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Group theoretical derivation of the minimal coupling principle

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The group theoretical methods worked out by Bargmann, Mackey and Wigner, which *deductively* establish the Quantum Theory of a free particle for which Galileian transformations form a symmetry group, are extended to the case of an interacting particle. In doing so, the obstacles caused by loss of symmetry are overcome. In this approach, specific forms of the wave equation of an interacting particle, including the equation derived from the minimal coupling principle, are implied by particular first-order invariance properties that characterize the interaction with respect to specific subgroups of Galileian transformations; moreover, the possibility of yet unknown forms of the wave equation is left open.

1. Introduction

In Quantum Theory, a complex and separable Hilbert space $\mathcal{H}$ can be associated to the physical system under investigation, in such a way that the self-adjoint operators of $\mathcal{H}$ represent the quantum observables of the system and the density operators represent quantum states [1]. The formulation of the effective Quantum Theory of a *specific* system requires two tasks: to explicitly identify the self-adjoint operator of $\mathcal{H}$ representing each relevant observable of the system, and secondly to specify the dynamical law.

Canonical quantization is a primary method for accomplishing these tasks; it obtains the Quantum Theory of a particle, for instance, 'quantizing' the position coordinates and their conjugate momenta, q_j, p_j as operators Q_j, P_j and then replacing q_j and p_j with Q_j and P_j in the Poisson brackets, transformed into commutation brackets, of the equations of the *classical theory* of that particle [2].

This method has played a decisive role for the development of the Quantum Theories of specific systems. However, it has intrinsic limits, as explained for instance in [3,4]. Indeed, canonical quantization can be implemented only if the observables to be 'quantized' already exist in a classical form; this is not always the case, as in the case of spin observables. Another problem, of an epistemological nature, is that while Quantum Theory aims to be more fundamental than the Classical Theory, canonical quantization is based on the latter [3].

There are, however, *group theoretical methods* that do not suffer from these shortcomings. In fact, they attain a formulation of the Quantum Theory of a free particle through a purely deductive development based on symmetry principles. Put simply, these approaches proceed according to the following scheme. First, they enforce the condition that Galilei's group $\mathcal{G}$ (or Poincaré's group $\mathcal{P}$, for a relativistic theory) is a group of symmetry transformations for an isolated particle, so that Wigner's theorem [5,6] on the representation of symmetries implies the existence of a projective representation of that group. Then, since the position observables $(Q_1, Q_2, Q_3) = \mathbf{Q}$ determine the existence of an imprimitivity system with respect to the restriction of the projective representation to the Euclidean group, Mackey's imprimitivity theory [7,8] can be applied to derive the explicit Quantum Theory of a free particle [9]. In this way, the drawbacks of canonical quantization are obviated. For instance, the spin observables, having no classical analogue, are correctly predicted by the approach [10,11]; no pre-existing classical theory is required.

This state of affairs makes it worthwhile to extend the group theoretical methods to develop an analogous approach to the Quantum Theory of more general physical systems than the free particle.

A line of research, effectively reviewed in [4], is devoted to generalizing the quantization via Mackey's imprimitivity theorem to the case of configuration manifolds M with topologies different from the trivial topology of $\mathbb{R}^n$; the physical interest of this line of research is also driven by the fact that non-trivial topologies are related to new non-classical effects, such as Dirac's magnetic monopole [12] and the Aharanov–Bohm effect [13]. To this end, a notion of *quantum Borel kinematics* has been introduced, investigated and classified, that generalizes the imprimitivity systems of Mackey [4,14–18] in this direction. A weakening of the notion of quantum Borel kinematics leads to the notion of *generalized imprimitity systems* that allows for the *description* of particle in external gauge fields, as for instance magnetic fields [4].

In this work, we specifically address the problem of the development of a group theoretical approach to the Quantum Theory of an *interacting* particle. In fact, the extension of the group theoretical methods, so satisfactory for a free particle, to an interacting particle encounters serious problems; the main obstacle is the fact that for a non-isolated system the Galileian transformations, or Poincaré's transformations in the relativistic case, do not form a group of symmetry transformations [19], so that neither Wigner's theorem nor Mackey's imprimitivity theorem can apply directly. One very special case was treated by Hoogland [20], who derived the wave equation of a spin-0 charged particle subjected to an interaction having the particular feature of leaving unaltered the symmetry condition of a rich subgroup of the whole transformations group. We shall discuss this work in remark 4.3.

In fact, in the literature several general approaches extend the group theoretical methods to the case of interacting particles. However, many of these proposals [9,19,21–23], in order to overcome the difficulty raised by the loss of symmetry, have to introduce certain assumptions; yet, as we argue in §2d, these assumptions lead to an empirically inadequate theory, unable, in particular, to describe particles interacting with electromagnetic fields.

We show how a group theoretical approach to the Quantum Theory of an interacting particle can be successfully pursued without introducing assumptions such as those required in [9,19,21–23], which restrict the empirical domain of the theory too drastically. In fact, our approach derives the known non-relativistic wave equations, and opens up the possibility of yet unknown equations. The approach applies for interactions which leave the $\mathbb{R}^3$ topology of the localization space of the particle unaltered; hence, we do not make use of generalized imprimitivity systems [4].

Let us now describe how the article is organized. First, we find preliminary results which hold both for a non-relativistic and for a relativistic theory, i.e. independently of whatever group, $\Upsilon = \mathcal{G}$ or $\Upsilon = \mathcal{P}$, is taken into account. The basic concept (QT) of *quantum transformation* corresponding to a space–time transformation $g \in \Upsilon$, which is viable also in the absence of the condition of symmetry, is introduced in §2b as a transformation S_g^Σ defined on the whole set of quantum observables. Three conditions (S.1), (S.2) and (S.3) required for this notion of quantum transformation are identified in §§2b and 3a, where we show that they, together with a continuity condition for $g \to S_g^\Sigma$, imply that every transformation $g \in \Upsilon$ can be assigned a *unitary* operator U_g that realizes the quantum transformation of a quantum observable A as $S_g^\Sigma[A] = U_g A U_g^{-1}$, even if g is not a symmetry; moreover, the correspondence $g \to U_g$ is proved to be a continuous mapping.

However, there is another obstacle: the properties (S.1), (S.2) and (S.3) are insufficient to imply that $g \to U_g$ is a *projective representation*, which is one of the conditions required for the imprimitivity theorem to apply.

To address this problem in §3c we introduce the notion of σ-conversion, which is a straight mathematical procedure that converts each U_g into another unitary operator $\hat{U}_g$ in such a way that $g \to \hat{U}_g$ is a *projective representation*. In the non-relativistic case, we prove that the imprimitivity theorem for the *Euclidean* group $\mathcal{E}$—not for the whole Galileian group $\mathcal{G}$—can be applied to identify a mathematical formalism of the theory explicitly; but in general the position operators are not explicitly identified, so that the identified formalism turns out to be devoid of physical significance.

In order to arrive at an effective theory it is necessary to determine which operators represent position and to determine the dynamical law. In §3d, we show how the operators that *physically* represent the position of the particle are explicitly represented for a particular class of interactions, fully characterized by admitting 'Q-covariant' σ-conversions, i.e. σ-conversions that leave the covariance properties of the position with respect to $\mathcal{G}$ unaltered. For this class of interactions, the general dynamical law is determined in §3e.

This law does not specify the explicit form of the Hamiltonian operator H; in fact, different specific forms of the wave equation are compatible with it. So we face the problem of singling out conditions related to the interaction, which determine the different wave equations.

In §4, we identify these conditions as *invariance properties related to the interaction*. More precisely, we single out the *specific forms the wave equation takes* if the σ-conversion admitted by the interaction leaves unaltered, at the first order, the covariance properties of $\mathbf{Q}^{(t)}$ (i.e. of position at time t) *with respect to specific subgroups of* $\mathcal{G}$. If this subgroup is the subgroup of boosts, then in the spin-0 case the wave equation turns out to coincide with the equation obtained by canonical quantization, or by means of the *minimal electromagnetic coupling principle*, that is to say by replacing, in the free particle Schroedinger equation $\mathrm{i}(\mathrm{d}/\mathrm{d}t)\psi_t(\mathbf{x}) = (1/2\mu)(\partial^2/\partial x_1^2) + \partial^2/\partial x_2^2 + \partial^2/\partial x_3^2)\psi_t(\mathbf{x})$, the operator $\mathrm{i}(\mathrm{d}/\mathrm{d}t)$ by $(\mathrm{i}(\mathrm{d}/\mathrm{d}t) - \Phi(\mathbf{x}))$ and $-\mathrm{i}(\partial/\partial x_\alpha)$ by $(-\mathrm{i}(\partial/\partial x_\alpha) + a_\alpha(\mathbf{x}))$. By taking into account other subgroups, the known wave equations are recovered, and also yet unknown ones could be derived.

In the final §4d, the relation of the present approach with other methods for quantizing the interaction are briefly discussed, and apparent conflicts with some results of the approach reviewed in [4] are clarified.

2. Space–time and quantum transformations

In this section, we establish the basic concepts and express them in the quantum formalism. In §2a, the necessary mathematics is outlined. In §2b, we introduce a concept of *quantum transformation*, corresponding to Galilei's or Poincaré's transformations, that is viable also in the case where the system is *interacting*, i.e. when the transformations are not symmetries. A general property (S.1) of these quantum transformations, entailed in their very meaning, is identified. The presence of the *symmetry* condition of the transformations implies more marked properties; they are established in §2c, where we outline how these further properties can be used to obtain the explicit Quantum Theory of a free particle by mathematically deducing it from the principles of symmetry. This outline allows us to identify, in §§2d, the obstacles raised by the loss of the symmetry condition to a similar deduction in the case of an interacting particle.

(a) Mathematical tools

Let us begin with the notation for the mathematical structures involved in the work. The Quantum Theory of a physical system, formulated in a complex and separable Hilbert space $\mathcal{H}$, needs the following mathematical structures.

— The set $\Omega(\mathcal{H})$ of all self-adjoint operators of $\mathcal{H}$, which represent *quantum observables*.
— The complete, ortho-complemented lattice $\Pi(\mathcal{H})$ of all projections operators of $\mathcal{H}$, i.e. quantum observables with possible outcomes in $\{0, 1\}$.
— The set $\Pi_1(\mathcal{H})$ of all rank one orthogonal projections of $\mathcal{H}$.
— The set $\mathcal{S}(\mathcal{H})$ of all density operators of $\mathcal{H}$, which represent *quantum states*.
— The set $\mathcal{U}(\mathcal{H})$ of all unitary operators of Hilbert space $\mathcal{H}$.

In the group theoretical approach, a key role is played by the *imprimitivity theorem* of Mackey, which classifies representations of *imprimitivity* systems relative to projective representations [8]. The following definition recalls the notion of projective representation.

Definition 2.1. Let G be a separable, locally compact group with identity element e. A correspondence $U : G \to \mathcal{U}(\mathcal{H})$, $g \to U_g$, with $U_e = \mathbb{1}$, is a projective representation of G if the following conditions hold.

(i) A complex function $\sigma : G \times G \to \mathbb{C}$, called multiplier, exists such that $U_{g_1 g_2} = \sigma(g_1, g_2) U_{g_1} U_{g_2}$;
(ii) for all $\phi, \psi \in \mathcal{H}$, the mapping $g \to \langle U_g \phi \mid \psi \rangle$ is a Borel function in g.

A projective representation with multiplier σ is also called σ-representation.

A projective representation is said to be continuous if for any fixed $\psi \in \mathcal{H}$ the mapping $g \to U_g \psi$ from G to $\mathcal{H}$ is continuous with respect to g.

Let $\mathcal{E}$ be the Euclidean group, i.e. the semi-direct product $\mathcal{E} = \mathbb{R}^3 ⓢ \mathrm{SO}(3)$ between the group of spatial translations $\mathbb{R}^3$ and the group of spatial proper rotations SO(3); each transformation $g \in \mathcal{E}$ bi-univocally corresponds to the pair $(\mathbf{a}, R) \in \mathbb{R}^3 \times \mathrm{SO}(3)$ such that $R^{-1}\mathbf{x} - R^{-1}\mathbf{a} \equiv \mathfrak{g}(\mathbf{x})$ is the result of the passive transformation of the spatial point $\mathbf{x} = (x_1, x_2, x_3)$ by g. The general imprimitivity theorem is an advanced mathematical result [8] with valuable generalization (see [4,24] and reference therein); in this article we shall make use of this theorem with relation to the Euclidean group $\mathcal{E}$ only. Then we introduce the concept of *imprimitivity system* and the theorem for this specific case [7,9].

Definition 2.2. Let $\mathcal{H}$ be the Hilbert space of a σ-representation $g \to U_g$ of Euclidean group $\mathcal{E}$. A *projection valued* (PV) measure $E : \mathcal{B}(\mathbb{R}^3) \to \Pi(\mathcal{H})$, $\Delta \to E(\Delta)$ is an *imprimitivity system* for the σ-representation $g \to U_g$ if the relation

$$U_g E(\Delta) U_g^{-1} = E(\mathfrak{g}^{-1}(\Delta)) \equiv E(R(\Delta) + \mathbf{a}) \tag{2.1}$$

holds for all $(\mathbf{a}, R) \in \mathcal{E}$.

Mackey's theorem of imprimitivity for $\mathcal{E}$. If a PV measure $E : \mathcal{B}(\mathbb{R}^3) \to \Pi(\mathcal{H})$ is an imprimitivity system for a continuous σ-representation $g \to U_g$ of the Euclidean group $\mathcal{E}$, then a σ-representation $L : \mathrm{SO}(3) \to \mathcal{U}(\mathcal{H}_0)$ exists such that, modulo a unitary isomorphism,

(M.1) $\mathcal{H} = L_2(\mathbb{R}^3, \mathcal{H}_0)$,
(M.2) $(E(\Delta)\psi)(\mathbf{x}) = \chi_\Delta(\mathbf{x})\psi(\mathbf{x})$, where χ_Δ is the characteristic functional of Δ,
(M.3) $(U_g \psi)(\mathbf{x}) = L_R \psi(\mathfrak{g}(\mathbf{x})) \equiv L_R \psi(R^{-1}\mathbf{x} - R^{-1}\mathbf{a})$, for every $g = (\mathbf{a}, R) \in \mathcal{E}$.

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Furthermore, the σ-representation U is irreducible if and only if the 'inducing' representation L is irreducible.

(b) Basic concepts

In this subsection, we formulate a concept of quantum transformation, which is also viable for space–time transformations that are not symmetry transformations.

For the sake of synthesis, in the following by Υ we denote the group $\mathcal{G}$ of Galileian transformations without time translations and space–time inversions, or the group $\mathcal{P}$ of Poincaré's transformations without space–time inversions; therefore, whatever holds for Υ must hold for $\mathcal{G}$ and $\mathcal{P}$. In the present work, group Υ is interpreted as a group of changes of reference frame in a class $\mathcal{F}$ of frames which move uniformly with respect to each other. So, given any reference frame Σ in $\mathcal{F}$, a transformation $g \in \Upsilon$ univocally singles out the reference frame Σ_g related to Σ by g.

Let us consider the Quantum Theory of a *localizable particle*, that is to say of a physical system which can be localized in a point of physical space, so that its Quantum Theory contains a unique triple $(Q_1, Q_2, Q_3) \equiv \mathbf{Q}$ of commuting self-adjoint operators representing the three coordinates of the position. Now, the point of space, where the particle is localized by a measurement of the position observables, is identified only if the frame the values of the coordinates refer to is specified. For instance, if $(Q_1, Q_2, Q_3) \equiv \mathbf{Q}$ are the three self-adjoint operators which represent the three coordinates of the position with respect to Σ and if $g \in \mathcal{E}$, then the α-th coordinate of the position with respect to another frame Σ_g, related to Σ by g, must be represented by $[\mathfrak{g}(\mathbf{Q})]_\alpha$, where $\mathfrak{g}(\mathbf{x}) = (y_1, y_2, y_3)$ is the triple of the coordinates, with respect to Σ_g, of the spatial point represented by $\mathbf{x}$ with respect to Σ. In the non-relativistic case, a pure Galileian boost $g \in \mathcal{G}$ characterized by a velocity $\mathbf{u} = (u, 0, 0)$, does not change the instantaneous position at all; hence $\mathfrak{g}(\mathbf{x}) = \mathbf{x}$ and $S_g^\Sigma[\mathbf{Q}] = \mathfrak{g}(\mathbf{Q}) = \mathbf{Q}$, so that the operators which represent the coordinates of the 'position with respect to Σ_g' coincide with the operators representing the position coordinates with respect to Σ. In order to transform the position quantum observables at time t, i.e. the operators $\mathbf{Q}^{(t)} = \mathrm{e}^{\mathrm{i}Ht}\mathbf{Q}^{(t)}\,\mathrm{e}^{-\mathrm{i}Ht}$, by a Galileian boost g, a function $\mathfrak{g}_t$ other than $\mathfrak{g}$ must be used. Indeed, $\mathbf{Q}^{(t)}$ represents the position measured with a delay t, therefore the operators which represent the 'position at time t with respect to Σ_g' must be $S_g^\Sigma[\mathbf{Q}^{(t)}] = (Q_1^{(t)} - ut, Q_2, Q_3) \equiv \mathfrak{g}_t(\mathbf{Q}^{(t)})$, where $\mathfrak{g}_t(\mathbf{x}) = (x_1 - ut, x_2, x_3)$.

In general, we can state that for every $g \in \mathcal{G}$ the following covariance relations hold for all $g \in \mathcal{G}$,

$$\text{(i) } S_g^\Sigma[\mathbf{Q}] = \mathfrak{g}(\mathbf{Q}) \quad \text{and} \quad \text{(ii)} \quad S_g^\Sigma[\mathbf{Q}^{(t)}] = \mathfrak{g}_t(\mathbf{Q}^{(t)}), \tag{2.2}$$

where $\mathfrak{g}_t$ is a function, in general different from $\mathfrak{g}$. In fact, relations (2.2) are the conditions which *define* the position operators of a localizable particle.

We cannot exclude *a priori* that observables other than position change their representation according to the frame they are referred to; so, in order that the Quantum Theory of our particle can account for such a possibility, it must extend transformations S_g^Σ to all quantum observables. To this end, given two reference frames Σ_1 and Σ_2 in $\mathcal{F}$, we introduce the following concept of *relative indistinguishability* between measuring procedures.

(Ind) Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two measuring procedures. If $\mathcal{M}_1$ *with respect to* Σ_1 is identical to $\mathcal{M}_2$ *with respect to* Σ_2, then we shall say that $\mathcal{M}_1$ and $\mathcal{M}_2$ are indistinguishable with relation to (Σ_1, Σ_2).

Then, for every $g \in \Upsilon$ and every Σ in $\mathcal{F}$, we introduce the mapping

$$S_g^\Sigma : \Omega(\mathcal{H}) \to \Omega(\mathcal{H}), \quad A \to S_g^\Sigma[A] \tag{2.3}$$

with the following conceptually explicit interpretation.

(QT) *The self-adjoint operators* A *and* $S_g^\Sigma[A]$ *represent two measuring procedures* $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *indistinguishable with relation to* (Σ, Σ_g).

For instance, if A represents a detector placed in the origin of Σ with a given orientation relative to Σ, then $S_g^{\Sigma}[A]$ is the operator that represents an identical detector placed in the origin of Σ_g with that orientation relative to Σ_g. It must be noted that (QT) presupposes that for each quantum observable $A \in \Omega(\mathcal{H})$ and every $g \in \Upsilon$, two measuring procedures with the required indistinguishability exist, at least in principle.

We call S_g^{Σ} the *quantum transformation* corresponding to g.

Relations (2.2) explicitly specify the action of the transformations S_g^{Σ} on the position operators $\mathbf{Q}^{(t)}$; for an arbitrary observable no such kind of explicit specification can be established *a priori*. However, the authentic meaning (QT) of the notion of quantum transformation is sufficient to infer, at a conceptual level, the following general constraint.

(S.1) *For every frame Σ in $\mathcal{F}$ the following statement holds.*

$$S_{gh}^{\Sigma}[A] = S_g^{\Sigma_h}[S_h^{\Sigma}[A]], \quad \text{for all } A \in \Omega(\mathcal{H}). \tag{2.4}$$

This general statement stresses how, without further conditions, the mapping S_g^{Σ}, with g fixed, can change by changing the 'starting' frame Σ.

(c) Symmetry transformations

Let us now briefly outline the particularly important implications of the existence of conditions of symmetry. A transformation $h \in \Upsilon$ is a symmetry transformation for the physical system under investigation if a class $\mathcal{F}$ exists such that for every frame Σ in $\mathcal{F}$, the frames Σ and Σ_h are equivalent for the formulation of the empirical theory of the system; for an isolated system, all $g \in \Upsilon$ are symmetry transformations.

The symmetry property allows us to apply Wigner's theorem, and in doing so the following implication is obtained [5,6,25,26].

SYM.1. *If $g \in \Upsilon$ is a symmetry transformation then a unitary or an anti-unitary operator U_g^{Σ}, unique modulo a phase factor, exists such that*

$$S_g^{\Sigma}[A] = U_g^{\Sigma} A [U_g^{\Sigma}]^*. \tag{2.5}$$

Moreover, according to the Principle of Relativity, for an isolated system all $g \in \Upsilon$ are symmetry transformations. Therefore, a class $\mathcal{F}$ exists such that the following statement holds.

SYM.2. *In the Quantum Theory of an isolated system, for each $g \in \Upsilon$ the quantum transformation S_g^{Σ} must be independent of Σ, i.e. $S_g^{\Sigma} = S_g^{\Sigma_h} \equiv S_g$ and $U_g^{\Sigma} = e^{i\lambda} U_g^{\Sigma_h}$ (with $\lambda \in \mathbb{R}$), so that (2.4) and (2.5) imply*

$$S_{gh}[A] = S_g[S_h[A]]. \tag{2.6}$$

Therefore, $U_{gh} = \sigma(g,h) U_g U_h$ holds, which implies that each U_g is unitary [9,25]; in particular, $U_g^* = U_g^{-1}$. Thus, if Υ is a group of symmetry transformations, the correspondence $g \to U_g$ such that $S_g[A] = U_g A U_g^*$ is a projective representation [8,9,27].

A *free localizable particle* is just a particular kind of isolated system, so that according to SYM.2 for every $g \in \Upsilon$ a unitary operator U_g exists such that $S_g[A] = U_g A U_g^{-1}$. The restriction of $g \to U_g$ to the Euclidean group $\mathcal{E}$ is a projective representation of $\mathcal{E}$ [9]. Thus, from (2.2) and SYM.1, the relation $U_g \mathbf{Q} U_g^{-1} = \mathfrak{g}(\mathbf{Q})$ holds; this implies that the common spectral PV spectral measure of $\mathbf{Q} = (Q_1, Q_2, Q_3)$ is an imprimitivity system for $U\mid_{\mathcal{E}}$ [9]; we can apply therefore Mackey's imprimitivity theorem. In so doing, to each choice of the representation $L : \mathrm{SO}(3) \to \mathcal{U}(\mathcal{H}_0)$ in Mackey's theorem there corresponds a different theory. Accordingly, the Hilbert space of the theory can be identified as $L_2(\mathbb{R}^3, \mathcal{H}_0)$ and the position operators are $(Q_\alpha \psi)(\mathbf{x}) = x_\alpha \psi(\mathbf{x})$. Furthermore, in a non-relativistic theory, by making use of Galileian invariance, valid for a free particle, it can be proved [9,28] that the form of the Hamiltonian operator must be $H = -(1/2\mu) \sum_{\alpha=1}^{3} (\partial^2/\partial x_\alpha^2)$. By choosing L

as an irreducible σ-representation of SO(3) of dimension $2s+1$ ($s \in \frac{1}{2}\mathbb{N}$), the Standard Quantum Theory of a spin-s free particle is obtained.

(d) The interacting particle

If the system under investigation is not isolated, e.g. if it is an *interacting* particle, then neither SYM.1 nor SYM.2 apply, so that we find an obstacle in extending the group theoretical approach to the non-relativistic interacting particle. However, in the literature several proposals can be found [9,19,22,23] where the group theoretical methods are extended to the interacting case. Each proposal overcomes the aforesaid obstacles by introducing particular assumptions which we can reformulate in the following statement.

PROJ. *Each Galileian transformation g is represented in the formalism of the Quantum Theory by a unitary operator U_g in such a way that*

(i) $S_g^{\Sigma}[\mathbf{Q}^{(t)}] = U_g \mathbf{Q}^{(t)} U_g^{-1}$ *is the quantum transformation of the 'position at time t' observables corresponding to g;*
(ii) *the correspondence $g \to U_g$ is a continuous projective representation.*

Statement PROJ is introduced as an assumption in [9, p. 201]; the conditions assumed by Jauch in [22, p. 236] are implied by PROJ; Ekstein, instead, essentially derives it from another assumption, namely from the 'empirical statement that it is possible to give an operational definition of any initial state intrinsically', i.e. independently of the presence or absence of the interaction [19, p. 1401].

By making use of PROJ, some of the cited approaches [9,22] deduce that, in the non-relativistic Quantum Theory of a spin-0 particle, undergoing an interaction homogeneous in time, the Hamiltonian operator H must have the form below, and hence it can describe interactions with electromagnetic fields [9,22].

$$H = \frac{1}{2\mu}\sum_{\alpha=1}^{3}\left(-\mathrm{i}\frac{\partial}{\partial x_\alpha} + a_\alpha(\mathbf{x})\right)^2 + \Phi(\mathbf{x}). \tag{2.7}$$

On the other hand, we shall prove the following statement.

STAT. *Assumption* PROJ *implies that the Hamiltonian of the Quantum Theory of a spin-0 particle undergoing an interaction homogeneous in time must have the form*

$$H = \frac{1}{2\mu}\sum_{\alpha=1}^{3}\left(-\mathrm{i}\frac{\partial}{\partial x_\alpha}\right)^2 + \Phi(\mathbf{x}).$$

To prove the sentence STAT we shall make use of the following results.

MP.1. The general evolution law of quantum observables with respect to a homogeneous time can be obtained [25] as implication of Wigner's theorem: a self-adjoint operator H exists, called the Hamiltonian operator, such that

$$A^{(t)} = \mathrm{e}^{\mathrm{i}Ht} A\, \mathrm{e}^{-\mathrm{i}Ht} \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}A^{(t)} \equiv \dot{A}^{(t)} = \mathrm{i}[H, A^{(t)}]. \tag{2.8}$$

MP.2. Let $g \to \hat{U}_g$ be every continuous non-trivial projective representation of Galilei group $\mathcal{G}$, i.e. the group generated by the Euclidean group $\mathcal{E}$ and by Galileian velocity boosts. Now, the nine one-parameter Abelian subgroups $\mathcal{T}_\alpha, \mathcal{R}_\alpha, \mathcal{B}_\alpha$ of spatial translations, spatial rotations and Galileian velocity boosts, relative to axis x_α, are all additive; then, from Stone's theorem [25], there exist nine self-adjoint generators $\hat{P}_\alpha, \hat{J}_\alpha, \hat{G}_\alpha$ of the nine one-parameter unitary subgroups $\{\mathrm{e}^{-\mathrm{i}\hat{P}_\alpha a_\alpha}, a \in \mathbb{R}\}$, $\{\mathrm{e}^{-\mathrm{i}\hat{J}_\alpha \theta_\alpha}, \theta_\alpha \in \mathbb{R}\}$, $\{\mathrm{e}^{\mathrm{i}\hat{G}_\alpha u_\alpha}, u_\alpha \in \mathbb{R}\}$ that represent the subgroups $\mathcal{T}_\alpha, \mathcal{R}_\alpha, \mathcal{B}_\alpha$ according to the projective representation $g \to \hat{U}_g$ of Galilei group $\mathcal{G}$. The structural properties of $\mathcal{G}$ as a Lie group imply the following commutation relations [29].

$$\begin{aligned}&\text{(i) } [\hat{P}_\alpha, \hat{P}_\beta] = \mathbb{O}, \quad \text{(ii) } [\hat{J}_\alpha, \hat{P}_\beta] = \mathrm{i}\hat{\epsilon}_{\alpha\beta\gamma}\hat{P}_\gamma, \quad \text{(iii) } [\hat{J}_\alpha, \hat{J}_\beta] = \mathrm{i}\hat{\epsilon}_{\alpha\beta\gamma}\hat{J}_\gamma, \\ &\text{(iv) } [\hat{J}_\alpha, \hat{G}_\beta] = \mathrm{i}\hat{\epsilon}_{\alpha\beta\gamma}\hat{G}_\gamma, \quad \text{(v) } [\hat{G}_\alpha, \hat{G}_\beta] = \mathbb{O} \quad \text{and} \quad \text{(vi) } [\hat{G}_\alpha, \hat{P}_\beta] = \mathrm{i}\delta_{\alpha\beta}\mu\mathbb{1},\end{aligned} \tag{2.9}$$

where $\hat{\epsilon}_{\alpha,\beta,\gamma}$ is the Levi–Civita symbol $\epsilon_{\alpha,\beta,\gamma}$ restricted by the condition $\alpha \neq \gamma \neq \beta$, and μ is a non-zero real number which characterizes the projective representation.

Proof of STAT. Now we explicitly prove STAT. Since $g \to U_g$ in PROJ is a projective representation, according to (**MP.2**) the subgroups $\mathcal{T}_\alpha$, $\mathcal{R}_\alpha$, $\mathcal{B}_\alpha$ can be represented by the one-parameter unitary subgroups $\{\mathrm{e}^{-\mathrm{i}P_\alpha a}\}_{a\in\mathbb{R}}$, $\{\mathrm{e}^{-\mathrm{i}J_\alpha \theta}\}_{\theta\in\mathbb{R}}$, $\{\mathrm{e}^{\mathrm{i}G_\alpha u}\}_{u\in\mathbb{R}}$, in such a way that the self-adjoint generators $P_\alpha, J_\alpha, G_\alpha$ satisfy (2.9). Once the self-adjoint operators $F_\alpha = G_\alpha/\mu$ are defined, it can be shown that relations (2.9) imply that the following relation holds for all $g \in \mathcal{G}$.

$$U_g \mathbf{F} U_g^{-1} = \mathfrak{g}(\mathbf{F}). \tag{2.10}$$

Since by (2.9(v)) the F_α's commute with each other, according to spectral theory, a unique PV measure $E : \mathcal{B}(\mathbb{R}^3) \to \Pi(\mathcal{H})$ exists such that $F_\alpha = \int \lambda\, \mathrm{d}E^{(\alpha)}_\lambda$, where $E^{(1)}_\lambda = E((-\infty, \lambda] \times \mathbb{R}^2)$, $E^{(2)}_\lambda = E(\mathbb{R} \times (-\infty, \lambda] \times \mathbb{R})$, $E^{(3)}_\lambda = E(\mathbb{R}^2 \times (-\infty, \lambda])$. Then (2.10) implies that $\Delta \to E(\Delta)$ satisfies (2.1) and hence it is an *imprimitivity* system for the *restriction* to $\mathcal{E}$ of $g \to U_g$; therefore, Mackey's theorem applies. Thus, the simplest choice for $\mathcal{H}_0$, i.e. $\mathcal{H}_0 = \mathbb{C}$, leads to identify $\mathcal{H}$, F_α, P_α and U_g for $g \in \mathcal{E}$ as

$$\mathcal{H} = L_2(\mathbb{R}^3), \quad (F_\alpha\psi)(\mathbf{x}) = x_\alpha\psi(\mathbf{x}), \quad P_\alpha = -\mathrm{i}\frac{\partial}{\partial x_\alpha} \quad \text{and} \quad (U_g\psi)(\mathbf{x}) = \psi(\mathfrak{g}(\mathbf{x})). \tag{2.11}$$

Now we can demonstrate that the position operators $\mathbf{Q}$ coincide with $\mathbf{F} = \mathbf{G}/\mu$. ■

Proposition 2.3. *If* PROJ *holds, then in the simplest Quantum Theory of a localizable interacting particle equality* $\mathbf{F} = \mathbf{Q}$ *holds for the position operators satisfying the covariance properties* (2.2).

Proof. If $g \in \mathcal{T}_\beta$ and PROJ holds, so that by (**MP.2**) $U_g = \mathrm{e}^{-\mathrm{i}P_\beta a}$, then (2.2(i)) implies $[Q_\alpha, P_\beta] = \mathrm{i}\delta_{\alpha\beta}\mathbb{1}$; since $[F_\alpha, P_\beta] = \mathrm{i}\delta_{\alpha,\beta}\mathbb{1}$ is implied by (2.9(iv)), also $[F_\alpha - Q_\alpha, P_\beta] = \mathbb{O}$ holds. On the other hand, (2.2(i)) for $U_g = \mathrm{e}^{\mathrm{i}G_\beta u}$ implies $[F_\alpha - Q_\alpha, F_\beta] = \mathbb{O}$, and hence $F_\alpha - Q_\alpha = c_\alpha\mathbb{1} \equiv$const. must hold for the irreducibility of (**F**, **P**). Finally, PROJ.i together with (2.9(iv)) and (2.2(i)) for $U_g = \mathrm{e}^{-\mathrm{i}J_\alpha\theta}$ imply $[J_\alpha, F_\beta - Q_\beta] = \mathrm{i}\hat{\epsilon}_{\alpha,\beta,\gamma}(F_\gamma - Q_\gamma) = \mathrm{i}\hat{\epsilon}_{\alpha,\beta,\gamma}c_\gamma\mathbb{1} = [J_\alpha, c_\beta\mathbb{1}] = \mathbb{O}$; thus, $F_\alpha - Q_\alpha = \mathbb{O}$.

Proposition 2.3 together with (2.2(ii)) is sufficient to determine the form of Hamiltonian operator H consistent with PROJ. First, we determine $[G_\alpha, \dot{Q}_\beta]$. Let us start with

$$\mathrm{e}^{\mathrm{i}G_\alpha u}\dot{Q}_\beta\mathrm{e}^{-\mathrm{i}G_\alpha u} = \dot{Q}_\beta + \mathrm{i}[G_\alpha, \dot{Q}_\beta]u + o(u), \tag{2.12}$$

where $o(u)$ is an infinitesimal operator of order greater than 1 with respect to u. By making use of $\dot{Q}_\beta = i[H, Q_\beta] = \lim_{t\to 0}((Q^{(t)}_\beta - Q_\beta)/t)$, and of $\mathrm{e}^{\mathrm{i}G_\alpha u}Q^{(t)}_\beta\,\mathrm{e}^{-\mathrm{i}G_\alpha u} = Q^{(t)}_\beta - \delta_{\alpha\beta}ut\mathbb{1}$, implied by (2.2(ii)), we also find

$$\mathrm{e}^{\mathrm{i}G_\alpha u}\dot{Q}_\beta\mathrm{e}^{-\mathrm{i}G_\alpha u} = \lim_{t\to 0}\mathrm{e}^{\mathrm{i}G_\alpha u}\frac{Q^{(t)}_\beta - Q_\beta}{t}\,\mathrm{e}^{-\mathrm{i}G_\alpha u} = \dot{Q}_\beta - \delta_{\alpha\beta}u\mathbb{1}. \tag{2.13}$$

The comparison between (2.12) and (2.13), and proposition 2.3 yields

$$[G_\alpha, \dot{Q}_\beta] = [Q_\alpha, \mu\dot{Q}_\beta] = \mathrm{i}\delta_{\alpha\beta}\mathbb{1}, \text{ which implies } [F_\alpha, \mu\dot{Q}_\beta - P_\beta] = \mathbb{O}. \tag{2.14}$$

This argument can be repeated with $U_g = \mathrm{e}^{-\mathrm{i}P_\alpha a}$ instead of $\mathrm{e}^{\mathrm{i}G_\alpha u}$, and also with $U_g = \mathrm{e}^{-\mathrm{i}J_\alpha\theta}$ instead of $\mathrm{e}^{\mathrm{i}G_\alpha u}$. In doing so we obtain, respectively, $[P_\alpha, \mu\dot{Q}_\beta - P_\beta] = \mathbb{O}$ and $[J_\alpha, \mu\dot{Q}_\beta] = \mathrm{i}\hat{\epsilon}_{\alpha,\beta,\gamma}\mu\dot{Q}_\gamma$; the first of these two equations, together with (2.14), imply $\mu\dot{Q}_\beta - P_\beta = b_\beta\mathbb{1}$; then, by making use of the second equation, we obtain $\mathrm{i}\hat{\epsilon}_{\alpha\beta\gamma}(\mu\dot{Q}_\gamma - P_\gamma) = [J_\alpha, \mu\dot{Q}_\beta - P_\beta] = [J_\alpha, b_\beta\mathbb{1}] = \mathbb{O}$, i.e. $\mu\dot{Q}_\beta = P_\beta$.

At this point the determination of H is straightforward. From (2.9(vi)), we obtain

$$\mathrm{i}[H, Q_\beta] = \dot{Q}_\beta = \frac{1}{\mu} P_\beta = \mathrm{i}\left[\frac{1}{2\mu}\sum_\gamma P_\gamma^2, \frac{G_\beta}{\mu}\right] \equiv \mathrm{i}\left[\frac{1}{2\mu}\sum_\gamma P_\gamma^2, Q_\beta\right]. \tag{2.15}$$

So the completeness of $\mathbf{Q}$ implies that the operator $H - (1/2\mu)\sum_\gamma P_\gamma^2$ is a function Φ of $\mathbf{Q}$. Thus,

$$H = -\frac{1}{2\mu}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right) + \Phi(\mathbf{Q}). \tag{2.16}$$

Thus, assumption PROJ excludes the description of electromagnetic interactions, because their physics is correctly described by the Hamiltonian in (2.7), inequivalent to (2.16). ■

3. Quantum Theory of an interacting particle

In line with the conclusion of the previous section, to develop a Quantum Theory of a particle able to describe also electromagnetic interactions, assumption PROJ must be abandoned. In this section, we undertake this task, under the hypothesis that the interaction does not destroy time homogeneity, so that according to (**MP.1**) Hamiltonian operator H exists such that (2.8) holds.

We begin by identifying two further properties (S.2) and (S.3) of quantum transformations, which add to the general property (S.1) already established.

(S.2) For every $g \in \Upsilon$, the mapping S_g^Σ is bijective.

(S.3) For every real Borel function f such that if A is a self-adjoint operator, then $B = f(A)$ is a self-adjoint operator too, the following equality holds:

$$f(S_g^\Sigma[A]) = S_g^\Sigma[f(A)]. \tag{3.1}$$

In fact, these further properties are implied by the authentic meaning of quantum transformation expressed by (QT). For instance, with regard to (S.3), one can argue as follows. Let f be any fixed real Borel function such that if A is a self-adjoint operator, then $B = f(A)$ is a self-adjoint operator too. Now, according to Quantum Theory a measurement of the quantum observable $f(A)$ can be performed by measuring A and then transforming the obtained outcome a by the purely mathematical function f into the outcome $b = f(a)$ of $f(A)$. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be the procedures that measure A and $S_g^\Sigma[A]$. Since $\mathcal{M}_1$ and $\mathcal{M}_2$ are indistinguishable with relation to (Σ, Σ_g), transforming the outcomes of both procedures by means of the same function f should not affect the relative indistinguishability of the thus modified procedures. So we should conclude that (3.1) holds.

Hence, the concept (QT) entails the validity of (S.2) and (S.3); for the time being, however, we formulate them as conditions which characterize a class of interactions, for reasons we shall explain in remark 3.11.

In §3a and 3b, we show that the further properties (S.2), (S.3) imply that if the correspondence $g \to S_g^\Sigma$ is continuous according to the metric adopted by Bargmann [27], then for every $g \in \Upsilon$ a *unitary* operator U_g must exists such that

(i) $g \to U_g$ is continuous;

(ii) $S_g^\Sigma[A] = U_g A U_g^{-1}$.

This result addresses one of the difficulties encountered by the extension of the group theoretical approach to an interacting particle; but other obstacles remain. Indeed, in order to identify the mathematical formalism of the theory explicitly we should apply the imprimitivity theorem; but this is not possible because, while the mapping $g \to U_g$ is continuous under a condition of continuity for $g \to S_g^\Sigma$, it is not a projective representation, and such a condition is required by the imprimitivity theorem.

To address this new obstacle we shall introduce in §3c, the notion of 'σ-conversion', which is a consistent mathematical process converting the mapping $U:\Upsilon\to\mathcal{U}(\mathcal{H})$, $g\to U_g$ into a mapping $\hat{U}:\Upsilon\to\hat{\mathcal{U}}(\mathcal{H})$, $g\to\hat{U}_g$ which is a projective representation.

The use of σ-conversions will allow us to proceed. In the non-relativistic case, where $\Upsilon=\mathcal{G}$, we prove that the position operators $\mathbf{Q}$ coincide with the multiplication operators endowed with the usual interpretation if and only if the interaction admits 'Q-covariant' σ-conversions, i.e. σ-conversions that leave unaltered the covariance properties of the position operators $\mathbf{Q}$ with respect to $\mathcal{G}$. For Q-covariant σ-conversions, we derive a general dynamical equation (3.11), in §3d.

(a) General implications for quantum transformations

Conditions (S.2) and (S.3) are sufficient to show further properties of the mappings S_g^Σ, according to the following propositions 3.1 and 3.3.

Proposition 3.1. *Let $S:\Omega(\mathcal{H})\to\Omega(\mathcal{H})$ be a bijective mapping such that $S[f(A)]=f(S[A])$ for every Borel real function f such that $f(A)\in\Omega(\mathcal{H})$ if $A\in\Omega(\mathcal{H})$. Then the following statements hold.*

(i) *If $E\in\Pi(\mathcal{H})$ then $S[E]\in\Pi[\mathcal{H}]$, i.e. the mapping S is an extension of a bijection of $\Pi(\mathcal{H}]$.*
(ii) *If $A,B\in\Omega(\mathcal{H})$ and $A+B\in\Omega(\mathcal{H})$, then $[A,B]=\mathbb{O}$ implies $S[A+B]=S[A]+S[B]$. This partial additivity implies $S[A]=\mathbb{O}$ if and only if $A=\mathbb{O}$.*
(iii) *For all $E,F\in\Pi(\mathcal{H})$, $EF=\mathbb{O}$ implies $S[E+F]=S[E]+S[F]\in\Pi(\mathcal{H})$; as a consequence, $E\le F$ if and only if $S[E]\le S[F]$.*
(iv) *$S[P]\in\Pi_1(\mathcal{H})$ if and only if $P\in\Pi_1(\mathcal{H})$.*

Proof. (i) If $E\in\Pi(\mathcal{H})$ and $f(\lambda)=\lambda^2$ then $f(E)=E$ holds; so $S[f(E)]=f(S[E])$ implies $(S[E])^2\equiv f(S[E])=S[E^2]\equiv S[E]$, i.e. $S^2[E]=S[E]$.
(ii) If $[A,B]=\mathbb{O}$ then a self-adjoint operator C and two functions f_a,f_b exist so that $A=f_a(C)$ and $B=f_b(C)$; once defined the function $f=f_a+f_b$, we have $S[A+B]\equiv S[f(C)]=f(S[C])=f_a(S[C])+f_b(S[C])=S[f_a(C)]+S[f_b(C)]\equiv S[A]+S[B]$.
(iii) If $EF=\mathbb{O}$, then $[E,F]=\mathbb{O}$ and $(E+F)\in\Pi(\mathcal{H})$ hold. Statements (i) and (ii) imply $S[E+F]=S[E]+S[F]\in\Pi(\mathcal{H}]$.
(iv) If $P\in\Pi_1(\mathcal{H})$ then $S[P]\in\Pi(\mathcal{H})$ by (i). If $Q\in\Pi_1(\mathcal{H})$ and $Q\le S[P]$ then $P_0\equiv S^{-1}[Q]\le P$ by (iii); but P is rank 1, therefore $P_0=P$ and $Q=S[P]$.

■

Corollary 3.2. *From proposition 3.1 immediately follows that the restriction of S to $\Pi(\mathcal{H})$ is a bijection that also satisfies $S[\mathbb{O}]=\mathbb{O}$, $S[\mathbb{1}]=\mathbb{1}$, $E\le F$ iff $S[E]\le S[F]$, $S[E^\perp]=(S[E])^\perp$.*

Different equivalent formulations of Wigner's theorem [6,30] have been demonstrated in the literature. The following version can be applied for the the mapping S of propostion 3.1.

Wigner's theorem. If $S:\Pi(\mathcal{H})\to\Pi(\mathcal{H})$ is an automorphism of $\Pi(\mathcal{H})$, i.e. if it is a bijective mapping such that

$$E_1\le E_2\Leftrightarrow S[E_1]\le S[E_2]\quad\text{and}\quad S[E^\perp]=(S[E])^\perp,\quad\forall E_1,E_2,E\in\Pi(\mathcal{H}),$$

then either a unitary operator or an anti-unitary operator U of $\mathcal{H}$ exists such that $S(E)=UEU^*$ for all $E\in\Pi(\mathcal{H})$, unique modulo a phase factor.

In virtue of corollary 3.2 and Wigner's theorem, the following proposition can be easily proved.

Proposition 3.3. *If a mapping S satisfies the hypothesis of proposition 3.1, then a unitary or an anti-unitary operator exists such that $S[A]=UAU^*$ for every $A\in\Omega(\mathcal{H})$; if another unitary or anti-unitary operator V satisfies $S[A]=VAV^*$ for every $A\in\Omega(\mathcal{H})$, then $V=\mathrm{e}^{\mathrm{i}\theta}U$ with $\theta\in\mathbb{R}$.*

Propositions 3.1 and 3.3 are proved for a mapping $S:\Omega(\mathcal{H})\to\Omega(\mathcal{H})$; therefore, they hold for every quantum transformation S_g^Σ of the Quantum Theory of a particle whose interaction is in the class for which (S.2), (S.3) hold. Then, for each $g\in\Upsilon$, according to proposition 3.3

each transformation $g \in \Upsilon$ is assigned a unitary or an anti unitary operator U_g which realizes the corresponding quantum transformation as the automorphism $S_g^\Sigma : \Pi(\mathcal{H}) \to \Pi(\mathcal{H})$, $S_g^\Sigma[A] = U_g A U_g^*$, even if g is not a symmetry transformation.

(b) Continuity and unitarity of $g \longrightarrow U_g$

Given $g \in \Upsilon$, the unitary or anti-unitary operator U_g such that $S_g^\Sigma[A] = U_g A U_g^*$ can be chosen arbitrarily within an equivalence class $\mathbf{U}_g$ of operators, all unitary or all anti-unitary, which differ from each other by a complex phase factor; this class $\mathbf{U}_g$ is called *operator ray* [27]; from Wigner's theorem, there is a bijective correspondence between operator rays and automorphisms of $\Pi(\mathcal{H})$. The possibility that the choice of U_g within $\mathbf{U}_g$ makes the correspondence $g \to U_g$ continuous has a decisive role in developing the Quantum Theory of a physical system; for instance, for the non-relativistic Quantum Theory of a free particle, it makes possible to apply Stone's theorem, and as a consequence the one-parameter subgroups $\mathcal{T}_\alpha$, $\mathcal{R}_\alpha$, $\mathcal{B}_\alpha$ can be represented as $\mathrm{e}^{-\mathrm{i}P_\alpha a}$, $\mathrm{e}^{-\mathrm{i}J_\alpha \theta}$, $\mathrm{e}^{\mathrm{i}G_\alpha u}$, respectively. According to results from Bargmann [27], a choice of U_g in $\mathbf{U}_g$ leading to a continuous correspondence $g \to U_g$ exists if the mapping $g \to S_g^\Sigma$ is *continuous*, where $S_g^\Sigma : \Pi(\mathcal{H}) \to \Pi(\mathcal{H})$ is the restriction to $\Pi(\mathcal{H})$ of the quantum transformation corresponding to g. However, in his proof Bargmann required *all operators U_g to be unitary*. Now we see how Bargmann's result also holds if such a restriction is removed.

The continuity notion of Bargmann[1] for $g \to S_g^\Sigma$ is based on the following metric of $\Pi_1(\mathcal{H})$.

Definition 3.4. Given two rank 1 projection operators $D_1, D_2 \in \Pi_1(\mathcal{H})$, the distance $d(D_1, D_2)$ is the minimal distance $\|\psi_1 - \psi_2\|$ between vectors ψ_1, ψ_2 such that $D_1 = |\psi_1\rangle\langle\psi_1|$ and $D_2 = |\psi_2\rangle\langle\psi_2|$, i.e. $d(D_1, D_2) = [2(1 - |\langle\psi_1 \mid \psi_2\rangle|]^{1/2}$.

Then, following Bargmann, the continuity of a mapping from a topological group G to the automorphisms of $\Pi(\mathcal{H})$, is defined as follows.

Definition 3.5. A correspondence $g \to S_g$ from a topological group G to the set of all automorphisms of $\Pi(\mathcal{H})$ is said to be continuous if for any fixed $D \in \Pi_1(\mathcal{H})$ the mapping from G to $\Pi_1(\mathcal{H})$, $g \to S_g[D]$ is continuous in g with respect to the distance d defined on $\Pi_1(\mathcal{H})$ by definition 3.4

Before proving the main result proposition 3.9, we formulate three lemmas. The first, lemma 3.6, was proved by Bargmann as lemma 1.1 in [27].

Lemma 3.6. *The real function $\kappa : \Pi_1(\mathcal{H}) \times \Pi_1(\mathcal{H}) \to \mathbb{R}$, $\kappa(D_1, D_2) = \mathrm{Tr}(D_1 D_2)$ is continuous in both variables D_1 and D_2 with respect to the metric of definition* 3.4.

Lemma 3.7. *Given a topological group G and a mapping $g \to S_g$ from G to the automorphisms of $\Pi(\mathcal{H})$, for every $g \in G$ let $\mathbf{U}_g$ denote the operator ray identified by S_g; for every $\varphi \in \mathcal{H}$ with $\|\varphi\| = 1$, let us define*

$$z_{h,g}(\varphi) = U_g\varphi - \langle U_h\varphi \mid U_g\varphi\rangle U_h\varphi,$$

where $h, g \in G$, $U_h \in \mathbf{U}_h$ and $U_g \in \mathbf{U}_g$. Then

$$\|z_{h,g}(\varphi)\|^2 = 1 - |\langle U_h\varphi \mid U_g\varphi\rangle|^2 \le d^2(S_h[D_\varphi], S_g[D_\varphi]);$$

where $D_\varphi = |\varphi\rangle\langle\varphi| \in \Pi_1(\mathcal{H})$.

Proof. The proof is identical to the proof of statement (1.9) in theorem 1.1 of [27]; indeed that proof can be successfully demonstrated independently of the unitary or anti-unitary character of U_g or U_h. ■

[1]In fact Bargmann's continuity refers to a correspondence $g \to \mathbf{U}_g$ from a topological group G to the set of all *unitary* operator rays $\mathbf{U}_g$; but, since an operator ray can be bijectively identified with an automorphism of $\Pi(\mathcal{H})$, Bargmann's continuity can be reformulated in terms of automorphisms; this reformulation immediately extends to all automorphisms, including those corresponding to anti-unitary operator rays, through our definition 3.5.

Lemma 3.8. *Let G be a topological group, let $g \to S_g$ be a continuous mapping from G to the automorphisms of $\Pi(\mathcal{H})$, and let us fix an operator $U_g \in \boldsymbol{U}_g$ for each $g \in G$.*

If $U_g\varphi_0$ is continuous in g as a function from G to $\mathcal{H}$ for a vector $\varphi_0 \in \mathcal{H}$ with $\|\varphi_0\| = 1$, then $U_g\varphi_1$ is continuous in g for every fixed $\varphi_1 \in \mathcal{H}$ with $\|\varphi_1\| = 1$, such that $\varphi_1 \perp \varphi_0$.

Proof. We prove the lemma by adapting a part of the proof of theorem 1.1 in [27]. Let us define $\varphi = \left(1/\sqrt{2}\right)(\varphi_0 + \varphi_1)$; so that $\langle U_g\varphi_0 \mid U_g\varphi\rangle = 1/\sqrt{2}$ for all $g \in G$ independently of the unitary or anti-unitary character of $U_g \in \mathbf{U}_g$. Then

$$\begin{aligned}\langle U_h\varphi_0 \mid z_{h,g}(\varphi)\rangle &= \langle U_h\varphi_0 - U_g\varphi_0 \mid U_g\varphi\rangle + \langle U_g\varphi_0 \mid U_g\varphi\rangle - \langle U_h\varphi \mid U_g\varphi\rangle\langle U_h\varphi_0 \mid U_h\varphi\rangle \\ &= \langle U_h\varphi_0 - U_g\varphi_0 \mid U_g\varphi\rangle + \frac{1}{\sqrt{2}}(1 - \langle U_h\varphi \mid U_g\varphi\rangle.\end{aligned}$$

So

$$(1 - \langle U_h\varphi \mid U_g\varphi\rangle) = \sqrt{2}\{\langle U_h\varphi_0 \mid z_{h,g}(\varphi)\rangle + \langle U_g\varphi_0 - U_h\varphi_0 \mid U_g\varphi\rangle\}. \tag{3.2}$$

Now,

$$\begin{aligned}\|U_g\varphi - U_h\varphi\|^2 &= 2|\mathbb{R}\mathrm{e}(1 - \langle U_h\varphi \mid U_g\varphi\rangle)| \le 2|1 - \langle U_h\varphi \mid U_g\varphi\rangle| \\ &\le 2\sqrt{2}\{|\langle U_h\varphi_0 \mid z_{h,g}(\varphi)\rangle| + 2\sqrt{2}|U_g\varphi_0 - U_h\varphi_0 \mid U_g\varphi\rangle|\} \\ &\le 2\sqrt{2}\| \mid z_{h,g}(\varphi)\| + 2\sqrt{2}\|U_g\varphi_0 - U_h\varphi_0\| \\ &\le 2\sqrt{2}(d(S_h[D_\varphi], S_g[D_\varphi]) + \|U_g\varphi_0 - U_h\varphi_0\|),\end{aligned}$$

where we made use of (3.2) in the second inequality, in the third inequality we use the Schwarz inequality, and in the fourth inequality lemma 3.7 is applied. These inequalities imply that $U_g\varphi$ is continuous in g; indeed, the distance $d(S_h[D_\varphi], S_g[D_\varphi])$ vanishes as $g \to h$ because the mapping $g \to S_g$ is continuous according to definition 3.5 from the first continuity hypothesis; but also $\|U_g\varphi_0 - U_h\varphi_0\|$ vanishes as $g \to h$, because $U_g\varphi_0$ is continuous in g by the second continuity hypothesis.

Now, $\varphi_1 = \sqrt{2}\varphi - \varphi_0$, so that $U_g\varphi_1 = \sqrt{2}U_g\varphi - U_g\varphi_0$ for all g such that U_g is unitary, but also for all g such that U_g is anti-unitary. Thus $U_g\varphi_1$ is continuous because $U_g\varphi$ and $U_g\varphi_1$ are continuous. ∎

Let us arbitrarily fix a vector $\varphi_0 \in \mathcal{H}$, with $\|\varphi_0\| = 1$. Given any mapping $g \to S_g$ from a topological group G to the automorphisms of $\Pi(\mathcal{H})$, we define the real function $\rho_{\varphi_0} : G \to \mathbb{R}$, $\rho_{\varphi_0}(g) = Tr^{1/2}(D_{\varphi_0}S_g[D_{\varphi_0}])$. Since $S_g[D_{\varphi_0}] = \tilde{U}_g D_{\varphi_0}\tilde{U}_g^*$, where $\tilde{U}_g$ is any operator in $\mathbf{U}_g$, we have $\rho_{\varphi_0}(g) = |\langle\varphi_0 \mid \tilde{U}_g\varphi_0\rangle|$. Hence, $\langle\varphi_0 \mid \tilde{U}_g\varphi_0\rangle = \mathrm{e}^{\mathrm{i}\alpha(g)}|\langle\varphi_0 \mid \tilde{U}_g\varphi_0\rangle| = \mathrm{e}^{\mathrm{i}\alpha(g)}\rho_{\varphi_0}(g)$, for some $\alpha(g) \in \mathbb{R}$. Then $\rho_{\varphi_0}(g) = |\langle\varphi_0 \mid \tilde{U}_g\varphi_0\rangle| = \mathrm{e}^{-\mathrm{i}\alpha(g)}\langle\varphi_0 \mid \tilde{U}_g\varphi_0\rangle$. Therefore, if for each $g \in G$ we choose $U_g = \mathrm{e}^{-\mathrm{i}\alpha(g)}\tilde{U}_g$ we obtain

$$\rho_{\varphi_0}(g) = \langle\varphi_0 \mid U_g\varphi_0\rangle; \text{ in particular, } U_e = \mathbb{1}. \tag{3.3}$$

Proposition 3.9. *Let G be a topological group, and let φ_0 be any fixed vector in $\mathcal{H}$ with $\|\varphi_0\| = 1$. Given a continuous mapping $g \to S_g$ from G to the automorphisms of $\Pi(\mathcal{H})$, if each $g \in G$ is assigned the operator $U_g \in \boldsymbol{U}_g$ such that* (3.3) *holds, then $U_g\psi$ is continuous in g, whatever the vector $\psi \in \mathcal{H}$.*

Proof. Bargmann showed that if $g \to S_g$ is continuous according to definition 3.5 and if U_g is the operator such that (3.3) holds, then $U_g\varphi_0$ is continuous.[2] Now, let ψ be any vector of $\mathcal{H}$.

If $\psi = 0$, then the continuity of $U_g\psi$ is obvious. It is sufficient, therefore, to prove the proposition for $\psi \neq 0$.

If $\psi = \lambda\varphi_0$ for $\lambda \in \mathbb{C} \setminus \{0\}$, then we can choose any $\varphi_1 \perp \varphi_0$, with $\|\varphi_1\| = 1$. According to lemma 3.8, $U_g\varphi_1$ is continuous. The same lemma implies that $U_g(\psi/\|\psi|)$ is continuous because $(\psi/\|\psi\|) \perp \varphi_1$. But $U_g\psi = \|\psi\|U_g(\psi/\|\psi\|$ for all $g \in G$. Therefore, $U_g\psi$ is continuous.

[2]In fact, Bargmann proved this statement for unitary U_g; but Bargmann's proof can be successfully carried out without assuming that all U_g are unitary.

If $\psi \neq \lambda\varphi_0$, define $\varphi = \psi/\|\psi\|$; then a vector $\varphi_1 \in \mathcal{H}$ exists, with $\|\varphi_1\| = 1$ and $\varphi_1 \perp \varphi_0$, such that

$$\varphi = a\varphi_0 + b\varphi_1 \quad \text{where } a \in \mathbb{C} \text{ but } b \in \mathbb{R}. \tag{3.4}$$

Now, a real number r and a vector φ_2, with $\|\varphi_2\| = 1$ exist such that $a\varphi_0 = r\varphi_2$; this implies $\varphi_2 \perp \varphi_1$ and $\varphi = r\varphi_2 + b\varphi_1$. Lemma 3.8 implies that $U_g\varphi_1$ is continuous because $\varphi_1 \perp \varphi_0$; but the same lemma implies that $U_g\varphi_2$ is continuous too, because $\varphi_2 \perp \varphi_1$. Therefore, since r and b are real numbers, $U_g\varphi = rU_g\varphi_2 + bU_g\varphi_1$ is continuous in g. Thus, $U_g\psi = \|\psi\|U_g\varphi$ is continuous too. ■

Another condition with useful implications is the unitary character of the operators U_g that realize the quantum transformations according to $S_g^\Sigma[A] = U_g A U_g^{-1}$. If the correspondence $g \to S_g^\Sigma$ satisfied $S_{g_1 g_2}^\Sigma = S_{g_1}^\Sigma \circ S_{g_2}^\Sigma$ so that $g \to U_g$ would be a projective representation, then it can be easily proved, from [7,9,25,27], that every U_g must be unitary. But in the presence of interaction $S_g^{\Sigma_1}$ can be different from $S_g^{\Sigma_2}$, so that only the more general statement (S.1) holds, and hence the unitary character of U_g cannot be implied from the cited proofs. Now we shall show that anti-unitary U_g can be excluded under the only hypothesis that the correspondence $g \to S_g^\Sigma$ is continuous according to definition 3.5.

Proposition 3.10. *If the mapping $g \to S_g^\Sigma$, that assigns each $g \in \Upsilon$ the quantum transformation of* (2.3)*, is continuous according to definition* 3.5*, then every operator U_g such that $S_g^\Sigma[A] = U_g A U_g^*$ for all $A \in \Omega(\mathcal{H})$ is unitary.*

Proof. According to proposition 3.9, for every $g \in \Upsilon$ a unitary or anti-unitary operator such that $S_g^\Sigma[A] = U_g A U_g^*$ exists which makes $U_g\psi$ continuous in g for all ψ. According to (3.3) $U_e = \mathbb{1}$ which is unitary. Hence, because of the continuity of $g \to U_g\psi$ for all ψ, a maximal-neighbourhood K_e of e must exist in Υ such that U_g is unitary for all $g \in K_e$; otherwise, a sequence $g_n \to e$ would exist with U_{g_n} anti-unitary, so that $\langle \psi \mid \varphi \rangle = \langle U_{g_n}\varphi \mid U_{g_n}\psi \rangle$ for all $\psi, \varphi \in \mathcal{H}$, and then $\langle \psi \mid \varphi \rangle = \lim_{n\to\infty} \langle U_{g_n}\varphi \mid U_{g_n}\psi \rangle = \langle U_e\varphi \mid U_e\psi \rangle = \langle \varphi \mid \psi \rangle$. This last equality cannot hold for all $\psi, \varphi \in \mathcal{H}$ unless $\mathcal{H}$ is real.

Now we prove that neighbourhood K_e has no boundary, and since Υ is a connected group, $K_e = \Upsilon$. If $g_0 \in \partial K_e$, two sequences $g_n \to g_0$ and $h_n \to g_0$ would exist with U_{g_n} unitary and U_{h_n} anti-unitary; therefore, the continuity of U_g would imply that U_{g_0} should simultaneously be unitary and anti-unitary. ■

Remark 3.11. The work of this subsection has shown that (S.2) and (S.3) imply that $S_g^\Sigma[A] = U_g A U_g^{-1}$, where U_g is unitary if $g \to S_g^\Sigma$ is continuous; as a consequence, the spectrum of any quantum observable is left unchanged by S_g^Σ. Now, for every translation $\mathbf{x} \to \mathbf{x} - \mathbf{a}$ we have $U_g \mathbf{Q} U_g^{-1} = S_g^\Sigma[\mathbf{Q}] = \mathbf{Q} - \mathbf{a}$. As a consequence the spectrum of $\mathbf{Q}$ must be the whole $\mathbb{R}^3$ because of the invariance of the spectrum. Hence the notion (QT) of quantum transformation satisfying (S.1)–(S.3) is inconsistent with some non-trivial topologies of the configurations manifold of the particle investigated in [4] and references therein. From another stand point, an interaction that spatially confines is inconsistent with (QT). Thus, for the time being we establish (S.1)–(S.3) as conditions which characterize the class of interactions investigated in this work. In the following, we shall see that such a class is large, enough, in particular, to encompass electromagnetic interaction.

(c) σ-Conversions

In §3b, we established, under a continuity condition for $g \to S_g^\Sigma$, that in the Quantum Theory of a physical system, even if it is not isolated, a continuous correspondence $U : \Upsilon \to \mathcal{U}(\mathcal{H})$ exists such that $S_g^\Sigma[A] = U_g A U_g^{-1}$. To assume that such a correspondence is a projective representation implies PROJ; therefore, according to §2d, the resulting theory is unable to describe particles interacting with electromagnetic fields. So, we must relinquish this condition in order to develop an empirically more adequate Quantum Theory of an interacting particle. But without such a 'projectivity' Mackey's imprimitivity theorem does not apply. Hence, the development of our group-theoretical approach encounters a further obstacle. We shall now address this obstacle.

The correspondence $g \to U_g$, can be *converted* into a continuous σ-representation if we multiply each operator U_g by a suitable unitary operator V_g of $\mathcal{H}$; namely, V_g is a unitary operator such that the correspondence $g \to \hat{U}_g = V_g U_g$ turns out to be a σ-representation. The transition from the correspondence $\{g \to U_g\}$ to $\{g \to \hat{U}_g = V_g U_g\}$ will be called *σ-conversion*; the mapping $V : \Upsilon \to \mathcal{U}(\mathcal{H})$, $g \to V_g$ that realizes the σ-conversion will be called *σ-conversion mapping*. If $g \to V_g$ is a σ-conversion mapping for $g \to U_g$ and $\theta : \Upsilon \to \mathbb{R}$ is a real function, then also $g \to e^{i\theta(g)} V_g$ is a σ-conversion mapping, provided that $e^{i\theta(e)} = 1$. In any case, $V_e = \mathbb{1}$ must hold.

Since non-trivial projective representations of Υ exist, we can state the following proposition.

Proposition 3.12. *A correspondence $V : \Upsilon \to \mathcal{U}(\mathcal{H})$ always exists such that $\hat{U} : \Upsilon \to \mathcal{U}(\mathcal{H})$, $g \to \hat{U}_g = V_g U_g$ is a non-trivial projective representation.*

The σ-conversion allows us to immediately identify a mathematical formalism for the Quantum Theory of the system, also in the case where the system is not isolated. In the case of a non-relativistic theory, where $\Upsilon = \mathcal{G}$, if $g \to V_g$ is a σ-conversion mapping for U_g then, according to (**MP.2**) in §2d, the σ-representation $g \to \hat{U}_g = V_g U_g$ has nine Hermitian generators $\hat{P}_\alpha$, $\hat{J}_\alpha$, $\hat{G}_\alpha$ for which (2.9) hold. Then, following the argument of the proof of STAT in §2d, the common spectral measure of the triple $\mathbf{F} = \hat{\mathbf{G}}/\mu$ turns out to be an imprimitivity system for the restriction of $g \to \hat{U}_g$ to $\mathcal{E}$. So, by applying Mackey's imprimitivity theorem [9], we can explicitly identify $\mathcal{H}$ as $L_2(\mathbb{R}^3, \mathcal{H}_0)$, modulo unitary isomorphisms, where the operators F_α, $\hat{P}_\alpha$, $\hat{J}_\alpha$ and $\hat{G}_\alpha$ are explicitly specified according to

$$\left.\begin{aligned} &\mathcal{H} = L_2(\mathbb{R}^3, \mathcal{H}_0), \quad (F_\alpha \psi)(\mathbf{x}) = x_\alpha \psi(\mathbf{x}), \quad \hat{P}_\alpha = -i\frac{\partial}{\partial x_\alpha} \\ \text{and} \qquad &\hat{J}_\alpha = F_\beta \hat{P}_\gamma - F_\gamma \hat{P}_\beta + S_\alpha, \quad \hat{G}_\alpha = \mu F_\alpha. \end{aligned}\right\} \tag{3.5}$$

Here (α, β, γ) is a cyclic permutation of (1,2,3); the S_α are operators that act on $\mathcal{H}_0$ only, i.e. their action is $(S_\alpha \psi)(\mathbf{x}) = \hat{s}_\alpha \psi(\mathbf{x})$ where the $\hat{s}_\alpha$ are self-adjoint operators of $\mathcal{H}_0$ which form a representation of the commutation rules $[\hat{s}_\alpha, \hat{s}_\beta] = i\hat{\epsilon}_{\alpha\beta\gamma}\hat{s}_\gamma$. Since the reducibility of the inducing representation $L : \mathrm{SO}(3) \to \mathcal{U}(\mathcal{H}_0)$ implies the reducibility of $\hat{U} : \mathcal{G} \to \mathcal{U}(\mathcal{H})$, if $\hat{U}$ is irreducible then also $(\hat{s}_1, \hat{s}_2, \hat{s}_3)$ must be an irreducible representation of $[\hat{s}_\alpha, \hat{s}_\beta] = i\hat{\epsilon}_{\alpha\beta\gamma}\hat{s}_\gamma$; in this case, modulo unitary isomorphisms, $\mathcal{H}_0$ is one of the finite-dimensional Hilbert spaces $\mathbb{C}^{2s+1}$, with $s \in \frac{1}{2}\mathbb{N}$: the $\hat{s}_\alpha$ are the familiar spin operators.

Hence, the mathematical formalism of the Quantum Theory of a localizable particle has been explicitly identified. However, the operators $\hat{U}_g$ concretely identified *are not* the unitary operators which realize the quantum transformations: given $g \in \mathcal{G}$, in general $S_g^\Sigma[A] = \hat{U}_g A \hat{U}_g^{-1}$ does not hold. As a consequence the operators $\mathbf{Q} = (Q_1, Q_2, Q_3)$ representing the position cannot be identified as illustrated in §2c or the argument of the proof of proposition 2.3. So, our explicit realization of the mathematical formalism of the theory would, in general, be devoid of physical significance.

Two tasks have to be accomplished for the formalism established by (3.5) to become the mathematical formalism of the effective Quantum Theory of an interacting particle.

First, the operators $\mathbf{Q}$ of the Hilbert space $\mathcal{H} = L_2(\mathbb{R}^3, \mathcal{H}_0)$ in (3.5), which represent the position of the particle, should be explicitly determined. We shall address this task in §3d.

Second, the wave equation ruling over the time evolution should be determined. In §3d, we derive a general dynamical law. Specific wave equations corresponding to specific features of the interaction are determined in §4.

(d) Q-covariant σ-conversions

The position operators $\mathbf{Q}$ can be determined for those interactions that have the particular feature of admitting a σ-conversion $U_g \to \hat{U}_g = V_g U_g$ that *leaves unaltered the covariance properties of the* *position operators* $\mathbf{Q}$, i.e. such that

$$\hat{U}_g \mathbf{Q} \hat{U}_g^{-1} = \mathfrak{g}(\mathbf{Q}), \quad \forall g \in \mathcal{G}. \tag{3.6}$$

A σ-conversion satisfying (3.6) is said to be *Q-covariant*. Indeed, the following proposition holds.

Proposition 3.13. *If a σ-conversion for a particle yields an irreducible projective representation $\hat{U}$, then it is a Q-covariant σ-conversion if and only if the position operators $\mathbf{Q}$ coincide with $\mathbf{F}$.*

Proof. If $\mathbf{Q} = \mathbf{F} = \hat{\mathbf{G}}/\mu$, then (2.9) and (3.6) imply $\hat{U}_g \mathbf{Q} \hat{U}_g^{-1} \equiv \hat{U}_g \mathbf{F} \hat{U}_g^{-1} = \mathfrak{g}(\mathbf{F}) = \mathfrak{g}(\mathbf{Q})$.

Conversely, if $\hat{U} : \mathcal{G} \to \mathcal{U}(\mathcal{H})$ is an irreducible projective representation obtained from $U : \mathcal{G} \to \mathcal{U}(\mathcal{H})$ through a Q-covariant σ-conversion, then (3.6) for $\hat{U}_g = \mathrm{e}^{\mathrm{i}\hat{G}_\beta u} = \mathrm{e}^{\mathrm{i}\mu F_\beta u}$ and (2.9(v)) imply $[Q_\alpha - F_\alpha, F_\beta] = [Q_\alpha, F_\beta] - [F_\alpha, F_\beta] = \mathbb{O} - \mathbb{O} = \mathbb{O}$; therefore $(Q_\alpha - F_\alpha)\psi(\mathbf{x}) = (f_\alpha(\mathbf{Q})\psi)(\mathbf{x}) = f_\alpha(\mathbf{x})\psi(\mathbf{x})$, where $f_\alpha(\mathbf{x})$ is a self-adjoint operator of $\mathcal{H}_0$. However, the Q-covariance and (2.9(vi)) also imply $[Q_\alpha - F_\alpha, \hat{P}_\beta] = [Q_\alpha, \hat{P}_\beta] - [Q_\alpha, \hat{P}_\beta] = \mathrm{i}\delta_{\alpha\beta}\mathbb{1} - \mathrm{i}\delta_{\alpha\beta}\mathbb{1} = \mathbf{0}$, i.e. $[f_\alpha(\mathbf{Q}), \hat{P}_\beta] = \mathbf{0}$ for all $\mathbf{x}$; this relation, since $\hat{P} = -\mathrm{i}(\partial/\partial x_\alpha)$, implies that $(\partial f_\alpha/\partial x_\alpha)(\mathbf{x}) = 0$, for all α, β; therefore $f_\alpha(\mathbf{x})$ is an operator $\hat{f}_\alpha$ of $\mathcal{H}_0$ which *does not depend* on $\mathbf{x}$. Now, since $\hat{f}_\alpha = Q_\alpha - F_\alpha$, also $[\hat{f}_\alpha, \hat{f}_\beta] = \mathbf{0}$ holds; moreover, from (2.2(i)) for a pure spatial rotation g about x_α and from (2.9(iv)) we obtain $[\hat{J}_\alpha, Q_\beta - F_\beta] = \mathrm{i}\hat{\epsilon}_{\alpha\beta\gamma}(Q_\gamma - F_\gamma) = \mathrm{i}\hat{\epsilon}_{\alpha\beta\gamma}\hat{f}_\gamma$; but the irreducibility of $\hat{U}$ implies the irreducibility of the inducing projective representation $L : \mathrm{SO}(3) \to \mathcal{U}(\mathcal{H}_0)$, so that $\mathcal{H}_0$ is finite dimensional; then $[\hat{f}_\alpha, \hat{f}_\beta] = \mathbf{0}$ and $[\hat{J}_\alpha, \hat{f}_\beta] = i\hat{\epsilon}_{\alpha\beta\gamma}\hat{f}_\gamma$ can hold only if $\hat{f}_\alpha = \mathbf{0}$, i.e. $F_\alpha = Q_\alpha$. ■

Hence, in the Quantum Theory of an interacting particle, where $\hat{U}$ is irreducible, the multiplication operators can be identified with the position operators if and only if the interaction has the particular regularity feature of admitting a σ-conversion which preserves the covariance properties of the position operators.

Extending a standard practice we say that a particle, whose interaction admits Q-covariant σ-conversion, is *elementary* if $\hat{U}$ is irreducible.

The following proposition specifies how in the Quantum Theory of an elementary particle each $\hat{U}_g$ is related to the unitary operator U_g that realizes the quantum transformation corresponding to g.

Proposition 3.14. *For every $g \in \mathcal{G}$, the operator V_g of a Q-covariant σ-conversion has the form $(V_g\psi)(\mathbf{x}) = (\mathrm{e}^{\mathrm{i}\theta(g,\mathbf{Q})}\psi)(\mathbf{x}) = \mathrm{e}^{\mathrm{i}\theta(g,\mathbf{x})}\psi(\mathbf{x})$, where $\theta(g, \mathbf{x})$ is a self-adjoint operator of $\mathcal{H}_0$ which depends on $\mathbf{x}$ and on g.*

Proof. Relations (3.6) and (2.2) imply $V_g U_g \mathbf{Q} U_g^{-1} V_g^{-1} = \mathfrak{g}(\mathbf{Q})$, which implies $V_g(\mathfrak{g}(\mathbf{Q}))V_g^{-1} = \mathfrak{g}(\mathbf{Q})$, i.e. $[V_g, \mathfrak{g}(\mathbf{Q})] = \mathbb{O}$. Then $[V_g, \mathsf{f}(\mathfrak{g}(\mathbf{Q}))] = \mathbb{O}$ for every sufficiently regular function f; by taking $\mathsf{f} = \mathfrak{g}^{-1}$ we have $[V_g, \mathbf{Q}] = \mathbf{0}$. Then $(V_g\psi)(\mathbf{x}) = \mathsf{h}_g(\mathbf{x})\psi(\mathbf{x})$, where $\mathsf{h}_g(\mathbf{x})$ is an operator of $\mathcal{H}_0$. Finally, the unitary character of V_g imposes that $h_g(\mathbf{x})$ must be unitary as an operator of $\mathcal{H}_0$; thus a self-adjoint operator $\theta(g, \mathbf{x})$ of $\mathcal{H}_0$ exists such that $\mathsf{h}_g(\mathbf{x}) = \mathrm{e}^{\mathrm{i}\theta(g,\mathbf{x})}$. ■

If $g \to S_g^\Sigma$ is continuous according to definition 3.5, then $g \to V_g$ must be continuous because $g \to \hat{U}_g = V_g U_g$ is continuous.

Remark 3.15. In the present approach the imprimitivity system for applying Mackey's theorem is identified within the abstract projective representation itself, namely it is the PV spectral measure of $\hat{\mathbf{G}}/\mu$. This is remarkably different from previous approaches, e.g. Mackey's, where the imprimitivity system is identified as the PV measure of the position operators.

(e) General dynamical law

We now derive a general dynamical equation governing the time evolution of an elementary particle, under the condition that the σ-conversion mapping $g \to V_g$ is differentiable with respect to the parameters $a_\alpha, \theta_\alpha, u_\alpha$ of the group $\mathcal{G}$.

Let us consider the pure velocity boost $g \in \mathcal{G}$ such that $\hat{U}_g = \mathrm{e}^{\mathrm{i}\hat{G}_\alpha u}$. According to §3c, the formalism of its Quantum Theory can be identified with that established by (3.5). Since $\hat{G}_\alpha =$

$\mu F_\alpha = \mu Q_\alpha$, we can write $\hat{U}_g = \mathrm{e}^{\mathrm{i}\mu Q_\alpha u}$; therefore

$$\hat{U}_g \dot{Q}_\beta \hat{U}_g^{-1} = \dot{Q}_\beta + i\mu[Q_\alpha, \dot{Q}_\beta]u + o_1(u). \tag{3.7}$$

On the other hand,

$$\hat{U}_g \dot{Q}_\beta \hat{U}_g^{-1} = \lim_{t\to 0} V_g U_g \frac{(Q_\beta^{(t)} - Q_\beta)}{t} U_g^{-1} V_g^{-1}. \tag{3.8}$$

By making use of $U_g Q_\beta^{(t)} U_g^{-1} = Q_\beta^{(t)} - \delta_{\alpha\beta} ut\mathbb{1}$, implied by (2.2), and of proposition 3.3, Proposition 3.9 in (3.8), and then comparing it with (3.7) we obtain

$$\hat{U}_g \dot{Q}_\beta \hat{U}_g^{-1} = V_g \dot{Q}_\beta V_g^{-1} - \delta_{\alpha\beta} u\mathbb{1} = \dot{Q}_\beta + \mathrm{i}\mu[Q_\alpha, \dot{Q}_\beta]u + o_1(u). \tag{3.9}$$

But proposition 3.14 implies that $V_g = \mathrm{e}^{\mathrm{i}\varsigma_\alpha(u,\mathbf{Q})}$, where $\varsigma_\alpha(u, \mathbf{x})$ is a self-adjoint operator of $\mathcal{H}_0$; replacing in (3.9) we obtain

$$\dot{Q}_\beta + \mathrm{i}[\varsigma_\alpha(u, \mathbf{Q}), \dot{Q}_\beta] + o_2(u) - \delta_{\alpha\beta} u\mathbb{1} = \dot{Q}_\beta + \mathrm{i}\mu[Q_\alpha, \dot{Q}_\beta]u + o_1(u). \tag{3.10}$$

Since $\mathrm{e}^{\mathrm{i}}\varsigma_\alpha(0, \mathbf{Q}) = \mathbb{1}$, the expansion of ς_α with respect to u yields $\varsigma_\alpha(u, \mathbf{Q}) = (\partial\varsigma_\alpha/\partial u)(0, \mathbf{Q})u + o_3(u)$; by replacing the latter relation in (3.10) we obtain

$$\mu[Q_\alpha, \dot{Q}_\beta] = [\eta_\alpha(\mathbf{Q}), \dot{Q}_\beta] + \mathrm{i}\delta_{\alpha\beta}\mathbb{1},$$

where $\eta_\alpha(\mathbf{Q}) = (\partial\varsigma_\alpha/\partial u)(0, \mathbf{Q})$. By replacing $\dot{Q}_\beta = \mathrm{i}[H, Q_\beta]$ in this equation we can apply Jacobi's identity, and in so doing we obtain $[Q_\beta, \mu\dot{Q}_\alpha] = [Q_\beta, \dot{\eta}_\alpha(\mathbf{Q})] + \mathrm{i}\delta_{\alpha\beta}\mathbb{1}$, i.e.

$$[Q_\beta, \dot{\eta}_\alpha(\mathbf{Q}) - \mu\dot{Q}_\alpha] = -\mathrm{i}\delta_{\alpha\beta}\mathbb{1} = [Q_\beta, -\hat{P}_\alpha].$$

Hence $[\dot{\eta}_\alpha(\mathbf{Q}) - \mu\dot{Q}_\alpha - \hat{P}_\alpha, Q_\beta] = \mathbf{0}$, from which it follows that for every $\mathbf{x} \in \mathbb{R}^3$ an operator $f_\alpha(\mathbf{x})$ of $\mathcal{H}_0$, must exist such that the equation $\{\dot{\eta}(\mathbf{Q}) - \mu\dot{Q}_\alpha + \hat{P}_\alpha\}\psi(\mathbf{x}) = f_\alpha(\mathbf{x})\psi(\mathbf{x})$ holds, which can be rewritten as

$$\mathrm{i}[H, \mu Q_\alpha - \eta_\alpha(\mathbf{Q})] = \hat{P}_\alpha - f_\alpha(\mathbf{Q}). \tag{3.11}$$

This is a general dynamical equation for a localizable particle whose interaction admits Q-covariant σ-conversions; according to this law, the effects of the interaction on the dynamics are encoded in the six 'fields' η_α, f_α.

(f) Electromagnetic interaction for spin-0 particles

Once the general dynamical law (3.11) for an elementary particle with homogeneous in time interaction has been derived, it is worth reviewing the wave equation currently adopted in quantum physics as a particular case of the general equation (3.11). In this subsection we do this for a spin-0 particle, for which $\mathcal{H}_0 = \mathbb{C}$ so that $\mathcal{H} = L_2(\mathbb{R}^3)$. The currently adopted Schroedinger equation for a spin-0 particle, obtained via the minimal electromagnetic coupling or canonical quantization, has the form

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\psi_t = \left\{\frac{1}{2m}\sum_{\alpha=1}^{3}[\hat{P}_\alpha + a_\alpha(\mathbf{Q})]^2 + \Phi(\mathbf{Q})\right\}\psi_t, \tag{3.12}$$

i.e. the Hamiltonian operator is $H = (1/2\mu)\sum_{\alpha=1}^{3}\{\hat{P}_\alpha + a_\alpha(\mathbf{Q})\}^2 + \Phi(\mathbf{Q})$, where $a_\alpha(\mathbf{Q})$ and $\Phi(\mathbf{Q})$ are self-adjoint operators of $L_2(\mathbb{R}^3)$ functions of $\mathbf{Q}$. Now we show that within our approach this specific Quantum Theory bi-univocally corresponds to the case where the functions η_α in the general law (3.11) are constant functions multiple of $\mathbb{1}$.

Proposition 3.16. *The Hamiltonian operator H of an interacting spin-0 particle which admits Q-covariant σ-conversion has the form $H = (1/2\mu)\sum_{\alpha=1}^{3}\{\hat{P}_\alpha + a_\alpha(\mathbf{Q})\}^2 + \Phi(\mathbf{Q})$ if and only if the functions η_α in (3.11) are constant functions. In this case $a_\alpha = -f_\alpha$.*

Proof. If η_α is a constant function, then (3.11) becomes $i[H,\mu Q_\alpha]=\hat{P}_\alpha - f_\alpha(\mathbf{Q})$ which holds if $H_0=(1/2\mu)\sum_{\alpha=1}^{3}\{\hat{P}_\alpha - f_\alpha(\mathbf{Q})\}^2$ replaces H. Hence the operator $H-H_0$ must be a function Φ of $\mathbf{Q}$ because of the completeness of $\mathbf{Q}$. Then $\eta_\alpha(\mathbf{Q})=c_\alpha\mathbb{1}$ implies $H=(1/2\mu)\sum_{\alpha=1}^{3}[\hat{P}_\alpha - f_\alpha(\mathbf{Q})]^2+\Phi(\mathbf{Q})$.

We shall now prove the converse. Let us suppose that $H=(1/2\mu)\sum_{\alpha=1}^{3}\{\hat{P}_\alpha + a_\alpha(\mathbf{Q})\}^2+\Phi(\mathbf{Q})$; by replacing this H in (3.11) we obtain

$$\begin{aligned}
&\mathrm{i}[H,\mu Q_\alpha - \eta_\alpha(\mathbf{Q})]=\hat{P}_\alpha - f_\alpha(\mathbf{Q})\\
&\quad=\frac{\mathrm{i}}{2\mu}\sum_\beta[\hat{P}_\beta^2,\mu Q_\alpha]+\frac{\mathrm{i}}{2\mu}\sum_\beta[a_\beta\hat{P}_\beta,\mu Q_\alpha]+\frac{\mathrm{i}}{2\mu}\sum_\beta[\hat{P}_\beta a_\beta,\mu Q_\alpha]+\frac{\mathrm{i}}{2\mu}\sum_\beta[a_\beta^2,\mu Q_\alpha]\\
&\qquad-\frac{\mathrm{i}}{2\mu}\sum_\beta[\hat{P}_\beta^2,\eta_\alpha]-\frac{\mathrm{i}}{2\mu}\sum_\beta[a_\beta\hat{P}_\beta,\eta_\alpha]-\frac{i}{2\mu}\sum_\beta[\hat{P}_\beta a_\beta,\eta_\alpha]-\frac{i}{2\mu}\sum_\beta[a_\beta^2,\eta_\alpha]\\
&\qquad+\mathrm{i}[\Phi(\mathbf{Q}),\mu Q_\alpha-\eta_\alpha].
\end{aligned}$$

In the final member of these equalities, the fourth, the eighth and the last term are zero. Thus we have

$$\begin{aligned}
\mathrm{i}[H,\mu Q_\alpha - \eta_\alpha(\mathbf{Q})]&=\hat{P}_\alpha - f_\alpha(\mathbf{Q})\\
&=\hat{P}_\alpha+\frac{\mathrm{i}}{2}\sum_\beta(a_\beta\hat{P}_\beta Q_\alpha - Q_\alpha a_\beta\hat{P}_\beta+\hat{P}_\beta a_\beta Q_\alpha - Q_\alpha\hat{P}_\beta a_\beta)\\
&\quad-\frac{\mathrm{i}}{2\mu}\sum_\beta[\hat{P}_\beta^2,\eta_\alpha]-\frac{\mathrm{i}}{2\mu}\sum_\beta(a_\beta\hat{P}_\beta\eta_\alpha-\eta_\alpha a_\beta\hat{P}_\beta+\hat{P}_\beta a_\beta\eta_\alpha-\eta_\alpha\hat{P}_\beta a_\beta) \qquad (3.13)\\
&=\hat{P}_\alpha+\frac{\mathrm{i}}{2}\sum_\beta(a_\beta[\hat{P}_\beta,Q_\alpha]+[\hat{P}_\beta,Q_\alpha]a_\beta)-\frac{\mathrm{i}}{2\mu}\sum_\beta[\hat{P}_\beta^2,\eta_\alpha]\\
&\quad-\frac{\mathrm{i}}{2\mu}\sum_\beta(a_\beta[\hat{P}_\beta,\eta_\alpha]+[\hat{P}_\beta,\eta_\alpha]a_\beta)\\
&=\hat{P}_\alpha+\frac{\mathrm{i}}{2}(-2\mathrm{i}a_\alpha)-\frac{\mathrm{i}}{2\mu}\sum_\beta[\hat{P}_\beta^2,\eta_\alpha]-\frac{\mathrm{i}}{2\mu}\sum_\beta\left(-2\mathrm{i}a_\beta\frac{\partial\eta_\alpha}{\partial q_\beta}\right)\\
&=\hat{P}_\alpha+a_\alpha-\frac{1}{\mu}\sum_\beta a_\beta\frac{\partial\eta_\alpha}{\partial q_\beta}-\frac{\mathrm{i}}{2\mu}\sum[\hat{P}_\beta^2,\eta_\alpha]. \qquad (3.14)
\end{aligned}$$

From the second and final member of this chain of equations we obtain $-f_\alpha(\mathbf{Q})=a_\alpha-(1/\mu)\sum_\beta a_\beta(\partial\eta_\alpha/\partial q_\beta)-(\mathrm{i}/2\mu)\sum[\hat{P}_\beta^2,\eta_\alpha]$, which implies that $\sum_\beta[\hat{P}_\beta^2,\eta_\alpha]$ is a function of $\mathbf{Q}$. Therefore, we have

$$\begin{aligned}
\sum_\beta[\hat{P}_\beta^2,\eta_\alpha]=\phi_\alpha(\mathbf{Q})&=\sum_\beta(\hat{P}_\beta[\hat{P}_\beta,\eta_\alpha]+[\hat{P}_\beta,\eta_\alpha]\hat{P}_\beta)=(-\mathrm{i})\sum_\beta\left(\hat{P}_\beta\frac{\partial\eta_\alpha}{\partial q_\beta}+\frac{\partial\eta_\alpha}{\partial q_\beta}\hat{P}_\beta\right)\\
&=(-\mathrm{i})\sum_\beta\left(\left[\hat{P}_\beta,\frac{\partial\eta_\alpha}{\partial q_\beta}\right]+2\frac{\partial\eta_\alpha}{\partial q_\beta}\hat{P}_\beta\right)\\
&=(-\mathrm{i})\sum_\beta\left((-\mathrm{i})\frac{\partial^2\eta_\alpha}{\partial q_\beta^2}+2\frac{\partial\eta_\alpha}{\partial q_\beta}\hat{P}_\beta\right).
\end{aligned}$$

As a consequence $\sum_\beta(\partial\eta_\alpha/\partial q_\beta)\hat{P}_\beta$ must be a function of $\mathbf{Q}$, so that for every γ we have $\sum_\beta[Q_\gamma,(\partial\eta_\alpha/\partial q_\beta)\hat{P}_\beta]=\mathbb{O}=(\partial\eta_\alpha/\partial q_\gamma)[Q_\gamma,\hat{P}_\gamma]=\mathrm{i}(\partial\eta_\alpha/\partial q_\gamma)$; therefore $\partial\eta_\alpha/\partial q_\gamma=\mathbb{O}$; thus η_α is a constant function. By using this result in the equality between the second and the final members of (3.13) we obtain $a_\alpha=-f_\alpha$. ■

4. Implying wave equations

From §3f, for a spin-0 particle the interaction described by (3.12), which encompasses the electromagnetic interaction, is determined by the fact that each operator $\eta_\alpha(\mathbf{Q})$ appearing in the general dynamical law (3.11) is a real multiple of the identity operator: $\eta_\alpha(\mathbf{Q}) = \lambda_\alpha \mathbb{1}$, with $\lambda_\alpha \in \mathbb{R}$. Hence, according to proposition 3.14, $\mathrm{e}^{\mathrm{i}\hat{G}_\alpha u} = V_g U_g = \mathrm{e}^{\mathrm{i}\varsigma(u,\mathbf{Q})} U_g = \mathrm{e}^{\mathrm{i}\{\eta_\alpha(\mathbf{Q})u + o_\alpha(u,\mathbf{Q})\}} U_g = \mathrm{e}^{\mathrm{i}\lambda_\alpha u}\,\mathrm{e}^{\mathrm{i}o_\alpha(u,\mathbf{Q})} U_g$, where $o_\alpha(u,\mathbf{Q})$ is an operator infinitesimal of order greater than 1 in u with respect to the topology of $\mathcal{H}$, so that $\mathrm{e}^{\mathrm{i}\hat{G}_\alpha u} Q_\beta^{(t)}\,\mathrm{e}^{-\mathrm{i}\hat{G}_\alpha u} = \mathrm{e}^{\mathrm{i}o_\alpha(u,\mathbf{Q})} U_g Q_\beta^{(t)} U_g^{-1}\,\mathrm{e}^{-\mathrm{i}o_\alpha(u,\mathbf{Q})} = S_g[Q_\beta^{(t)}] + \omega(u,\mathbf{Q})$, where $\omega(u,\mathbf{Q})$ is an operator infinitesimal of order greater than 1 in u and t. Therefore, the σ-conversion leaves the transformation properties of $\mathbf{Q}^{(t)}$ invariant with respect to Galileian boosts at the first order.

This result suggests that several possible forms of the wave equation, i.e. of the Hamiltonian operator H, could be similarly determined by this kind of invariance relative to specific subgroups of $\mathcal{G}$, also for arbitrary values of the spin.

This is, in fact, the case. In this section we shall determine the specific form the Hamiltonian operator H must take as a consequence of the fact that the covariance properties of $\mathbf{Q}^{(t)}$ with respect to specific subgroups of $\mathcal{G}$ are left unaltered at the first order by the σ-conversion permitted by the interaction. In §4a we address the case where such a subgroup is the subgroup of boosts, for every value of the spin, and in §4b we shall tackle the task for the subgroup of spatial translations.

(a) Invariance under boosts: minimal coupling

The covariance properties of $\mathbf{Q}^{(t)}$ with respect to Galileian boosts g are expressed by $S_g[Q_\beta^{(t)}] = U_g Q_\beta^{(t)} U_g^{-1} = Q_\beta^{(t)} - \delta_{\alpha\beta} u t \mathbb{1}$; therefore the equality

$$\mathrm{e}^{\mathrm{i}\hat{G}_\alpha u} Q_\beta^{(t)}\,\mathrm{e}^{-\mathrm{i}\hat{G}_\alpha u} = Q_\beta^{(t)} - \delta_{\alpha\beta} u t \mathbb{1} + o_1^{(t)}(u), \tag{4.1}$$

where $o_1^{(t)}(u)$ is an operator infinitesimal of order greater than 1 is the necessary and sufficient condition so that the σ-conversion leaves the covariance properties of $\mathbf{Q}^{(t)}$ with respect to boosts unaltered, at the first order in u.

Proposition 4.1. *A Q-covariant σ-conversion leaves unaltered the covariance properties of $\mathbf{Q}^{(t)}$ under Galileian boosts at the first order if and only if*

$$[\eta_\alpha(\mathbf{Q}), Q_\beta^{(t)}] = \mathbb{O}. \tag{4.2}$$

If (4.2) holds, then the following relations must hold.

$$\text{(i) } [\hat{G}_\alpha, Q_\beta^{(t)}] = \mathrm{i}\delta_{\alpha\beta} t, \quad \text{(ii) } [\hat{G}_\alpha, \dot{Q}_\beta] = \mathrm{i}\delta_{\alpha\beta}; \tag{4.3}$$

and

$$\text{(i) } \mu Q_\beta^{(t)} - \hat{P}_\beta t = \varphi_\beta^{(t)}(\mathbf{Q}), \quad \text{(ii) } \dot{Q}_\beta = \frac{1}{\mu}(\hat{P}_\beta + a_\beta(\mathbf{Q})), \tag{4.4}$$

where $\varphi_\beta^{(t)}(\mathbf{x})$ and $a_\beta(\mathbf{x}) = (\mathrm{d}/\mathrm{d}t)\varphi_\beta^{(t)}(\mathbf{x})\,|_{t=0}$ are self-adjoint operators of $\mathcal{H}_0$.

Proof. Let $\hat{U}_g = \mathrm{e}^{\mathrm{i}\hat{G}_\alpha u} = V_g U_g$ be the σ-converted unitary operator associated with the Galileian boost g, where $V_g = \mathrm{e}^{\mathrm{i}\varsigma_\alpha(u,\mathbf{Q})}$ from proposition 3.14. By starting from (4.1) and expanding $\mathrm{e}^{\pm\mathrm{i}\varsigma_\alpha(u,\mathbf{Q})}$ with respect to u we obtain

$$\mathrm{e}^{\mathrm{i}\hat{G}_\alpha u} Q_\beta^{(t)}\,\mathrm{e}^{-\mathrm{i}\hat{G}_\alpha u} = V_g U_g Q_\beta^{(t)} U_g^{-1} V_g^{-1} = Q_\beta^{(t)} + i[\eta_\alpha(\mathbf{Q}), Q_\beta^{(t)}]u - \delta_{\alpha\beta} u t \mathbb{1} + o_2^{(t)}(u). \tag{4.5}$$

Comparison with (4.1) shows that such a condition holds if and only if (4.2) holds.

By expanding $\mathrm{e}^{\pm\mathrm{i}\hat{G}_\alpha u}$ with respect to u we find $\mathrm{e}^{\mathrm{i}\hat{G}_\alpha u} Q_\beta^{(t)}\,\mathrm{e}^{-\mathrm{i}\hat{G}_\alpha u} = Q_\beta^{(t)} + \mathrm{i}[\hat{G}_\alpha, Q_\beta^{(t)}]u + o_3^{(t)}(u)$, so that (4.1) holds if and only if $\mathrm{i}[\hat{G}_\alpha, Q_\beta^{(t)}] = -\delta_{\alpha\beta} t \mathbb{1}$; therefore (4.3) holds. Finally, since $\hat{G}_\alpha = \mu Q_\alpha$, (4.3(i)) implies $[\mu Q_\alpha, Q^{(t)}_\beta] = [\hat{G}_\alpha, Q^{(t)}_\beta] = [Q_\alpha, \hat{P}_\beta]t$, and then a self-adjoint operator $\varphi^{(t)}_\beta$ of $\mathcal{H}_0$ must exist for every $\mathbf{x}$ such that (4.4) hold. ■

If we put $H_0 = (1/2\mu)\sum_\gamma(\hat{P}_\gamma + a_\gamma(\mathbf{Q}))^2$, then a simple calculation yields $\mathrm{i}[H_0, Q_\beta] = (1/\mu)(\hat{P}_\beta + a_\beta(\mathbf{Q}))$. Whenever (4.2) holds, proposition 4.1 implies $\mathrm{i}[H_0, Q_\beta] = \dot{Q}_\beta$, i.e. $[H, Q_\beta] = [H_0, Q_\beta]$; therefore,

$$H = H_0 + \Phi(\mathbf{Q}) = \frac{1}{2\mu}\sum_\gamma(\hat{P}_\gamma + a_\gamma(\mathbf{Q}))^2 + \Phi(\mathbf{Q}), \tag{4.6a}$$

where $\Phi(\mathbf{x})$ is a self-adjoint operator of $\mathcal{H}_0$. Then the wave equation is

$$\mathrm{i}\frac{\partial}{\partial t}\psi_t = \left\{\frac{1}{2\mu}\sum_\gamma(\hat{P}_\gamma + a_\gamma(\mathbf{Q}))^2 + \Phi(\mathbf{Q})\right\}\psi_t. \tag{4.6b}$$

Thus, the minimal electromagnetic coupling principle has been derived from group theoretical invariance properties.

According to (4.6), the dynamics of the particle is determined by the $\Omega(\mathcal{H}_0)$-valued functions a_α, Φ. We can call them the 'fields' which describe the effects of the interaction; in so doing, however, we must not confuse them with other notions of field involved in Quantum Physics. Now we shall see how these fields are related to the fields η_α, f_α entering the general dynamical law (3.11).

From (4.4(ii)) we imply $[\eta_\alpha(\mathbf{Q}), \dot{Q}_\beta] = (1/\mu)[\eta(\mathbf{Q}), \hat{P}_\beta] + (1/\mu)[\eta_\alpha(\mathbf{Q}), a_\beta(\mathbf{Q})]$. By making use of (4.2), we obtain

$$\frac{\partial \eta_\alpha}{\partial x_\beta}(\mathbf{Q}) = \frac{\mathrm{i}}{2}[\eta_\alpha(\mathbf{Q}), a_\beta(\mathbf{Q})]. \tag{4.7a}$$

Now, by replacing the form (4.6) of H in (3.11), we obtain

$$\begin{aligned}\hat{P}_\alpha - f_\alpha(\mathbf{Q}) &= \mathrm{i}[H, \mu Q_\alpha - \eta_\alpha(\mathbf{Q})]\\ &= \mathrm{i}\left[\frac{1}{2\mu}\sum_\beta \mu^2\dot{Q}^2_\beta + \Phi(\mathbf{Q}), \mu Q_\alpha\right] - \mathrm{i}\left[\frac{1}{2\mu}\sum_\beta \mu^2\dot{Q}^2_\beta + \Phi(\mathbf{Q}), \eta_\alpha(\mathbf{Q})\right]\\ &= \mathrm{i}\left\{\frac{1}{2}\mu\sum_\beta[\dot{Q}^2_\beta, \mu Q_\alpha] + [\Phi(\mathbf{Q}), \mu Q_\alpha]\right\} - \mathrm{i}\left\{\frac{1}{2}\mu\sum_\beta[\dot{Q}^2_\beta, \eta_\alpha(\mathbf{Q})] + [\Phi(\mathbf{Q}), \eta_\alpha(\mathbf{Q})]\right\}.\end{aligned} \tag{4.8}$$

By making use of (4.3(ii)), which implies $[\mu Q_\alpha, \dot{Q}^2_\beta] = 2\mathrm{i}\delta_{\alpha\beta}\dot{Q}_\beta$, of (4.2) and of (4.4(ii)), we find

$$\begin{aligned}\hat{P}_\alpha - f_\alpha(\mathbf{Q}) &= \frac{1}{2}\mu\sum_\beta(-2\mathrm{i}\delta_{\alpha\beta}\dot{Q}_\beta) + \mathbb{O} - \frac{\mathrm{i}}{2}\mu\,\mathbb{O} - \mathrm{i}[\Phi(\mathbf{Q}), \eta_\alpha(\mathbf{Q})]\\ &= \mu\dot{Q}_\alpha - \mathrm{i}[\Phi(\mathbf{Q}), \eta_\alpha(\mathbf{Q})] = \hat{P}_\alpha + a_\alpha(\mathbf{Q}) - \mathrm{i}[\Phi(\mathbf{Q}), \eta_\alpha(\mathbf{Q})] = \hat{P}_\alpha - f_\alpha(\mathbf{Q}).\end{aligned}$$

We have, therefore, shown that

$$f_\alpha(\mathbf{Q}) = \mathrm{i}[\Phi(\mathbf{Q}), \eta_\alpha(\mathbf{Q})] - a_\alpha(\mathbf{Q}). \tag{4.9a}$$

Hence, whenever (4.2) holds, the fields η_α and f_α in the general law (3.11) are determined, according to (4.7), by the fields a_α, Φ.

In the particular case of a spin-0 particle, we can show the following further characterization.

Proposition 4.2. *In the simplest Quantum Theory of an interacting particle, corresponding to the case $\mathcal{H}_0 = \mathbb{C}$ in (3.5), the Q-covariant σ-conversions for which $\eta_\alpha(\mathbf{Q}) =$ const. are those and only those which leave the covariant properties of $\mathbf{Q}^{(t)}$ with respect to the Galileian boosts unaltered, at the first order in the boost's velocity.*

Proof. If η_α =const. then (4.2) holds. Therefore, in order to prove the proposition, it is sufficient to prove the inverse implication. Hence we assume that (4.2) holds, which implies the condition

$[\eta_\alpha(\mathbf{Q}), Q_\alpha] = \mathbb{O}$. On the other hand, (4.4(i)) implies $Q^{(t)}_\beta = (t/\mu)(\varphi^{(t)}_\beta(\mathbf{Q}) + \hat{P}_\beta)$, which replaced in (4.2) yields $[\eta_\alpha(\mathbf{Q}), \hat{P}_\beta] = \mathbb{O}$; therefore $\eta_\alpha(\mathbf{Q})$ is a constant operator $\lambda_\alpha \mathbb{1}$. ■

Remark 4.3. In a highly particular case, Hoogland [20] obtained a different derivation of the minimal electromagnetic coupling. He considered a non-isolated particle for which the subgroup $\mathcal{G}_0$ generated by space–time translations, rotations around the z-axis and boosts along the z-direction continues to be a symmetry group; this is a very particular kind of interaction, corresponding for instance to a charged particle interacting with a uniform electric field and a uniform magnetic field, both oriented along z. Since $\mathcal{G}_0$ is a symmetry group, the Quantum Theory of the system must contain a projective representation of it; Hoogland derived the commutation rules for the self-adjoint generators of this projective representation, which were used to derive the form (3.12) of the wave equation, that coincides with that dictated by the minimal coupling principle; Hoogland was also able to explicitly determine the 'fields' Φ and a_α for this case.

Hoogland's derivation has the merit of avoiding σ-conversion, thanks to the fact that *residual* symmetry group $\mathcal{G}_0$ is sufficiently rich; it is hard therefore to extend this argument to more general kind of interaction. Moreover, it must be noted that Hoogland's argument requires the assumption that the state vectors can be expressed as wave functions $\psi(\mathbf{x})$ in such a way that $|\psi(\mathbf{x})|^2$ is the position probability density of the particle, which amounts to assuming that the position operators are the multiplication operators. Our approach, on the other hand, establishes general conditions that determine the form (4.6*b*) of the wave equation.

(b) Invariance under spatial translations

Let us now suppose that the interaction admits a Q-covariant σ-conversion such that if $\hat{U}_g = \mathrm{e}^{-\mathrm{i}\hat{P}_\alpha a}$ then

$$\mathrm{e}^{-\mathrm{i}\hat{P}_\alpha a} Q^{(t)}_\beta \, \mathrm{e}^{\mathrm{i}\hat{P}_\alpha a} = Q^{(t)}_\beta - \delta_{\alpha\beta} a \mathbb{1} + o^{(t)}_1(a), \tag{4.10}$$

where $o^{(t)}_1(a)$ is an infinetisimal operator of order greater than 1. In fact, we are studying the case where the σ-conversion leaves the covariance properties of $\mathbf{Q}^{(t)}$ with respect to spatial translations unaltered at the first order. Now, by expanding $\mathrm{e}^{-\mathrm{i}\hat{P}_\alpha a}$ with respect to the translation parameter a, (4.10) yields

$$\text{(i) } [Q^{(t)}_\beta, \hat{P}_\alpha] = \mathrm{i}\delta_{\alpha\beta} \quad \text{which implies} \quad \text{(ii) } [\dot{Q}_\beta, \hat{P}_\alpha] = \mathbb{O}. \tag{4.11}$$

Therefore, we can state that

$$\dot{Q}_\beta = v_\beta(\hat{\mathbf{P}}), \tag{4.12}$$

where $v_\beta(\mathbf{p})$ is a self-adjoint operator of $\mathcal{H}_0$. Since $[Q_\alpha, v_\beta(\hat{\mathbf{P}})] = \mathrm{i}(\partial v_\beta/\partial p_\alpha)(\hat{\mathbf{P}})$, by making use of the Jacobi identity for $[Q_\alpha, [H, Q_\beta]]$ we obtain

$$\mathrm{i}\frac{\partial v_\beta}{\partial p_\alpha}(\hat{\mathbf{P}}) = [Q_\alpha, \dot{Q}_\beta] = \mathrm{i}[Q_\alpha, [H, Q_\beta]] = [Q_\beta, \dot{Q}_\alpha] = \mathrm{i}\frac{\partial v_\alpha}{\partial p_\beta}(\hat{\mathbf{P}}).$$

This equality shows that $\mathbf{v}(\mathbf{p}) = (v_1(\mathbf{p}), v_2(\mathbf{p}), v_3(\mathbf{p}))$ is an irrotational field; hence a function F of $\mathbf{p}$ exists such that $v_\alpha(\mathbf{p}) = (\partial F/\partial p_\alpha)(\mathbf{p})$, where $F(\mathbf{p})$ is a self-adjoint operator of $\mathcal{H}_0$. Therefore, we can establish the following equalities.

$$\dot{Q}_\alpha = v_\alpha(\hat{\mathbf{P}}) = \frac{\partial F}{\partial p_\alpha}(\hat{\mathbf{P}}) = \mathrm{i}[F(\hat{\mathbf{P}}), Q_\alpha] = \mathrm{i}[H, Q_\alpha]. \tag{4.13}$$

The last equation implies that a function Ψ of $\mathbf{x}$ exists such that $H - F(\hat{\mathbf{P}}) = \Psi(\mathbf{Q})$, i.e.

$$H = F(\hat{\mathbf{P}}) + \Psi(\mathbf{Q}), \tag{4.14}$$

where $\Psi(\mathbf{x})$ is a self-adjoint operator of $\mathcal{H}_0$. Then the wave equation is

$$\mathrm{i}\frac{\partial}{\partial t}\psi_t = \{F(\hat{\mathbf{P}}) + \Psi(\mathbf{Q})\}\psi_t.$$

(c) Invariance under both

Let us suppose the interaction admits a σ-conversion that leaves unaltered the covariance properties of $\mathbf{Q}^{(t)}$ under both subgroups of boosts and of spatial translations. Accordingly, the following equality holds

$$H = F(\hat{\mathbf{P}}) + \Psi(\mathbf{Q}) = \frac{1}{2\mu}\sum_\gamma (\hat{P}_\gamma + a_\gamma(\mathbf{Q}))^2 + \Phi(\mathbf{Q})$$

$$= \frac{1}{2\mu}\sum_\gamma (\hat{P}_\gamma^2 + a_\gamma(\mathbf{Q})\hat{P}_\gamma + \hat{P}_\gamma a_\gamma(\mathbf{Q}) + a_\gamma^2(\mathbf{Q})) + \Phi(\mathbf{Q}).$$

Since $a_\gamma(\mathbf{Q})\hat{P}_\gamma + \hat{P}_\gamma a_\gamma(\mathbf{Q}) = [a_\gamma(\mathbf{Q}), \hat{P}_\gamma] + 2\hat{P}_\gamma a_\gamma(\mathbf{Q}) = \mathrm{i}(\partial a_\gamma/\partial x_\gamma)(\mathbf{Q}) + 2\hat{P}_\gamma a_\gamma(\mathbf{Q})$ the equality above implies

$$\frac{1}{\mu}\sum_\gamma \hat{P}_\gamma a_\gamma(\mathbf{Q}) = \left(F(\hat{\mathbf{P}}) - \frac{1}{2\mu}\sum_\gamma \hat{P}_\gamma^2\right) + \Psi(\mathbf{Q}) - \frac{\mathrm{i}}{2\mu}\sum_\gamma \frac{\partial a_\gamma}{\partial x_\gamma}(\mathbf{Q}) - \Phi(\mathbf{Q}) - \sum_\gamma a_\gamma^2(\mathbf{Q}).$$

Then

$$\frac{1}{\mu}\sum_\gamma \hat{P}_\gamma a_\gamma(\mathbf{Q}) = F_1(\hat{\mathbf{P}}) + F_2(\mathbf{Q}),$$

where $F_1(\mathbf{p}) = (F(\mathbf{p}) - (1/2\mu)\sum_\beta p_\beta^2)$ and $F_2(\mathbf{x}) = \Psi(\mathbf{x}) - (\mathrm{i}/2\mu)\sum_\beta(\partial a_\beta/\partial x_\beta)(\mathbf{x}) - \Phi(\mathbf{x}) - \sum_\beta a_\beta^2(\mathbf{x})$. Therefore,

$$\left[Q_\gamma, \frac{1}{\mu}\sum_\beta \hat{P}_\beta a_\beta(\mathbf{Q})\right] = \frac{\mathrm{i}}{\mu} a_\gamma(\mathbf{Q}) = \frac{\partial F_1}{\partial p_\gamma}(\hat{\mathbf{P}}).$$

Then

$$[\hat{P}_\alpha, a_\gamma(\mathbf{Q}] = -\mathrm{i}\frac{\partial a_\gamma}{\partial x_\alpha}(\mathbf{Q}) = -\mathrm{i}\mu\left[\hat{P}_\alpha, \frac{\partial F_1}{\partial p_\gamma}(\hat{\mathbf{P}})\right] = \mathbb{O}.$$

Therefore, $a_\gamma(\mathbf{Q})$ is an operator that acts as follows:

$$(a_\gamma(\mathbf{Q})\psi)(\mathbf{x}) = \hat{a}_\gamma\psi(\mathbf{x}),$$

where $\hat{a}_\gamma$ is an operator of $\mathcal{H}_0$ which does not depend on $\mathbf{x}$.

Thus, if (4.2) and (4.10) hold, then $H = (1/2\mu)\sum_\gamma(\hat{P}_\gamma + \hat{a}_\gamma)^2 + \Phi(\mathbf{Q})$, and the wave equation is

$$\mathrm{i}\frac{\partial}{\partial t}\psi_t = \left\{\frac{1}{2\mu}\sum_\gamma(\hat{P}_\gamma + \hat{a}_\gamma)^2 + \Phi(\mathbf{Q})\right\}\psi_t. \tag{4.15}$$

In the spin-0 case, if the σ-conversion also leaves unaltered the covariance properties of $\mathbf{Q}^{(t)}$ with respect to the rotations subgroup, then $\hat{a}_\gamma = 0$.

(d) Concluding remarks

Our work has been successful in deriving the known forms (4.6*b*) and (4.15) of the non-relativistic wave equation of an interacting particle, through a deductive development based on group theoretical methods. However, the present approach does not exclude the possibility of wave equations, and hence of interactions, different from those already known. In fact, the existence of interactions besides those described by equations (4.6) and (4.15) is manifestly implied by the phenomenological reality. It is sufficient to recall that in its classical limit (4.6) describes a charged particle *slowly* accelerated by the electromagnetic field; for *strongly* accelerated particles, the wave equation must be different from (4.6) and from (4.15). This implies that the corresponding σ-conversion cannot leave unaltered the covariance properties of $\mathbf{Q}^{(t)}$ with respect to the subgroup of boosts. The present theoretical framework, therefore, lays the groundwork for such an investigation.

This is not possible with the other methods for quantizing the interaction; for instance, those based on the gauge principle—without entering Quantum Field Theories—immediately lead to wave equations of the form (4.6). On the other hand, the method of canonical quantization is inherently constrained to the wave equations implied by the classical equations.

Now we can explain some relations of our results with those of the approach involving Quantum Borel kinematics (QBk), cited in the introduction [4,15–18], we shall refer to as the (QBk) approach.

In the (QBk) approach the notion of *generalized system of imprimitivity* that allows for the description of a particle in an external field is introduced and investigated (e.g. [4, ch. 4] and references therein). In these generalized imprimitivity systems, the quantum transformations of the projection operator $E(S)$ representing the localization of the particle in the subset S are realized by unitary operators W_g (denoted by $V^X(t)$ in [4, §3.1]) according to

$$E(S) \to S_g[E(S)] = W_g E(S) W_g^{-1}, \tag{4.16}$$

where g belongs to a one-parameter subgroup of transformations; moreover, these operators W_g form a unitary representation and hence a *projective representation* of this subgroup.

On the other hand, in this work, it is proved that the unitary operators U_g that realize the quantum transformation $A \to S_g[A] = U_g A U_g^{-1}$, in general, *do not form a projective representation*, otherwise electromagnetic interaction could not be encompassed by the theory.

At first sight these results of the two approaches seem to contradict each other. We shall see now that this not the case. In fact, in the present approach a *projective representation* of the transformations group exists; indeed, the σ-converted operators $\hat{U}_g$ *do form* a projective representation. Yet, in general, they do not realize the quantum transformations: $\hat{U}_g A \hat{U}_g^{-1} \neq S_g[A]$ does not hold for all quantum observables A; for convenience we can call the $\hat{U}_g$ *pseudo quantum transformers*. However, for Q-covariant σ-conversions, according to §3d, a pseudo-quantum transformer $\hat{U}_g$ becomes a true transformer if restricted to position observables: $\hat{U}_g Q_\alpha \hat{U}_g^{-1} = S_g[Q_\alpha]$. Hence, if $E : \mathcal{B}(\mathbb{R}^3) \to \Pi(\mathcal{H})$ is the common spectral PV-measure of the position operators (Q_1, Q_2, Q_3), then

$$\hat{U}_g E(\Delta) \hat{U}_g^{-1} = S_g[E(\Delta)] \quad \text{holds for all } \Delta \in \mathcal{B}(\mathbb{R}^3) \tag{4.17}$$

and $g \to \hat{U}_g$ is a projective representation.

Now, the projection operators $E(S)$ of the (QBk) approach in (4.16) are just localization quantum observables; thus the contradiction disappears: the operators W_g of the (QBk) approach are pseudo quantum transformers; in particular, they correctly realize the quantum transformation of position operators according to (4.16); but in general, they cannot be used for the quantum transformation of quantum observables other than localization observables.

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