text{ are in cyclic order} \\ -1 & \text{if } j,k \text{ are in anti-cyclic order} \end{cases}$$

Now we consider the product $\hat{h}_4(s)\hat{h}_2(s)\hat{h}_4(-s)\hat{h}_2(-s)$. By making use of the expansion of each matrix up to the second order in s, we find

$$\hat{h}_4(s)\,\hat{h}_2(s)\,\hat{h}_4(-s)\,\hat{h}_2(-s) = \mathbb{1} + \hat{a}_3\, s^2 + \hat{o}(s^2),$$

so that prop. 3.3 implies

$$[A_2, A_4] = i(\alpha_3 + A_3). \qquad (5)$$

Analogously we find

$$[A_3, A_5] = i(\alpha_2 + A_2) \qquad (5)$$

$$[A_1, A_6] = i(\alpha_1 + A_1). \qquad (5)$$

Once again we redefine the generators corresponding to translations as

$$P_x = -\alpha_1 - A_1, \quad P_y = -\alpha_2 - A_2, \quad P_z = -\alpha_3 - A_3,$$

so that $U_g = e^{-iP_x x_1} e^{-iP_y x_2} e^{-iP_z x_3} e^{-iJ_x x_4} e^{-iJ_y x_5} e^{-iJ_z x_6} \cdot$
$\cdot\, e^{iA_7 x_7} e^{iA_8 x_8} e^{iA_9 x_9}.$

is the unitary operator that realizes

$$S_g[A] = U_g A U_g^{-1}.$$

(5) imply $[J_x, P_y] = iP_z$, $[J_y, P_z] = iP_x$, $[J_z, P_x] = iP_y$. (6)
Now, given any triplet of linear operators A, B, C the following Jacobi identity holds

$$[[A,B],C] + [[B,C],A] + [[C,A],B] = \mathbb{0}$$

holds.

Since $h_4(x), h_1(y)$ ((resp., $h_5(x), h_2(y)$, $h_6(x), h_3(y)$)) commute, according to the corollary of prop. 3.3 we have $[J_x, P_x] = i\xi$, $[J_y, P_y] = i\eta$, $[J_z, P_z] = i\zeta$.

Now, $[J_y, P_x] = -i[[J_z, J_x], P_x] =$

$$= i[[J_x, P_x], J_z] + i[[P_x, J_z], J_x] \quad (Z)$$

$$= -[J_x, P_y]$$

Now we prove that the constants ξ, η, ζ are 0.

$$i\xi = [J_x, P_x] = [J_x, -i[J_y, P_z]]$$

$$= i[J_y, \underbrace{[P_z, J_x]}_{-iP_y}] + i[P_z, [J_x, J_y]]$$

$$= [J_y, P_y] + [J_z, P_z].$$

therefore $\xi - \eta - \zeta = 0$ holds.

Starting from $[J_y, P_y]$ and $[J_z, P_z]$ we find

$$\eta - \zeta - \eta = 0 \quad \text{and} \quad \zeta - \xi - \eta = 0,$$

therefore the equations

$$\xi - \eta - \zeta = 0$$
$$\eta - \zeta - \xi = 0$$
$$\zeta - \xi - \eta = 0 \quad \text{must hold.}$$

This system of equation is not singular, i.e. it has only one solution: $\xi = \eta = \zeta = 0$.

Collecting these result we can state that

$$[J_j, P_k] = i\hat{\epsilon}_{jkl} P_l . \qquad (8)$$

Since $h_1(x), h_2(y), h_3(z)$ commute, then

$$[P_j, P_k] = i\alpha_{jk} . \text{ However}$$

$$\begin{aligned} [P_x, P_y] &= -i\,[[J_y, P_z], P_y] = \\ &= i\,[[P_z, P_y], J_y] + i\,[[P_y, J_y], P_z] \qquad (9) \\ &= 0 . \end{aligned}$$

Now we consider the product $\hat{h}_4(s)\hat{h}_8(s)\hat{h}_4(-s)\hat{h}_8(-s)$.
Proceeding as in the previous case we find

$$\hat{h}_4(s)\hat{h}_8(s)\hat{h}_4(-s)\hat{h}_8(-s) = 1\!\!1 + \hat{\alpha}_y s^2 + \hat{o}(s^2),$$

so that $[A_8, A_4] = i\,(\alpha_9 + A_9)$, i.e. $[J_x, A_8] = i(\alpha_9 + A_9)$.

Once redefined the generators of boosts as

$$G_x = \alpha_7 + A_7, \quad G_y = \alpha_8 + A_8, \quad G_z = \alpha_9 + A_9,$$

we have $[J_j, G_k] = i\hat{\epsilon}_{jkl}\, G_l$ (10).

By making use of the methods used for P_j,
we find $[G_j, G_k] = 0$. (11)

Hence, the commutation relations to be determined are those of the kind $[G_j, P_k]$.

Since translations and boost commute $[G_j, P_k] = i\delta_{jk}$.
For $[G_x, P_y]$ we find

$$[G_x, P_y] = -i\left[[J_y, G_z], P_y\right] = i\left[[G_z, P_y], J_y\right] + i\left[[P_y, J_y], G_z\right] = 0.$$

So $[G_j, P_k] = i\delta_{jk}$.

If $j = k = x$, we have

$$\begin{aligned}[G_x, P_x] &= -i[G_x, [J_y, P_z]] \\ &= i[J_y, [P_z, G_x]] + i[G_x, [J_y, P_z]] \\ &= [G_z, P_z].\end{aligned}$$

Analogously we find $[G_z, P_z] = [G_y, P_y]$.
Thus $[G_j, P_k] = i\delta_{jk}\mu$,

where μ is a real constant that characterize the projective representations: different value of μ corresponds to inequivalent projective representations of $\mathcal{G}$.

Concluding, the generators of a projective representation of $\mathcal{G}$, which realizes the quantum symmetry transformations, must satisfy

$$[J_j, J_k] = i\hat{\epsilon}_{jkl} J_l, \quad [J_j, P_k] = i\hat{\epsilon}_{jkl} P_l$$
$$[J_j, G_k] = i\hat{\epsilon}_{jkl} G_l, \quad [P_j, P_k] = 0; \quad [G_j, G_k] = 0, \quad (12)$$
$$[G_j, P_k] = i\delta_{jk}\mu.$$

3.4. DERIVING THE QUANTUM THEORY OF ISOLATED SYSTEMS

From a physical theoretical point of view the commutation relations (12) are constraints for the generators of the projective representation realizing the quantum symmetry transformations in the theory of an isolated system, constraints implied by the symmetry principles that specify an isolated system. These constraints are sufficient to explicitly determine the quantum theories of an isolated system as particular structures stemming from a kind of "induced representations". The key role in such a determination is played by the so called "imprimitivity theorem" of G.W. Mackey. Before formulate this theorem we introduce the needed concept of induced representation.

DEF. 3.2. Let $L : SO(3) \to \mathcal{U}(\mathcal{H}_0)$ a projective representation of the rotation group $SO(3)$.
The projective representation of the euclidean group $\mathcal{E}$, i.e. the subgroup of $\mathcal{G}$ generated by rotations and translations is the mapping
$U^L : \mathcal{E} \to \mathcal{U}(\mathcal{H})$ such that
(i) $\mathcal{H}$ is the Hilbert space of the functions
$$\psi : \mathbb{R}^3 \to \mathcal{H}_0,$$
such that $\int_{\mathbb{R}^3} \langle \psi(\vec{x}) | \psi(\vec{x}) \rangle_{\mathcal{H}_0} \, d\vec{x}^3 < +\infty$.

(ii) Given $g \in \mathcal{E}$, let $R \in SO(3)$ and $\vec{a} \in \mathbb{R}^3$ such that $g = h_1(a_x) h_2(a_y) h_3(a_z) R$, so that
$$\underline{g}(\vec{x}) = R\vec{x} + \vec{a}$$
where $\underline{g}$ is the bijection of $\mathbb{R}^3$ that transform vector of $\mathbb{R}^3$ according to g. Then
$$(U^L_g \psi)(\vec{x}) = L_R \psi(\underline{g}^{-1}(\vec{x})).$$

EXERCIZE. Prove that U^L is a projective representation.

Hence, any projective representation L of $SO(3)$ induces a projective representation of $\mathcal{E}$.

EXAMPLES. If the Hilbert space $\mathcal{H}_0$ of the inducing representation L is $\mathcal{H} = \mathbb{C}$, then $L: SO(3) \to \mathcal{U}(\mathcal{H}_0)$, $L_R = 1$ is a projective representation. Of course $\mathcal{H} = L_2(\mathbb{R}^3)$.
The induced representation U^L is
$$U^L: \mathcal{E} \to \mathcal{U}(L_2(\mathbb{R}^3)),\ (U_g\psi)(\vec{x}) = \psi(\underline{g}^{-1}(x));$$
if $g = h_1(a)$, $U_g\psi(\vec{x}) = \psi(x-a, y, z)$.
Every projective representation of $SO(3)$ with $\mathcal{H}_0 = \mathbb{C}$ is of the form $\tilde{L}_R z = e^{i\alpha(R)} z$, so that $e^{-i\alpha(R)}\tilde{L}_R = 1$; i.e. it can be converted into $L_R = 1$ by multiplication for a complex number of modulus 1.

If $\mathcal{H}_0 = \mathbb{C}^2$, then every projective representation of $SO(3)$ is equivalent to (converted into by a factor of modulus 1)

$$L_R = e^{-iS_x\alpha}e^{-iS_y\beta}e^{-iS_z\gamma}$$

where $S_x = \frac{1}{2}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$, $S_y = \frac{1}{2}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}$, $S_z = \frac{1}{2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$,

and $R = h_4(\alpha)h_5(\beta)h_6(\gamma)$. In this case

$$\mathcal{H} = L_2(\mathbb{R}^3, \mathbb{C}^2), \quad \psi \in \mathcal{H} \Rightarrow \psi(\vec{x}) = \begin{bmatrix}\psi_1(\vec{x})\\ \psi_2(\vec{x})\end{bmatrix};$$

if $\underline{g}(\vec{x}) = R\vec{x} + \vec{a}$ then

$$(U^L_{\underline{g}}\psi)(\vec{x}) = L_R\,\psi(\underline{g}^{-1}(\vec{x})),$$

where L_R is the matrix

$$L_R = e^{-\frac{i}{2}\begin{bmatrix}0 & \alpha\\ \alpha & 0\end{bmatrix}}\, e^{-\frac{i}{2}\begin{bmatrix}0 & -i\beta\\ i\beta & 0\end{bmatrix}}\, e^{-\frac{i}{2}\begin{bmatrix}\gamma & 0\\ 0 & -\gamma\end{bmatrix}}.$$

EXERCIZE. Prove that if $\mathcal{H}_0 = \mathbb{C}$, the generators of U^L are $P_x = -i\frac{\partial}{\partial x}$, $P_y = -i\frac{\partial}{\partial y}$, $P_z = -i\frac{\partial}{\partial z}$,

$J_z = -i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$, $J_x = -i\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$, $J_y = -i\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$.

(hint: exploit $\left.\frac{d\,U_{h_j(x)}}{dx}\right|_{x=0}\psi = iA\psi$ and $(U_{\underline{g}}^L\psi)(\vec{x}) = \psi(\underline{g}^{-1}(\vec{x}))$.

Prove that if $\mathcal{H}_0 = \mathbb{C}^2$ then

$$J_x = -i\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) + S_x,$$

$$J_y = -i\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) + S_y,$$

$$J_z = -i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) + S_z$$

DEF. 3.3. Let $U : \mathcal{E} \to \mathcal{U}(\mathcal{H})$ a projective representation of Euclide group $\mathcal{E}$. A triplet $(S_x, S_y, S_z) \equiv \vec{S}$ of self-adjoint operators such that $[S_j, S_k] = 0$ is said to be an imprimitivity system for U if
$$U_g(\vec{S})U_g^{-1} = \underline{g}^{-1}(\vec{S}).$$

In any projective representation of $\mathcal{E}$ induced by a projective representation L of $SO(3)$ the triplet $\vec{F} = (F_x, F_y, F_z)$ of the multiplication operators is an imprimitivity system. For instance, if $g = h_1(a)$ then $(U_g^L \psi)(\vec{x}) = \psi(x-a, y, z)$ and $\underline{g}^{-1}(\vec{F}) = \begin{bmatrix} F_x - a \\ F_y \\ F_z \end{bmatrix}$.

Then $\left\{\left(U_g^L F_x U_g^{L^{-1}}\right)\psi\right\}(\vec{x}) = \left(U_g^L F_x \varphi\right)(\vec{x})$

when $\varphi(\vec{x}) = \left(U_g^{L^{-1}}\psi\right)(\vec{x}) = \psi(\underline{g}(\vec{x})) = \psi(x+a, y, z)$.

Therefore $\left\{\left(U_g^L F_x U_g^{L^{-1}}\right)\psi\right\}(\vec{x}) = \left(U_g^L \phi\right)(\vec{x})$, where $\phi(\vec{x}) = x\,\varphi(\vec{x})$,

so that $\left\{\left(U_g^L F_x U_g^{L^{-1}}\right)\psi\right\}(\vec{x}) = \phi(\underline{g}^{-1}(x))$

$$= (x-a)\,\varphi(\underline{g}^{-1}(\vec{x})) = (x-a)\,\psi(\underline{g}(\underline{g}^{-1}(\vec{x})))$$
$$= (x-a)\,\psi(\vec{x}) = \left\{(F_x - a)\,\psi\right\}(\vec{x}).$$

An analogous computation shows that
$$U_g^L F_y U_g^{L^{-1}} = F_y, \quad U_g^L F_z U_g^{L^{-1}} = F_z.$$

EXERCIZE. Prove that $U_g^L \vec{F} U_g^{L^{-1}} = \underline{g}^{-1}(\vec{F})$ if $\underline{g} = h_6(0)$.

IMPRIMITIVITY THEOREM. Let $U: \mathcal{E} \to \mathcal{U}(\mathcal{H})$ a projective representation of Euclide group $\mathcal{E}$.

If $\vec{S}$ is an imprimitivity system for U then

(IM.1) a projective representation $L: SO(3) \to \mathcal{U}(\mathcal{H}_0)$ exists such that U^L is unitarily isomorphic to U, i.e.
a unitary transformation $W: \mathcal{H} \to L_2(\mathbb{R}^3, \mathcal{H}_0)$ exists such that $U_g^L = W U_g W^{-1}$, $\forall g \in \mathcal{E}$;

(IM.2) the tiple $\vec{F}$ of $L_2(\mathbb{R}^3, \mathcal{H}_0)$ is unitarily isomorphic to the imprimitivity system $\vec{S}$ of $\mathcal{H}$
$$\vec{F} = W \vec{S} W^{-1},$$
through the same unitary transformation W of (IM.1).

Our aim is to determine the quantum theories of an isolated system, that is to say the projective representations of Galilei group $\mathcal{G}$. The theorem of imprimitivity would seem to be not helpful, because it establishes conditions to identify projective representations of $\mathcal{E}$, i.e. only a subgroup of $\mathcal{G}$. In fact, the part (IM.2) of the theorem solves this lack: it can be proved that in every

projective representation $U: \mathcal{G} \to \mathcal{U}(\mathcal{H})$,
$U_g = e^{iP_x a_x} \cdots e^{iJ_z \gamma} e^{iG_x u_x} e^{iG_y u_y} e^{iG_z u_z}$,
the triplet $\vec{S} = \frac{\vec{G}}{\mu}$ always is an imprimitivity system for the restriction $U|_{\mathcal{E}}$ of U to the euclidean group $\mathcal{E}$; therefore the theorem of imprimitivity applies establishing that in every projective representation of $\mathcal{G}$, its restriction to $\mathcal{E}$ is unitarily isomorphic to the projective representation of $\mathcal{E}$ induced by some projective representation L of $SO(3)$, but also the generators G_j are isomorphic to μF_j on $L_2(\mathbb{R}^3, \mathcal{H}_0)$; therefore, the entire projective representation of $\mathcal{G}$ is unitarily isomorphic to

$$g \to \hat{U}_g = U^L_{h_E(a_x) \cdots h_G(\gamma)} e^{i\mu F_x u_x} e^{i\mu F_y u_y} e^{i\mu F_z u_z};$$

Thus, the quantum theories of an isolated system are explicitly determined.
Now we prove indeed that $\vec{S} = \vec{G}/\mu$ is an imprimitivity system for $U|_{\mathcal{E}}$, after the following technical lemmas.

LEMMA 3.1. Let $A, B \in \mathcal{R}(\mathcal{H})$ be such that $[A, B] = i$. If $f(B)$ is an analytic function of B then

$$[A, f(B)] = i\frac{df}{dB}(B).$$

PROOF. If $[A,B]=i$ then $[A,B^n]=i n B^{n-1} = i\frac{dB^n}{dB}$;

indeed, by induction,

$$
\begin{aligned}
[A,B^n] &= AB^n - BAB^{n-1} + BAB^{n-1} - BB^{n-1}A \\
&= [A,B]B^{n-1} + B[A,B^{n-1}] \\
&= [A,B]B^{n-1} + B\, i(n-1)B^{n-2} \\
&= i n B^{n-1}.
\end{aligned}
$$

The extension to an analytic function $f(B)=\sum_n a_n B^n$ (weak convergence) is straightforward

$$
\begin{aligned}
\langle\psi|[A,f(B)]\psi\rangle &= \lim_{N\to\infty} \langle\psi|\sum_n^N a_n [A,B^n]\psi\rangle \\
&= \lim_{N\to\infty} \langle\psi|\sum_n^N i a_n n B^{n}\psi\rangle \\
&= \lim_{N\to\infty} \langle\psi| i\frac{d}{dB}\sum_n^N a_n B^n\psi\rangle = \langle\psi| i\frac{d}{dB}f(B)\psi\rangle.
\end{aligned}
$$

LEMMA 3.2. Let $\vec{J}=(J_x,J_y,J_z)$ a triplet of self-adjoint operators such that $[J_j,J_k]=i\epsilon_{jkl}J_l$. If $\vec{D}=(D_x,D_y,D_z)$ is a triplet of self-adjoint operators such that $[D_j,D_k]=0$ and $[J_j,D_k]=i\epsilon_{jkl}$, then
$$[J_j,\phi(\vec{D})] = i\left(\frac{\partial\phi(\vec{D})}{\partial D_k}D_l - \frac{\partial\phi(\vec{D})}{\partial D_l}D_k\right),$$
for (j,k) in cyclic order, and ϕ analytic.

PROOF. If (j,k) are in cyclic order, then it can be proved by induction that (see proof of lemma 3.1)
$$[J_j, D_k^n] = i n D_k^{n-1} D_l = i\frac{\partial D_k^n}{\partial D_k}D_l, \quad j\neq l\neq k.$$
If (j,k) are in anti-cyclic order $[J_j, D_k^n] = -i\frac{\partial D_k^n}{\partial D_k}D_l$.

So, $[J_j, D_j^{m_1} D_k^{m_2} D_l^{m_3}] = D_j^{m_1} J_j D_k^{m_2} D_l^{m_3} - D_j^{m_1} D_k^{m_2} D_l^{m_3} J_j$

$$= D_j^{m_1} J_j D_k^{m_2} D_l^{m_3} - D_j^{m_1} D_k^{m_2} J_j D_l^{m_3} + D_j^{m_1} D_k^{m_2} J_j D_l^{m_3} - D_j^{m_1} D_k^{m_2} D_l^{m_3} J_j$$

$$= D_j^{m_1} [J_j, D_k^{m_2}] D_l^{m_3} + D_j^{m_1} D_k^{m_2} [J_j, D_l^{m_3}]$$

$$= D_j^{m_1} (i m_2 D_k^{m_2-1} D_l) D_l^{m_3} + D_j^{m_1} D_k^{m_2} (i m_3 D_l^{m_3-1}) D_k,$$

i.e. $$[J_j, D_j^{m_1} D_k^{m_2} D_l^{m_3}] = i\left(\frac{\partial (D_j^{m_1} D_k^{m_2} D_l^{m_3})}{\partial D_k} D_l - \frac{\partial (D_j^{m_1} D_k^{m_2} D_l^{m_3})}{\partial D_l} D_k\right).$$

Now, if $\phi(\vec{D}) = \sum_{\vec{m}} a_{\vec{m}} D_j^{m_1} D_k^{m_2} D_l^{m_3}$ is analytic, then the previous equation applies to each term of the series, so that

$$[J_j, \phi(\vec{D})] = i\left(\frac{\partial \phi(\vec{D})}{\partial D_k} D_l - \frac{\partial \phi(\vec{D})}{\partial D_l} D_k\right) \quad \left(\text{resp.} = -i\left(\frac{\partial \phi}{\partial D_k} D_l - \frac{\partial \phi}{\partial D_l} D_k\right)\right)$$

if (j,k,l) are in cyclic order (resp., in anti-cyclic order).

<u>PROPOSITION 3.4.</u> Let $U: \mathcal{G} \to \mathcal{U}(\mathcal{H})$, $U_g = e^{-iP_x a_x} \cdots e^{-iJ_z \gamma} \cdots e^{iG_z u_z}$ a projective representation of G.

The triplet $\vec{S} = \left(\frac{G_x}{\mu}, \frac{G_y}{\mu}, \frac{G_z}{\mu}\right)$ is an imprimitivity system for the restriction $U|_E$.

<u>PROOF.</u> Let g be a translation, say $g = h_1(a)$. In such a case $$g^{-1}\left(\begin{bmatrix} S_x \\ S_y \\ S_z \end{bmatrix}\right) = \begin{bmatrix} S_x - a \\ S_y \\ S_y \end{bmatrix}.$$

Now, $U_g S_x U_g^{-1} = e^{-iP_x a} S_x e^{iP_x a} = e^{-iP_x a}\left([S_x, e^{iP_x a}] + e^{iP_x a} S_x\right)$.

Since $[S_x, P_x] = \frac{1}{\mu}[G_x, P_x] = i$ by (12), Lemma 3.1 implies $U_g S_x U_g^{-1} = e^{-iPa}\left(i \cdot i a e^{iP_x} + e^{iP_x a} S_x\right) = S_x - a$,

i.e., $U_g S_x U_g^{-1} = \underline{g}^{-1}(S_x)$.
Since $[S_y, P_x] = 0$, we shall find

$$U_g S_y U_g^{-1} = S_y = \underline{g}^{-1}(S_y).$$

Repeating the method for $g = h_2(a)$ and $g = h_3(a)$ we find that $U_g \vec{S} U_g^{-1} = \underline{g}^{-1}(\vec{S})$ for all $g \in \mathcal{G}$.

Now we prove the relation $U_g \vec{S} U_g^{-1} = \underline{g}^{-1}(\vec{S})$ for rotations. Let us start with $g = h_6(\theta)$, so that $U_g = e^{-iJ_z\theta}$. In such a case

$$\underline{g}^{-1}(\vec{S}) = \begin{bmatrix} \cos\theta\, S_x + \sin\theta\, S_y \\ -\sin\theta\, S_x + \cos\theta\, S_y \\ S_z \end{bmatrix}. \qquad (13)$$

Since $[J_j, S_k] = [J_j, \frac{G_k}{\mu}] = i\hat{\epsilon}_{jkl} \frac{G_l}{\mu} = i\hat{\epsilon}_{jkl} S_l$ by (12), we apply Lemma 3.2 to $\phi(\vec{S}) = \operatorname{arctg} \frac{S_y}{S_x}$.

So we have

$$[\phi(\vec{S}), J_z] = -[J_z, \phi(\vec{S})] = -i\left(S_y \frac{\partial\phi}{\partial S_x} - S_x \frac{\partial \phi}{\partial S_y}\right) = i, \qquad (14a)$$

$$[\sqrt{S_x^2+S_y^2}, J_z] = -i\left(S_y \frac{\partial\sqrt{S_x^2+S_y^2}}{\partial S_x} - S_x \frac{\partial \sqrt{S_x^2+S_y^2}}{\partial S_y}\right) = 0. \qquad (14b)$$

Then $e^{-iJ_z\theta} S_x e^{iJ_z\theta} = e^{-iJ_z\theta}\left(\sqrt{S_x^2+S_y^2}\cos\phi(\vec{S})\right) e^{iJ_z\theta}$

$$= \sqrt{S_x^2+S_y^2}\,\cos\left(e^{-iJ_z\theta}\phi(\vec{S})e^{iJ_z\theta}\right)$$

by (14b) $$= \sqrt{S_x^2+S_y^2}\,\cos\left(e^{-iJ_z\theta}\{[\phi(\vec{S}), e^{iJ_z\theta}] + e^{iJ_z\theta}\}\right)$$

by lemma 3.2 and (14a) $$= \sqrt{S_x^2+S_y^2}\,\cos\left(e^{-iJ_z\theta}\, i \cdot i\theta\, e^{iJ_z\theta} + e^{iJ_z\theta}\right)$$

$$= \sqrt{S_x^2+S_y^2}\,\cos(-\phi(S)+\theta) = \sqrt{S_x^2+S_y^2}\,(\cos\theta\cos\phi(S) + \sin\theta\sin\phi(S));$$

Therefore $e^{-iJ_z\theta} S_x e^{iJ_z\theta} = \cos\theta S_x + \sin\theta S_y = g^{-1}(S_x)$.
The same method proves (13).
Hence, the relation $U_g \vec{S} U_g^{-1} = g^{-1}(\vec{S})$ holds for elementary rotation and translation. For every $g \in \mathcal{E}$ we have
$U_g = e^{-iP_x a_x} e^{-iP_y a_y} e^{-iP_z a_z} e^{-iJ_x\alpha} e^{-iJ_y\beta} e^{-iJ_z\gamma}$, therefore
$U_g \vec{S} U_g^{-1} = g^{-1}(\vec{S})$ extends to all $g \in \mathcal{E}$. Thus, since $[S_x, S_y] = 0$, $\vec{S} = \vec{G}/\mu$ is an imprimitivity system for $U|_{\mathcal{E}}$.

Every quantum theory of an isolated system has as Hilbert space the Hilbert space of a projective representation $g \to e^{-iP_x a_x} \cdots e^{-iJ_z\gamma} e^{iG_x u_x} e^{iG_y u_y} e^{iG_z u_z}$ of $\mathcal{G}$.
Since $\vec{S} = \frac{\vec{G}}{\mu}$ is an imprimitivity system for $U|_{\mathcal{E}}$, the theorem of imprimitivity implies that a projective representation $L : SO(3) \to \mathcal{U}(\mathcal{H}_0)$ and a unitary transformation W exist such that

- $\mathcal{H}$ is unitarily isomorphic to $L_2(\mathbb{R}^3, \mathcal{H}_0)$,
- for $g = g_1 g_2$, with $g_1 = h_1(a_x)h_2(a_y)h_3(a_z)h_4(\alpha)h_5(\beta)h_6(\gamma)$ and $g_2 = h_7(u_x)h_8(u_y)h_9(u_z)$, U_g is unitarily isomorphic to $\mathcal{U}^L_{g_1} e^{i\mu F_x u_x} e^{i\mu F_y u_y} e^{i\mu F_z u_z}$,

through the same unitary transformation W.
Thus, by collecting all these projective representations for all μ and L, we determine all quantum theories of an isolated system.

APPENDIX

PROOF OF WIGNER THEOREM

2.1 Transformations and Wigner's theorem

Definition 2.1. Given two Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$, both complex and separable, by Wigner transformation we mean a mapping

$$S:\Pi_1(\mathcal{H})\rightarrow \Pi_1(\mathcal{H}')$$

bijective, such that

$$Tr(P_1 P_2) = Tr(S(P_1)S(P_2))$$

for all $P_1, P_2 \in \Pi_1(\mathcal{H})$.

Remark 2.1. Every unitary operator U: $\mathcal{H}\rightarrow \mathcal{H}'$ induces a Wigner transormation. Given any $P \in \Pi_1(\mathcal{H})$, we can define

$$S^U(P) = UPU^{-1}.$$

Then $S^U : \Pi_1(\mathcal{H})\rightarrow \Pi_1(\mathcal{H}')$ is a bijective transformation;
in fact $[S^U]^{-1}(P) = U^{-1}PU$, and also $S^U(P) \in \Pi_1(\mathcal{H}')$, if $P \in \Pi_1(\mathcal{H})$ ($U^*=U^{-1}$) and multiplying by unitary operators does not imply alterations of rank. The map S^U is a Wigner transformation: $\mathrm{Tr}(S^U(P_1)S^U(P_2)) = Tr(UP_1U^{-1}UP_2U^{-1}) = Tr(UP_1P_2U^{-1}) = \mathrm{Tr}(P_1P_2)$, because two similarity transformations leave the trace unchanged. We can observe that if two unitary operators U and V differ

by a phase factor, such that $V = e^{i\alpha}U$, then they generate the same Wigner's transformation: $S^V(P) = VPV^{-1} = e^{i\alpha}UPe^{-i\alpha}U^{-1} = UPU^{-1} = S^U(P)$.

Note 2.1. We can also adapt the same transformations to the case of anti unitary operators.

Now we enunciate Wigner's theorem in two equivalent formulations.

Theorem 2.1 (Wigner Theorem 1) Let S: $\Pi_1(\mathcal{H}) \rightarrow \Pi_1(\mathcal{H}')$ be a Wigner's transformation. Then an operator U:$\mathcal{H} \rightarrow \mathcal{H}'$ exists, unitary or antiunitary, such that S(P)= UPU^{-1} =S$^U(P)$ for all P $\in \Pi_1(\mathcal{H})$. Moreover if an operator V exists, unitary or antiunitary, such that S=S^V, then $\alpha \in \mathbb{R}$ exists for which V= $e^{i\alpha}U$. Indeed, the existence of an operator U that realizes the transformation, implies the existence of an equivalence class **U**=$\{e^{i\alpha}U\}_{\alpha}$ called the ray of operators, and every V $\in \mathcal{V}$ realises the same transformation.

Let's denote by $\Pi(\mathcal{H})$={E: $\mathcal{H} \rightarrow \mathcal{H}'$, such that E=E*, E$^2 = E\}$ the set of all projection operators of $\mathcal{H}$.

Theorem 2.2 (Wigner Theorem 2) Let S: $\Pi(\mathcal{H}) \rightarrow \Pi(\mathcal{H}')$ a bijective mapping, such that for all E$_1, E_2, E \in \Pi(\mathcal{H})$:

1) E$_1 \leq E_2 \Longrightarrow S(E_1) \leq S(E_2)$;

2) S(E$^\perp$) $= [S(E)]^\perp$.

Then there exists an unitary or anti-unitary operator U of $\mathcal{H}$, such that

$$\text{S(E)=UEU}^{-1} \text{ for all E } \in \Pi_1(\mathcal{H}).$$

Furthermore, the operator U is unique up to a phase factor.

Definition 2.2. A ray is a family of vectors that differ between them by a phase factor: if $\varphi \in \mathcal{H}$, the ray $\hat{R}_\varphi$ generated by φ is the set of all vectors of the form $\tau\varphi$ where τ is a scalar of modulus one. If φ is an unitary vector, $\hat{R}_\varphi$ will also be said to be unitary.

We denote by $\mathcal{R}(\mathcal{H})$ the set of all unit rays of the space $\mathcal{H}$ and a generic φ, belonging to it, is a representative of $\hat{R}_\varphi$.

Definition 2.3. Given two rays $\hat{R}_\varphi$ and $\hat{R}_\psi$, their product is given by

$$\hat{R}_\varphi \cdot \hat{R}_\psi = |\langle\varphi|\psi\rangle|.$$

This definition is independent of the choice of representatives since a ray is uniquely determined by any of its representatives.

Observation 2.2. Given a ray $\hat{R} \in \mathcal{R}(\mathcal{H})$, if $\varphi, \psi \in \hat{R}$, then

$$|\varphi\rangle\langle\varphi| = |\psi\rangle\langle\psi| \ .$$

Therefore, each unit ray corresponds biunivocally to a rank-one projection operator.

Theorem 2.3. (Wigner's Theorem 1 (reformulation)) Given a Wigner transformation S: $\Pi_1(\mathcal{H}) \to \Pi_1(\mathcal{H}')$, we define

$$\delta : \mathcal{R}(\mathcal{H}) \to \mathcal{R}(\mathcal{H}')$$

and

$$T_\psi = \delta(\hat{R}_\psi) = \hat{R}(S(|\psi\rangle\langle\psi|)) \ ;$$

where T_ψ is the ray associated to the projector $S(|\psi\rangle\langle\psi|)$.
The transformation S is bijective and such that , if $\psi_1 \in \hat{R}_{\psi_1}, \psi_2 \in \hat{R}_{\psi_2}$,

$$|\langle\psi_1|\psi_2\rangle| = |\langle\psi_1'|\psi_2'\rangle| \tag{2.1}$$

for all $\psi_1' \in T_{\psi_1}, \psi_2' \in T_{\psi_2}$.
Then the following statements hold:

1. There exists a unitary or anti-unitary operator $U : \mathcal{H} \to \mathcal{H}'$, such that $U\psi \in T_\psi$, for all $\psi \in \mathcal{H}$.

2. The operator U is unique up to a phase factor: If $V : \mathcal{H} \to \mathcal{H}$ is a unitary or anti-unitary operator such that $V\psi \in T_\psi$ for all $\psi \in H$ with $\|\psi\| = 1$, then there exists $\alpha \in \mathbb{R}$ such that $V = e^{i\alpha}U$.

Proof. The proof proceeds by lemmas.

Lemma 2.1. Consider an orthonormal basis $\{\psi_n\}_n$ of $\mathcal{H}$; $\{\tilde{\psi}_k\}_k$ is an orthonormal basis of $\mathcal{H}'$; if $\tilde{\psi}k \in T\psi_k$. Let $\varphi_k = \frac{1}{\sqrt{2}}(\psi_1 + \psi_k)$ for every $k \in \mathbb{N}$. Then, there exists a real number $\theta_k \in \mathbb{R}$ such that

$$\phi'_k = \frac{1}{\sqrt{2}}(\tilde{\psi}_1 + e^{i\theta_k}\tilde{\psi}_k) \in T_{\phi_k}.$$

Proof. If $\tilde{\psi}_j \in T\psi_j$, from hypothesis 2.1 of the theorem it follows that $|\langle\psi_j|\phi_k\rangle| = |\langle\tilde{\psi}_j|\tilde{\psi}\rangle|$ for every $\tilde{\psi} \in T\phi_k$, that is to say

$$|\langle\psi_j|\varphi_k\rangle| = \begin{cases} \frac{1}{\sqrt{2}} & \text{if } j = 1, k, \\ 0 & \text{otherwise.} \end{cases}$$

We can deduce that $\tilde{\psi} = \sum_j \langle\tilde{\psi}_j|\tilde{\psi}\rangle\tilde{\psi}_j$ has only the terms corresponding to $\tilde{\psi}_1$ and $\tilde{\psi}_k$, hence

$$\tilde{\psi} = \sum_j \langle\tilde{\psi}_j|\tilde{\psi}\rangle\tilde{\psi}_j = \frac{1}{\sqrt{2}}(e^{i\alpha_1}\tilde{\psi}_1 + e^{i\alpha_k}\tilde{\psi}_k) \in T_{\phi_k}.$$

Multiplying the vector $\tilde{\psi}$ by a phase factor, the product remains in the same ray, so we have

$$\phi' = e^{-i\alpha_1}\tilde{\psi} = \frac{1}{\sqrt{2}}(\tilde{\psi}_1 + e^{i(\alpha_k - \alpha_1)}\tilde{\psi}_k) \in T_{\phi_k}.$$

Defining $\theta_k = \alpha_k - \alpha_1$ as the phase of $\tilde{\psi}k$, we obtain

$$\frac{1}{\sqrt{2}}(\tilde{\psi}_1 + e^{i\theta_k}\tilde{\psi}k) \in T_{\phi_k}.$$

Note 2.3. We choose a particular basis of $\mathcal{H}$ since the proved lemma allows us to introduce the orthonormal basis $\{\psi'_n\}_n$ of $\mathcal{H}$ by setting $\psi'_k = e^{i\theta_k}\tilde{\psi}_k$ and $\psi'_1 = \tilde{\psi}_1$. So

$$\frac{1}{\sqrt{2}}(\psi'_1 + \psi'_k) \in T_{\phi_k}, \forall k \in \mathbb{N}.$$

Lemma 2.2. Given $\phi_k(\theta) = \frac{1}{\sqrt{2}}(\psi_1 + e^{i\theta}\psi_k)$, then for every $k \in \mathbb{N}$ and $\theta \in \mathbb{R}$, there exists $y_k(\theta) \in \{-1, +1\}$ such that

$$\phi_k' = \frac{1}{\sqrt{2}}(\psi_1' + e^{iy_k(\theta)\theta}\psi_k') \in T_{\phi_k(\theta)}.$$

Proof. Let us fix k and θ. If $\phi = \frac{1}{\sqrt{2}}(\psi_1 + e^{i\theta}\psi_k)$, then by the previous lemma there must exist $\alpha \in \mathbb{R}$ such that $\phi' = \frac{1}{\sqrt{2}}(\psi_1' + e^{i\alpha}\psi_k') \in T_\phi$. If $\phi_k = \frac{1}{\sqrt{2}}(\psi_1 + \psi_k)$, from hypothesis 2.1 of the theorem it follows that $|\langle\phi_k|\phi\rangle| = |\langle\phi_k'|\phi\rangle|$, where $\phi_k' = \frac{1}{\sqrt{2}}(\psi_1' + \psi_k')$. Then, since

$$\langle\phi_k|\phi\rangle = \frac{1}{2}(1 + e^{i\theta}) = \frac{e^{i\theta/2}}{2}(e^{-i\theta/2} + e^{i\theta/2}) = e^{i\theta/2}\cos(\theta/2),$$

$$\langle\phi_k'|\phi\rangle = \frac{1}{2}(1 + e^{i\alpha}) = \frac{e^{i\alpha/2}}{2}(e^{-i\alpha/2} + e^{i\alpha/2}) = e^{i\alpha/2}\cos(\alpha/2),$$

the equality $|\langle\phi_k|\phi\rangle| = |\langle\phi_k'|\phi'\rangle|$ implies that $|\cos(\theta/2)| = |\cos(\alpha/2)|$. The previous equation has solutions $\alpha/2 = n\pi \pm \theta/2$ for every $n \in \mathbb{Z}$, i.e. $\alpha = 2n\pi \pm \theta$. Therefore, $e^{i\alpha} = e^{i2n\pi}e^{\pm i\theta}$, which implies that $y_k(\theta) \in$ {-1,+1}.

Lemma 2.3. Fixed k, let $\theta_0 = \frac{\pi}{6}$, then

$$\frac{1}{\sqrt{2}}\left(\psi_1' + e^{iy_k(\theta_0)\theta}\psi_k'\right) \in T_{\phi_k}(\theta), \text{ for all } \theta.$$

In other words, the dependence of $y_k(\theta)$ on θ can be eliminated, i.e.$y_k(\theta) = y_k(\theta_0) = y_k$.

Proof. Let $\phi_k'(\theta) = \frac{1}{\sqrt{2}}\left(\psi_1' + e^{iy_k\theta}\psi_k'\right)$ *be* $\in T_{\phi_k}(\theta)$ and suppose $y_k(\theta_0) = -1$. If $y_k(\theta) = 1$, then $y_k(\theta) \neq y_k(\theta_0)$, which would imply

$$|\langle\phi_k(\theta)|\phi_k(\theta_0)\rangle| = |\langle\phi_k'(\theta)|\phi_k'(\theta_0)\rangle|,$$

which implies

$$\left|\cos\left(\frac{\theta + \theta_0}{2}\right)\right| = \left|\cos\left(\frac{\theta - \theta_0}{2}\right)\right|.$$

The above equality holds only for $\theta = k\pi$, in which case we have

$$e^{in\pi} = e^{-in\pi} = e^{iy_k(\theta_0)n\pi}.$$

Lemma 2.4. Let $\Psi = \sum_k c_k e^{i\alpha_k}\psi_k$ such that

- $\alpha_1 = 0, c_1 > 0$,
- $c_k = |\langle\psi_k|\Psi\rangle| = |\langle\psi'_k|\Psi'\rangle| \geq 0$,
- $\sum_k |c_k|^2 = 1$, i.e., $\|\Psi\| = 1$

Then, $\sum_k c_k e^{iy_k\alpha_k}\psi'_k \in T_\Psi$.

Proof. For each k, let us define the auxiliary vector $\Phi_k(\alpha_k) = \frac{1}{\sqrt{2}}(\psi_1 + e^{i\alpha_k}\psi_k)$, so that $\Phi'_k(\alpha_k) = \frac{1}{\sqrt{2}}(\psi'_1 + e^{iy_k\alpha_k}\psi'_k) \in T_{\Phi_k(\alpha_k)}$. Let $\Psi' \in T_\Psi$, then we have

$$\Psi' = \sum_k \langle\psi'_k|\Psi'\rangle\psi'_k = \sum_k c_k e^{i\beta_k}\psi'_k$$

because, in the image ray, the coefficients must be equal up to a phase factor, given, by hypothesis, the equality of the modules. By the symmetry condition of Wigner's theorem 2.1, we have

$$|\langle\Phi_k(\alpha_k)|\Psi\rangle| = |\langle\Phi'_k(\alpha_k)|\Psi'\rangle|,$$

which implies the following equality

$$\frac{1}{\sqrt{2}}|c_1 + c_k| = \frac{1}{\sqrt{2}}|c_1 + c_k e^{-i(y_k\alpha_k - \beta_k)}|.$$

Since $c_1 > 0$ and $c_k \geq 0$, the equality holds when $e^{-i(y_k\alpha_k-\beta_k)} = 1$. Therefore,

$$e^{-i(y_k\alpha_k-\beta_k)} = e^{i2k\pi} \text{ , i.e. } e^{i\beta_k} = e^{iy_k\alpha_k}$$

which means that $\Psi' = \sum_k c_k e^{iy_k\alpha_k}\psi'_k \in T_\Psi$.

Lemma 2.5. The value of y_k in the previous lemma's statement is independent of k and only depends on the symmetry transformation: if $y_k = 1$ for all k, the transformation is linear (unitary operator). If $y_k = -1$ for all k, the transformation is anti-linear (antiunitary operator).

Proof. Consider the vectors

$$\Psi_0 = \tfrac{1}{\sqrt{3}}(\psi_1 + \psi_j + \psi_k) \in \mathcal{H},$$
$$\Psi'_0 = \tfrac{1}{\sqrt{3}}(\psi'_1 + \psi'_J + \psi'_K) \in T_{\Psi_0}.$$

Let $\Psi = \frac{1}{\sqrt{3}}(\psi_1 + e^{i\frac{2}{3}\pi}\psi_j + e^{i\frac{4}{3}\pi}\psi_k)$. Then, by the previous lemma,

$$\Psi' = \frac{1}{\sqrt{3}}(\psi'_1 + e^{iy_j\frac{2}{3}\pi}\psi'_j + e^{iy_k\frac{4}{3}\pi}\psi'_k) \in T_\Psi \ .$$

From condition 2.1, we have

$$|\langle\Psi_0|\Psi\rangle| = |\langle\Psi_0|\Psi'\rangle|,$$

from which it follows that

$$\frac{1}{\sqrt{3}}|1 + e^{i\frac{2}{3}\pi} + e^{i\frac{4}{3}\pi}| = \frac{1}{\sqrt{3}}|1 + e^{iy_j\frac{2}{3}\pi} + e^{iy_k\frac{4}{3}\pi}|.$$

However, $|1 + e^{i\frac{2}{3}\pi} + e^{i\frac{4}{3}\pi}| = |1 + e^{-i\frac{2}{3}\pi} + e^{i\frac{2}{3}\pi}| = |1 + 2\cos(\frac{2}{3}\pi)| = 0$, and so $|1 + e^{iy_j\frac{2}{3}\pi} + e^{iy_k\frac{4}{3}\pi}| = 0$. This implies that

$$\cos(y_j\frac{2}{3}\pi) + \cos(y_k\frac{4}{3}\pi) + i[\sin(y_j\frac{2}{3}\pi) + \sin(y_k\frac{4}{3}\pi)] = -1.$$

Furthermore, $\cos(y_j\frac{2}{3}\pi) + \cos(y_k\frac{4}{3}\pi) = 2\cos(y_k\frac{4}{3}\pi) = -1$, so $\sin(y_j\frac{2}{3}\pi) = -\sin(y_k\frac{4}{3}\pi)$. This last equality only holds if $y_j = y_k$, so the Wigner transformation determines a unique value y_0 for each k.

Lemma 2.6. Let $\Psi = \sum_k c_k e^{i\alpha_k}\psi_k$ where $\sum_k |c_k|^2 = 1$ and $c_k \geq 0$ for all k , then

$$\sum_k c_k e^{iy_0\alpha_k}\psi'_k \in T_\Psi.$$

Proof. If $c_1 > 0$, the thesis follows from Lemma 2.4. Therefore, it suffices to prove the thesis in the case where $c_1 = 0$, that is when $\Psi = \sum_k c_k e^{i\alpha_k}\psi_k$. It is clear that $\Psi \perp \psi_1$, since $\psi_k \perp \psi_1$ for $k \neq 1$. Let us define the unit vector

$$\Phi = \frac{1}{\sqrt{2}}(\psi_1 + \Psi) = \frac{1}{\sqrt{2}}\left(\psi_1 + \sum_k c_k e^{i\alpha_k}\psi_k\right).$$

Applying Lemma 2.4 to this vector, we have

$$\Phi' = \frac{1}{\sqrt{2}}\left(\psi'_1 + \sum_k e^{iy_k\alpha_k}\psi'_k\right) \in T_\Phi.$$

Let $\Psi' \in T_\Psi$ be a vector. Using condition 2.1, we have

$$|\langle\Psi|\psi_1\rangle| = |\langle\Psi'|\psi'_1\rangle| = 0;$$

and we can deduce that $\Psi' \perp \psi_1'$. Let us evaluate the moduli $|\langle\Psi|\Phi\rangle|$ and $|\langle\Psi'|\Phi'\rangle|$, which must coincide for the symmetry condition:

$$|\langle\Psi|\Phi\rangle| = \left|\langle\Psi|\frac{1}{\sqrt{2}}\left(\psi_1 + \sum_{k>1} c_k e^{i\alpha_k}\psi_k\right)\rangle\right| =$$

$$= \frac{1}{\sqrt{2}}\left|\langle\Psi|\psi_1\rangle + \frac{1}{\sqrt{2}}\langle\Psi|\Psi\rangle\right|$$

$$= \frac{1}{\sqrt{2}},$$

.

$$|\langle\Psi'|\Psi'\rangle| = \left|\langle\Psi'|\frac{1}{\sqrt{2}}\left(\psi_1' + \sum_{k>1} c_k e^{iy_0\alpha_k}\psi_k'\right)\rangle\right|$$

$$= \left|\frac{1}{\sqrt{2}}\langle\Psi'|\psi_1\rangle + \frac{1}{\sqrt{2}}\langle\Psi'|\sum_{k>1} c_k e^{y_0 i\alpha}\psi_k'\rangle\right|$$

$$= \frac{1}{\sqrt{2}}\left|\langle\Psi'|\sum_{k>1} c_k e^{y_0 i\alpha}\psi_k'\rangle\right|.$$

It follows that:

$$|\langle\Psi'|\sum_{k>1} c_k e^{iy_0\alpha_k}\psi_k'\rangle| = 1 = \|\Psi'\| \left\|\sum_{k>1} c_k e^{iy_0\alpha_k}\psi_k'\right\|$$

and by the theorem on Schwartz inequality,

$$\sum_{k>1} c_k e^{iy_0\alpha_k}\psi_k' = e^{i\lambda}\Psi' \in T_\Psi.$$

We have shown that, given a bijective transformation $\delta : \mathcal{R}(\mathcal{H}) \to \mathcal{R}(\mathcal{H}')$ satisfying condition 2.1, for every orthonormal basis $\{\psi_k\}_k \subseteq \mathcal{H}$, there exists an orthonormal basis $\{\psi_k'\}_k \subseteq \mathcal{H}$ with $\psi_k' \in T_{\psi k}$, such that one of the following conditions holds:

1. For every $\Psi = \sum_k \langle\psi_k|\Psi\rangle\psi_k = \sum_k c_k e^{-i\alpha_k}\psi_k$, we have $\Psi' = \sum_k \overline{\langle\psi_k|\Psi\rangle}\psi_k' = \sum_k c_k e^{-i\alpha_k}\psi_k' \in T_\Psi$.

2. For every $\Psi = \sum_k \langle \psi_k | \Psi \rangle \psi_k = \sum_k c_k e^{i\alpha_k} \psi_k$, we have

$\Psi' = \sum_k \langle \psi_k | \Psi \rangle \, \psi'_k = \sum_k c_k e^{i\alpha_k} \psi'_k \in T_\Psi$.

Then the correspondence $\psi_k \to \psi'_k \equiv U\psi_k \in T_{\psi_k}$ extends to a unitary or anti-unitary operator U such that:

- $U\Psi = \sum_k \overline{\langle \psi_k | \Psi \rangle} \psi'_k \in T\Psi$, for every $\Psi \in \mathcal{H}$, $||\Psi|| = 1$, in case 1;
- $U\Psi = \sum_k \langle \psi_k | \Psi \rangle \psi'_k \in T\Psi$, for every $\Psi \in \mathcal{H}$, $||\Psi|| = 1$, in case 2.

To complete the proof of the theorem, it is necessary to show the uniqueness, up to a phase factor, of the operator U.

Lemma 2.7. The unitary or anti-unitary operator : $\mathcal{H} \to \mathcal{H}'$ is unique up to a phase factor.

Proof. Let : $\mathcal{H} \to \mathcal{H}'$ be a unitary or anti-unitary operator such that VΨ $\in T_\psi$ for all unitary $\Psi \in \mathcal{H}$, then V$\psi_k = e^{i\gamma_k} \psi'_k$. We can define the auxiliary vector $\Phi_{jk} = \frac{1}{\sqrt{2}}(\psi_j + \psi_k)$, then

$$V\Phi_j k = \frac{1}{\sqrt{2}}(e^{\gamma_j}\psi'_j + e^{i\gamma_k}\psi'_k) \in \; T_{\phi jk}.$$

However

$$\frac{1}{\sqrt{2}}(\psi'_j + \psi'_k) = U\Phi_{jk} \in \; T_{\Phi jk},$$

there exists $\alpha \in \mathbb{R}$ such that e$^{i\alpha} U\Phi_{jk}$ =VΦ_{jk}. Indeed

$$e^{i\alpha}\frac{1}{\sqrt{2}}(\psi'_j + \psi'_k) = \frac{1}{\sqrt{2}}(e^{i\gamma_j}\psi'_j + e^{i\gamma_k}\psi'_k).$$

i.e. e$^{i\gamma_j} = e^{i\gamma_k}$. Therefore, $V\psi_j = e^{i\alpha} U\psi_j$ for all j, which means that $V\Psi = e^{i\alpha} U\Psi$ for all $\Psi \in H$.

Remark 2.1. Wigner showed that any bijective correspondence of rays f $\leftrightarrow$ f' can be extended to a bijective correspondence of vectors $f \leftrightarrow f' \equiv Uf$, where U is said to be a continuous extension of the correspondence of rays if it transforms any representative of a ray **f** into a representative of the corresponding ray **f'**. Moreover, the correspondence of rays uniquely determines the operator U, which can be either unitary or antiunitary, up to a phase factor (the ray of the operator **U** is defined). Conversely, every unitary or antiunitary operator uniquely determines a correspondence of rays.