TQ.6. QUANTUM THEORY OF A FREE PARTICLE

By **Particle** we mean a physical system whose quantum theory is endowed with the "position observable", i.e. a triplet $\vec{Q}=(Q_x, Q_y, Q_z)$ of self-adjoint operators representing three observables $\mathcal{Q}_x, \mathcal{Q}_y, \mathcal{Q}_z$ whose measurement yields three outcomes (q_x, q_y, q_z) which are the three coordinates of the point position of the system.

A particle is said to be "free" if it is an isolated system, and "elementary" if the triplet is unique.

REMARK. An elementary particle has no components. Indeed, if a system were compound by two or more components, then there would be a position triplet $(Q_x^{(i)}, Q_y^{(i)}, Q_z^{(i)})$ for each component and thus there would exist more than one triplet (Q_x, Q_y, Q_z) representing position.

Hence, the following definition can be established

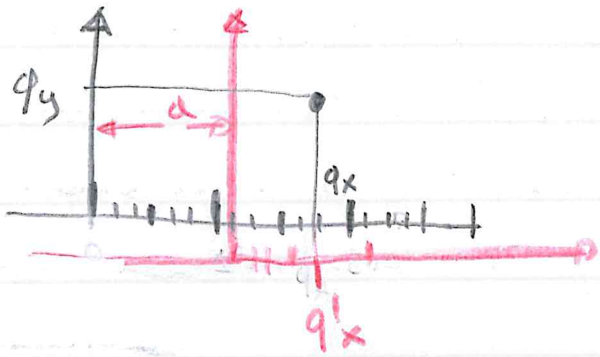
DEF. 1.1. An elementary free particle is an isolated system whose quantum theory contains a unique triplet $\vec{Q}=(Q_x, Q_y, Q_z)$ of selfadjoint operators that satisfy the following conditions.

(Q.1) $[Q_j, Q_k]=\mathbb{0}$ for all $j,k=x,y,z$.

This condition establishes that a measurement of the position yields all three values of the coordinates of the same specimens.

(Q.2) For every $g \in \mathcal{G}$, the triple $\vec{Q} = (Q_x, Q_y, Q_z)$ and the transformed position operator $S_g[\vec{Q}]$ satisfy the specific relations implied by the transformation properties of position with respect to g.

EXAMPLE. Let g be a translation along x, i.e. $g = h_1(a)$.



To transform the position observable according to g entails to displace the axes where the values of the coordinates are redd of a length a in the positive direction of x axis; so, if the outcome of a measurement of the Q_x is q_x, the outcome of $S_g[Q_x]$ on the same specimen must be $q'_x = q_x - a$, while $q'_y = q_y$ and $q'_z = q_z$.
Therefore, there is a functional relation between $\vec{Q}$ and $S_g[\vec{Q}]$, namely
$S_g[Q_x] = Q_x - a$, $S_g[Q_y] = Q_y$, $S_g[Q_z] = Q_z$,
i.e. $S_g[\vec{Q}] = U_g \vec{Q} U_g^{-1} = g^{-1}(\vec{Q})$, where $g\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+a \\ y \\ z \end{bmatrix}$.

More generally, condition (Q.2) implies,

$$e^{-iP_j a} Q_k e^{iP_j a} = Q_k - \delta_{jk} a; \qquad (1)$$

$$\left.\begin{array}{l} e^{-iJ_j\theta} Q_k e^{iJ_j\theta} = \cos\theta\, Q_k + \sin\theta\, Q_l \text{ and} \\ e^{-iJ_j\theta} Q_l e^{+iJ_j\theta} = -\sin\theta\, Q_k + \cos\theta\, Q_l \\ \text{if } j,k,l \text{ are in cyclic order, while} \\ e^{-iJ_j\theta} Q_j e^{iJ_j\theta} = Q_j\,; \end{array}\right\} \quad (2)$$

$$e^{+iG_j u} Q_k e^{-iG_j u} = Q_k . \qquad (3)$$

1. IDENTIFICATION OF POSITION OPERATOR

By making use of the implications (1), (2), (3) of (Q.2), we now determine the commutation relations between the position operators Q_k and the generators P_j, J_j, G_j.

These commutation relations are sufficient to explicitly determine the position operators in the simplest quantum theory of an elementary free particle.

1.1. COMMUTATION RELATIONS FOR $\vec{Q}$.

The expansion of $e^{-iP_x a}$ with respect to a yields

If $e^{-iP_x a} = \mathbb{1} - iP_x a + O_1(a)$.

then (1) becomes for $Q_j = Q_x$

$$\left(\mathbb{1} - iP_x a + O_1(a)\right) Q_x \left(\mathbb{1} + iP_x a + O_2(a)\right) = Q_x - a, \text{ i.e.}$$

$$Q_x + i[Q_x, P_x]a + O(a) = Q_x - a, \text{ that implies}$$

$$[Q_x, P_x] = i.$$

For $Q_j = Q_y$ (1) implies

$$Q_y + i[Q_y, P_x]a + O(a) = Q_y; \text{ therefore}$$

$$[Q_y, P_x] = 0. \text{ Thus}$$

$$[Q_j, P_k] = i\delta_{jk}. \tag{4}$$

The expansion of $e^{-iJ_z\theta}$ yields

$$e^{-iJ_z\theta} = \mathbb{1} - iJ_z\theta + O_1(\theta), \text{ so that (2) implies}$$

$$Q_x - i[J_z, Q_x]\theta + O_2(\theta) = \cos\theta\, Q_x + \sin\theta\, Q_y$$

$$= Q_x + \theta Q_y + O_3(\theta); \text{ i.e.}$$

$$[J_z, Q_x] = iQ_z.$$

The same calculus for all cases in (2) yields

$$[J_j, Q_k] = i\epsilon_{jkl} Q_l. \tag{5}$$

Analogously, starting from (3) we obtain

$$[G_j, Q_k] = 0. \tag{6}$$

1.2. IDENTIFICATION OF $\vec{Q}$.

The theory of elementary free particle, being an isolated system, must be formulated in a Hilbert space $\mathcal{H} = L_2(\mathbb{R}, \mathcal{H}_0)$, according to section 3.4 in TQ.4. The simplest theory corresponds to $\mathcal{H} = \mathcal{L}_2(\mathbb{R}^3)$, where

$P_i = -i\frac{\partial}{\partial x_i}$, $J_i = -i\left(x_k\frac{\partial}{\partial x_l} - x_l\frac{\partial}{\partial x_k}\right)$, i, k, l in cyclic order,

$G_i = \mu F_i$.

Now, the commutation rule (6) implies

$$[F_j, Q_k] = \mathbb{0}, \text{ i.e. } [F_j, Q_k - F_k] = \mathbb{0} \qquad (2)$$

therefore the completeness of $\vec{F}$ implies

$$Q_k - F_k = f_k(\vec{F}).$$

On the other hand, since $[P_j, F_k] = -i\delta_{jk}$, by (4)

$[P_j, f_k(\vec{F})] = [P_j, Q_k] - [P_j, F_k] = -i\delta_{jk} + i\delta_{jk} = \mathbb{0}$ holds too.

For every ψ we have

$$\mathbb{0} = \left([P_j, f_k(\vec{F})]\psi\right)(\vec{x}) = -i\frac{\partial}{\partial x_j}\left(f_k(\vec{x})\psi(\vec{x})\right) - \left(f_k(\vec{x})\left(-i\frac{\partial\psi}{\partial x_j}(\vec{x})\right)\right)$$

$$= -i\frac{\partial f_k(\vec{x})}{\partial x_j}\psi(\vec{x}) - if_k(\vec{x})\frac{\partial\psi}{\partial x_j}(\vec{x}) + if_k(\vec{x})\frac{\partial\psi}{\partial x_j}(\vec{x})$$

$$= -i\left(\frac{\partial f_k(\vec{x})}{\partial x_j}\right)\psi(\vec{x}) = 0.$$

this result implies $f_k(\vec{F}) = Q_k - F_k = i\alpha_k$. (constant)

A straight computation yields $[J_j, F_k] = i\hat{\epsilon}_{jkl} F_l$.
Now, $[J_j, f_k(\vec{F})] = \mathbb{0}$, because $f_k(\vec{F}) = i d_k$; but

$$[J_j, f_k(\vec{F})] = [J_j, Q_k] - [J_j, F_k] =$$

$$= i\hat{\epsilon}_{jkl} Q_k - i\hat{\epsilon}_{jkl} F_k = i\hat{\epsilon}_{jkl} d_l = 0.$$

Thus $Q_l - F_l = \mathbb{0}$, $\forall l$, i.e.

$$\vec{Q} = \vec{F}.$$

The simplest theory of an elementary free particle is now completely identified.
The Hilbert space of the theory can be chosen as $L_2(\mathbb{R}^3)$.
The selfadjoint generators of the projective representation realizing the quantum symmetry transformations associated to $\mathcal{G}$ are

$$P_{x_j} = -i\frac{\partial}{\partial x_j}, \quad J_j = -i\left(x_k\frac{\partial}{\partial x_l} - x_l\frac{\partial}{\partial x_k}\right) \left(\text{or } \vec{J} = \vec{F}\wedge\vec{P}\right), \quad G_j = \mu F_j.$$

The hamiltonian operator is

$$H = \frac{P^2}{2\mu}.$$

2. EMPIRICAL CONSISTENCY

In TQ.0 we saw that in some circumstances physical phenomena occur where it is not possible to consistently assign the values of different observables to the same specimen of the physical system. These facts originated the epistemological necessity of developing physical theories on conceptual basis quite different from that of "classical" theories.

In particular, this problem arises in double slit experiment: it is not possible to assign position to the same particle at the time of the final impact and at the time the particle crosses the screen supporting the slits, and therefore these two observables cannot be measured together on the same specimen.

Now we see that the quantum theory of a free particle, i.e. the physical system of double slit experiment, is consistent with this empirical phenomenon; indeed, for a free particle positions at different times of the same particle do not commute.

Position at time 0 is represented by $\vec{Q} = \vec{F}$.

Position at time t can be determined as

$$Q_j^{(t)} = e^{iHt} Q_j e^{-iHt} = e^{iHt}\left([F_j, e^{-iHt}] + e^{-iHt} F_j\right)$$

$$= e^{iHt}\left([F_j, e^{-i\frac{P^2}{2\mu}t}] + e^{-iHt} F_j\right)$$

$$= e^{iHt}\left(i \frac{\partial}{\partial P_j} e^{-i\frac{P^2}{2\mu}t} + e^{-iHt} F_j\right)$$

$$= e^{iHt}\left(i\left\{-i \frac{P_j}{\mu} t\, e^{-iHt}\right\} + e^{-iHt} F_j\right)$$

$$= F_j + \frac{P}{\mu} t \,.$$

Then $[Q_j, Q_j^{(t)}] = [F_j, F_j + \frac{P_j}{\mu} t]$

$$= i\frac{t}{\mu} \neq 0 \,.$$