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Abstract Group theoretical methods, worked out in particular by Mackey and Wigner, allow to attain the explicit Quantum Theory of a free particle through a purely deductive development based on symmetry principles. The extension of these methods to the case of an interacting particle finds a serious obstacle in the loss of the symmetry condition for the transformations of Galilei's group. The known attempts towards such an extension introduce restrictions which lead to theories empirically too limited. In the present article we show how the difficulties raised by the loss of symmetry can be overcome without the restrictions that affect tha past attempts. According to our results, the different specific forms of the wave equation of an interacting particle are implied by particular first order invariance properties that characterize the interaction with respect to specific sub-groups of galileian transformations. Moreover, the possibility of yet unknown forms of the wave equation is left open.

Keywords Quantum theory · Interaction of particles · Group theory · Wave equations

1 Introduction

The explicit non-relativistic Quantum Theory of a free particle can be attained, for instance following [1], through a purely deductive development based on symmetry principles, by making use in particular of Wigner theorem [2, 3] on the representation of symmetries and of Mackey's imprimitivity theorem [4, 5].

The extension of these group-theoretical methods to an interacting particle finds serious obstacles because the galileian transformations no longer form a group of symmetry

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transformations in this case, so that neither Wigner's theorem nor Mackey's imprimitivity theorem can directly apply. A strategy to go beyond the obstacle [6–8] is to restrict the investigation to a class of interactions, those for which the galileian transformations are represented through a *projective representation* [5], as in the case of the free particle. In so doing, however, the class of the interactions encompassed by the theory excludes a too large class of interactions, as for instance electromagnetic [9] but also non-uniform forces [10]. The present work pursues a different group-theoretical approach to the Quantum Theory of an interacting particle, without restrictions or assumptions that are not based on physical principles.

The basic concept, introduced in Section 2, that enforces our work is that of *quantum transformation* corresponding to a Galilei transformation $g \in \mathcal{G}$, viable also in absence of the condition of symmetry. The properties of these quantum transformations, inferred on a conceptual ground in Section 2.1, are used in Section 2.2, together with a continuity condition, to imply that every transformation $g \in \mathcal{G}$ can be assigned a *unitary* operator U_g, also if g is not a symmetry, that realizes the quantum transformation $S_g^{\Sigma}[A]$ of a quantum observable A as $S_g^{\Sigma}[A] = U_g A U_g^{-1}$.

In order that the imprimitivity theorem can be applied, we introduce in Section 3 the notion of σ-conversion, which is a straight mathematical procedure that converts each U_g into another unitary operator $\hat{U}_g$ in such a way that $g \to \hat{U}_g$ is a *projective representation*; this "projectivity" condition allows the application of the imprimitivity theorem to explicitly identify a mathematical formalism of the theory. But to attain an effective theory it is necessary to determine which operators represent position and to determine the dynamical law. In Section 3.1 we show how the operators that *physically* represent the position of the particle are explicitly identified if the interaction admits "Q-covariant" σ-conversions, i.e. σ-conversions that leave unaltered the covariance properties of the position with respect to $\mathcal{G}$. This class of interactions will result large enough to encompass also electromagnetic interactions. A general dynamical law is determined in Section 4 for a theory with Q-covariant σ-conversion.

Different specific forms of the hamiltonian H are compatible with this general law. Then we face the problem of singling out conditions related to the interaction, which determine the different specific wave equations. In Sections 4.1, 4.2, 4.3 we single out which *specific forms the wave equation must take* if the σ-conversion admitted by the interaction leaves unaltered, at the first order, the covariance properties of $\mathbf{Q}^{(t)}$, namely of position at time t, *with respect to specific sub-groups of* $\mathcal{G}$; in so doing, each known wave equation is obtained and characterized by specific sub-groups. In the conclusive Section 4.4 the relation of the present approach with other methods for obtaining the wave equation are briefly discussed.

2 Basic Concepts

Before introducing the founding concepts in Section 2.1 and developing some of their implications in Section 2.2, let us outline the necessary mathematical tools. We begin by listing the usual mathematical structures of a Quantum Theory formulated in a complex and separable Hilbert space $\mathcal{H}$ of non-finite dimension:

– the set $\Omega(\mathcal{H})$ of all self-adjoint (densely defined) operators of $\mathcal{H}$, which represent *quantum observables*;
– the complete, ortho-complemented lattice $\Pi(\mathcal{H})$ of all projections operators of $\mathcal{H}$, i.e. quantum observables with possible outcomes in $\{0, 1\}$;

- the set $\Pi_1(\mathcal{H})$ of all rank one orthogonal projections of $\mathcal{H}$;
- the set $\mathcal{S}(\mathcal{H})$ of all density operators of $\mathcal{H}$, which represent *quantum states*;
- the set $\mathcal{U}(\mathcal{H})$ of all unitary operators of the Hilbert space $\mathcal{H}$.

In this article we shall make use of the theorem of imprimitivity [1, 4] relatively to the euclidean group $\mathcal{E}$ only; then we formulate it for this specific case, after having introduced the involved notions of projective representation and of imprimitivity system.

Definition 2.1 Let G be a separable, locally compact group with identity element e. A correspondence $U : G \to \mathcal{U}(\mathcal{H})$, $g \to U_g$, with $U_e = \mathbb{I}$ and $g \to \langle U_g\phi \mid \psi\rangle$ being a Borel function in g, is a projective representation of G if $U_{g_1g_2} = \sigma(g_1, g_2)U_{g_1}U_{g_2}$, where $\sigma : G \times G \to \mathbb{C}$ is a complex function.

A projective representation $U : G \to \mathcal{U}(\mathcal{H})$ is said to be continuous if for any fixed $\psi \in \mathcal{H}$ the mapping $\Psi : G \to \mathcal{H}$, $g \to \Psi(g) = U_g\psi$ is continuous.

Let $\mathcal{E}$ be the Euclidean group, i.e. the semi-direct product $\mathcal{E} = \mathbb{R}^3 \circledS SO(3)$ between the group of spatial translations $\mathbb{R}^3$ and the group of spatial proper rotations $SO(3)$; each transformation $g \in \mathcal{E}$ bi-univocally corresponds to the ordered pair $(\mathbf{a}, R) \in \mathbb{R}^3 \times SO(3)$ such that $R^{-1}\mathbf{x} - R^{-1}\mathbf{a} \equiv \mathsf{g}(\mathbf{x})$ is the result of the passive transformation of the spatial point $\mathbf{x} = (x_1, x_2, x_3)$ by g.

Let $g \to \hat{U}_g$ be every continuous non trivial projective representation of Galilei's group $\mathcal{G}$, i.e. the group generated by the nine one-parameter abelian sub-groups $\mathcal{T}_\alpha$, $\mathcal{R}_\alpha$, $\mathcal{B}_\alpha$, $\alpha = 1, 2, 3$, of spatial translation, spatial rotations and galileian velocity boosts, relative to all axes x_α. Then there exist nine self-adjoint generators $\hat{P}_\alpha$, $\hat{J}_\alpha$, $\hat{G}_\alpha$ of the nine one-parameter unitary subgroups $\{e^{-i\hat{P}_\alpha a_\alpha}, a \in \mathbb{R}\}$, $\{e^{-i\hat{J}_\alpha\theta_\alpha}, \theta_\alpha \in \mathbb{R}\}$, $\{e^{i\hat{G}_\alpha u_\alpha}, u_\alpha \in \mathbb{R}\}$ that represent the sub-groups $\mathcal{T}_\alpha$, $\mathcal{R}_\alpha$, $\mathcal{B}_\alpha$ according to the projective representation $g \to \hat{U}_g$. The structural properties of $\mathcal{G}$ as a Lie group imply the following commutation relations [11].

$$\begin{aligned}&\text{(i) } [\hat{P}_\alpha, \hat{P}_\beta] = \mathbb{O}, \quad \text{(ii) } [\hat{J}_\alpha, \hat{P}_\beta] = i\hat{\epsilon}_{\alpha\beta\gamma}\hat{P}_\gamma, \quad \text{(iii) } [\hat{J}_\alpha, \hat{J}_\beta] = i\hat{\epsilon}_{\alpha\beta\gamma}\hat{J}_\gamma,\\ &\text{(iv) } [\hat{J}_\alpha, \hat{G}_\beta] = i\hat{\epsilon}_{\alpha\beta\gamma}\hat{G}_\gamma, \quad \text{(v) } [\hat{G}_\alpha, \hat{G}_\beta] = \mathbb{O}, \quad \text{(vi) } [\hat{G}_\alpha, \hat{P}_\beta] = i\delta_{\alpha\beta}\mu\, \mathbb{I},\end{aligned} \tag{1}$$

where $\hat{\epsilon}_{\alpha,\beta,\gamma}$ is the Levi-Civita symbol $\epsilon_{\alpha,\beta,\gamma}$ restricted by the condition $\alpha \neq \gamma \neq \beta$, and μ is a non-zero real number which characterizes the projective representation.

Definition 2.2 Let $\mathcal{H}$ be the Hilbert space of a projective representation $g \to U_g$ of the Euclidean group $\mathcal{E}$. A *projection valued* (PV) measure $E : \mathcal{B}(\mathbb{R}^3) \to \Pi(\mathcal{H})$, $\Delta \to E(\Delta)$ is an *imprimitivity system* for the projective representation $g \to U_g$ if

$$U_gE(\Delta)U_g^{-1} = E(\mathsf{g}^{-1}(\Delta)) \equiv E(R(\Delta) + \mathbf{a}) \tag{2}$$

holds for all $(\mathbf{a}, R) \in \mathcal{E}$ and for all $\Delta \in \mathcal{B}(\mathbb{R}^3)$.

Mackey's theorem of imprimitivity for $\mathcal{E}$. *If a PV measure $E : \mathcal{B}(\mathbb{R}^3) \to \Pi(\mathcal{H})$ is an imprimitivity system for a continuous projective representation $g \to U_g$ of the Euclidean group $\mathcal{E}$, then a projective representation $L : SO(3) \to \mathcal{U}(\mathcal{H}_0)$ exists such that, modulo a unitary isomorphism,*

(M.1) $\mathcal{H} = L_2(\mathbb{R}^3, \mathcal{H}_0)$,

(M.2) $(E(\Delta)\psi)(\mathbf{x}) = \chi_\Delta(\mathbf{x})\psi(\mathbf{x})$, *where χ_Δ is the characteristic functional of Δ,*

(M.3) $(U_g\psi)(\mathbf{x}) = L_R\psi(\mathsf{g}(\mathbf{x})) \equiv L_R\psi(R^{-1}\mathbf{x} - R^{-1}\mathbf{a})$, *for every $g = (\mathbf{a}, R) \in \mathcal{E}$.*

Furthermore, the projective representation U is irreducible if and only if the "inducing" representation L is irreducible.

2.1 Quantum Transformations

Now we formulate a concept of quantum transformation which is viable also for space-time transformations that are not symmetry transformations.

We stipulate to interpret the galileian group $\mathcal{G}$ as a group of changes of reference frame in a class $\mathcal{F}$ of frames which move uniformly with respect to each other: given any reference frame Σ in $\mathcal{F}$, by Σ_g we denote the frame related to Σ just by g.

Let us consider the Quantum Theory of a *localizable particle*, that is to say of a physical system which can be localized in a point of the physical space. If $(Q_1, Q_2, Q_3) \equiv \mathbf{Q}$ is the commuting triple of self-adjoint operators which represent the three coordinates of the position with respect to Σ and if $g \in \mathcal{E}$, then the α-th coordinate of the position with respect to the frame Σ_g must be represented by the operator $S_g^\Sigma[Q_\alpha] \equiv [\mathsf{g}(\mathbf{Q})]_\alpha$, where $\mathsf{g}(\mathbf{x}) = (y_1, y_2, y_3)$ is the triple of the coordinates, with respect to Σ_g, of the spatial point that is represented by $\mathbf{x}$ in Σ. Then the operators that represent the three coordinates of position depend on the frame they refer to.

In the case of a galileian boost $g \in \mathcal{G}$, characterized by a velocity $\mathbf{u} = (u, 0, 0)$ of Σ_g relative to Σ, it does not change the instantaneous position at all; hence $\mathsf{g}(\mathbf{x}) = \mathbf{x}$ and $S_g^\Sigma[\mathbf{Q}] = \mathsf{g}(\mathbf{Q}) = \mathbf{Q}$; but to transform the "position at time t", i.e. the operators $\mathbf{Q}^{(t)} = e^{iHt}\mathbf{Q}e^{-iHt}$, a function g_t different from g is required; namely, $S_g^\Sigma[\mathbf{Q}^{(t)}] = (Q_1^{(t)} - ut, Q_2, Q_3) \equiv \mathsf{g}_t(\mathbf{Q}^{(t)})$, where $\mathsf{g}_t(\mathbf{x}) = (x_1 - ut, x_2, x_3)$

In general, we can state that for every $g \in \mathcal{G}$ the following covariance relations hold,

$$(i) \quad S_g^\Sigma[\mathbf{Q}] = \mathsf{g}(\mathbf{Q}), \qquad (ii) \quad S_g^\Sigma[\mathbf{Q}^{(t)}] = \mathsf{g}_t(\mathbf{Q}^{(t)}), \tag{3}$$

where g_t is a suitable function, in general different from g. In fact, relations (3) are the conditions which *define* the position operators of a localizable particle.

In general, a Quantum Theory must account for the possibility that also operators representing observables other than position depend on the reference frame; therefore, the transformations S_g^Σ must be appropriately extended to all quantum observables. To do this, given two reference frames Σ_1 and Σ_2 in $\mathcal{F}$, we introduce the following concept of *relative indistinguishability* between measuring procedures:

If a measuring procedure $\mathcal{M}_1$ is relatively to Σ_1 identical to what is $\mathcal{M}_2$ relatively to Σ_2, we say that $\mathcal{M}_1$ and $\mathcal{M}_2$ are indistinguishable relatively to (Σ_1, Σ_2). Then, for every $g \in \mathcal{G}$ and every Σ in $\mathcal{F}$ we introduce the mapping

$$S_g^\Sigma : \Omega(\mathcal{H}) \to \Omega(\mathcal{H}), \quad A \to S_g^\Sigma[A] \tag{4}$$

with the following conceptually explicit interpretation.

(QT) *The self-adjoint operators A and $S_g^\Sigma[A]$ can be respectively measured by two measuring procedures $\mathcal{M}_1$ and $\mathcal{M}_2$ indistinguishable relatively to* (Σ, Σ_g).

We call the mapping S_g^Σ *Quantum Transformation* of g relative to Σ.
For instance, if A is a quantum observable measured by a detector placed in the origin of Σ with a given orientation relative to Σ, then $S_g^\Sigma[A]$ is the operator that represents an identical detector placed in the origin of Σ_g with that orientation relative to Σ_g. It must be noticed that (QT) presupposes that for each quantum observable $A \in \Omega(\mathcal{H})$ and every $g \in \mathcal{G}$, two measuring procedures with the required relative indistinguishability exist, at least in principle.

The authentic meaning (QT) of the notion of quantum transformation immediately infers, at a conceptual level, the following general property.

(S.1) *For every frame Σ in $\mathcal{F}$ the following statement holds.*

$$S^{\Sigma}_{gh}[A] = S^{\Sigma_h}_{g}\left[S^{\Sigma}_{h}[A]\right], \text{ for all } A \in \Omega(\mathcal{H}). \tag{5}$$

This statement stresses how in general, i.e. without further particular conditions, the mapping S^{Σ}_{g}, with g fixed, can change by changing the "starting" frame Σ. Two further properties (S.2) and (S.3) can be inferred according to the meaning of quantum transformation expressed by (QT).

(S.2) For every $g \in \mathcal{G}$, the mapping S^{Σ}_{g} is bijective.

(S.3) For every real Borel function f, if A and $B = f(A)$ are self-adjoint operators, then the following equality holds:

$$f(S^{\Sigma}_{g}[A]) = S^{\Sigma}_{g}[f(A)]. \tag{6}$$

While (S.2) is straightforward, to infer (S.3), one can argue as follows. Let f be any fixed real Borel function, and let A and $B = f(A)$ be self-adjoint operators. According to Quantum Theory a measurement of the quantum observable $f(A)$ can be performed by measuring A and then transforming the obtained outcome a by the purely mathematical function f into the outcome $b = f(a)$ of $f(A)$. The measurement procedures of A and $S^{\Sigma}_{g}[A]$ are indistinguishable relatively to (Σ, Σ_g); then transforming the outcomes of both procedures by means of the same function f should not affect the relative indistinguishability of the so modified procedures. So we should conclude that (6) holds.

2.2 Peliminary Results

Now we show that conditions (S.2) and (S.3), together with a continuity condition, are sufficient to prove that each transformation $g \in \mathcal{G}$ is assigned a *unitary* operator U_g which realizes the corresponding quantum transformation S^{Σ}_{g} as $S^{\Sigma}_{g}[A] = U_g A U_g^{-1}$, also if g is not a symmetry transformation.

Proposition 2.1 *Let $S : \Omega(\mathcal{H}) \to \Omega(\mathcal{H})$ be a bijective mapping such that $S[f(A)] = f(S[A])$ for every Borel real function f, whenever $A, f(A) \in \Omega(\mathcal{H})$. Then the following statements hold.*

i) *If $E \in \Pi(\mathcal{H})$ then $S[E] \in \Pi[\mathcal{H}]$, i.e., the mapping S is an extension of a bijection of $\Pi(\mathcal{H}]$.*

ii) *If $A, B \in \Omega(\mathcal{H})$ and $A + B \in \Omega(\mathcal{H})$, then $[A, B] = \mathbb{O}$ implies $S[A + B] = S[A] + S[B]$.*
This partial additivity immediately implies $S[A] = \mathbb{O}$ if and only if $A = \mathbb{O}$.

iii) *For all $E, F \in \Pi(\mathcal{H})$, $EF = \mathbb{O}$ implies $S[E + F] = S[E] + S[F] \in \Pi(\mathcal{H})$; as a consequence, $E \leq F$ if and only if $S[E] \leq S[F]$.*

iv) *$S[P] \in \Pi_1(\mathcal{H})$ if and only if $P \in \Pi_1(\mathcal{H})$.*

Proof (i) If $E \in \Pi(\mathcal{H})$ and $f(\lambda) = \lambda^2$ then $f(E) = E$ holds; so $S[f(E)] = f(S[E])$ implies $(S[E])^2 \equiv f(S[E]) = S[E^2] \equiv S[E]$, i.e. $S^2[E] = S[E]$.

(ii) If $[A, B] = \mathbb{O}$ then a self-adjoint operator C and two functions f_a, f_b exist so that $A = f_a(C)$ and $B = f_b(C)$; once defined the function $f = f_a + f_b$, we have $S[A+B] \equiv S[f(C)] = f(S[C]) = f_a(S[C]) + f_b(S[C]) = S[f_a(C)] + S[f_b(C)] \equiv S[A] + S[B]$.

(iii) If $EF = \mathbb{O}$, then $[E, F] = \mathbb{O}$ and $(E+F) \in \Pi(\mathcal{H})$ hold. Statements (i) and (ii) imply $S[E+F] = S[E] + S[F] \in \Pi(\mathcal{H}]$.
(iv) If $P \in \Pi_1(\mathcal{H})$ then $S[P] \in \Pi(\mathcal{H})$ by (i). If $Q \in \Pi_1(\mathcal{H})$ and $Q \leq S[P]$ then $P_0 \equiv S^{-1}[Q] \leq P$ by (iii); but P is rank 1, therefore $P_0 = P$ and $Q = S[P]$.

□

Properties (S.2), (S.3) imply that every quantum transformation S_g^Σ satisfies the hypotheses of Prop. 2.1; therefore, it is an automorphism of the lattice $\Pi(\mathcal{H})$. Hence, according to an equivalent version of Wigner theorem [3], a unitary or anti-unitary operator U_g exists such that $S_g^\Sigma[E] = U_g E U_g^*$ for every $E \in \Pi(\mathcal{H})$; Prop. 2.1 can be re-used for extending this equality to $S_g^\Sigma[A] = U_g A U_g^*$ for all $A \in \Omega(\mathcal{H})$. Moreover, under the condition that the correspondence $g \to U_g$ is continuous, U_g is forced to be unitary because U_g is connected with $U_e = \mathbb{I}$ which is unitary [9].

Remark 2.1 The unitarity of each U_g implies that the spectrum of any quantum observable A is invariant under S_g^Σ: $\sigma(S_g^\Sigma[A]) = \sigma(A)$. An important consequence of such an invariance is that an interaction having the effect of sharply confining a particle in a bounded region of the physical space, hence such that $\sigma(\mathbf{Q}) \subset \mathbb{R}^3$, is not consistent with the theory; indeed, (3.i) implies $\mathsf{g}(\sigma(\mathbf{Q})) = \sigma(\mathsf{g}(\mathbf{Q})) = \sigma(S_g^\Sigma[\mathbf{Q}]) \equiv \sigma(\mathbf{Q})$, which cannot hold for all $g \in \mathcal{G}$ unless $\sigma(\mathbf{Q}) = \mathbb{R}^3$. For this reason we find appropriate, for the time being, to re-establish (S.2) and (S.3) as conditions characterizing the class of interactions investigated in the present work. In the following we shall see that such a class is a very large one, enough to encompass also electromagnetic interaction.

3 σ-Conversions

In Section 2.2 we concluded, under a continuity condition for $g \to U_g$, that in the Quantum Theory of a physical system, also if it is not isolated, a continuous correspondence $U : \mathcal{G} \to \mathcal{U}(\mathcal{H})$ exists such that $S_g^\Sigma[A] = U_g A U_g^{-1}$. Elsewhere [9] it has been shown that if the mapping $g \to U_g$, such that $S_g[A] = U_g A U_g^{-1}$, is a *projective representation*, then the Hilbert space of the simplest Quantum Theory for a localizable particle can be identified with $L_2(\mathbb{R}^3)$, where the position operators Q_α form a complete system, the translations' and boosts' generators are $P_\alpha = \mu \dot{Q}_\alpha$ and $G_\alpha = \mu Q_\alpha$. However, the Hamiltonian must be $H = (2\mu)^{-1} \sum_\alpha P_\alpha^2 + \Phi(\mathbf{Q})$, where $\Phi(q_1, q_2, q_3)$ is a real function of $\mathbf{q} \in \mathbb{R}^3$; therefore, electromagnetic interaction cannot be described.

But the empirical inadequateness of this 'projectivity' condition is much stronger, because, as argued in [10], it excludes a much larger class of interactions. Let us outline the argument. If $U_g = e^{iG_\alpha u}$, then (3.ii) implies $e^{iG_\alpha u} Q_\beta^{(t)} e^{iG_\alpha u} = Q_\beta^{(t)} - \delta_{\alpha\beta} u t$; this equality can be used to compute $e^{iG_\alpha u} \dot{Q}_\beta^{(t)} e^{iG_\alpha u} = \dot{Q}_\beta^{(t)} - \delta_{\alpha\beta} u$ and, in cascade, $e^{iG_\alpha u} \ddot{Q}_\beta^{(t)} e^{iG_\alpha u} = \ddot{Q}_\beta^{(t)}$, $e^{iG_\alpha u} \dddot{Q}_\beta^{(t)} e^{iG_\alpha u} = \dddot{Q}_\beta^{(t)}$. Last equality implies $[G_\alpha, \dddot{Q}_\beta] = \mathbb{O}$, i.e. $\dddot{Q}_\beta = h_\beta(\mathbf{Q})$ by the completeness of $\mathbf{Q}$. Yet, since $H = (2\mu)^{-1} \sum_\alpha P_\alpha^2 + \Phi(\mathbf{Q})$, we have $\ddot{Q}_\beta = i[H, \dot{Q}_\beta] = i[H, \frac{P_\beta}{\mu}] = -\frac{\partial \Phi}{\partial q_\beta}(\mathbf{Q})$; hence $\dddot{Q}_\beta = i[H, \ddot{Q}_\beta] = i\left[H, -\frac{\partial \Phi}{\partial q_\beta}(\mathbf{Q})\right] =$
$= -\frac{i}{2\mu} \sum_\alpha \left[P_\alpha^2, \frac{\partial \Phi}{\partial q_\beta}(\mathbf{Q})\right] = -\frac{i}{2\mu} \sum_\alpha \left(P_\alpha \left[P_\alpha, \frac{\partial \Phi}{\partial q_\beta}(\mathbf{Q})\right] + \left[P_\alpha, \frac{\partial \Phi}{\partial q_\beta}(\mathbf{Q})\right] P_\alpha\right) =$
$= -\frac{1}{2\mu} \sum_\alpha \left(P_\alpha \frac{\partial^2 \Phi}{\partial q_\alpha \partial q_\beta} + \frac{\partial^2 \Phi}{\partial q_\alpha \partial q_\beta} P_\alpha\right) = -\frac{1}{2\mu} \sum_\alpha \left(\left[P_\alpha, \frac{\partial^2 \Phi}{\partial q_\alpha \partial q_\beta}\right] + 2 \frac{\partial^2 \Phi}{\partial q_\alpha \partial q_\beta} P_\alpha\right)$.

Now, $\dddot{Q}_\beta = h_\beta(\mathbf{Q})$ implies $\left[\sum_\alpha\left(\left[P_\alpha, \frac{\partial^2\Phi}{\partial q_\alpha \partial q_\beta}\right] + 2\frac{\partial^2\Phi}{\partial q_\alpha \partial q_\beta} P_\alpha\right), Q_\gamma\right] = \mathbb{O}$, and then $2\left[\sum_\alpha \frac{\partial^2\Phi}{\partial q_\alpha \partial q_\beta} P_\alpha, Q_\gamma\right] = 2\frac{\partial^2\Phi}{\partial q_\alpha \partial q_\beta}\sum_\alpha\left[P_\alpha, Q_\gamma\right] = -2i\frac{\partial^2\Phi}{\partial q_\gamma \partial q_\beta} = \mathbb{O}$, i.e. $\frac{\partial\Phi}{\partial q_\beta} = \lambda_\beta \mathbb{I}$.

Therefore, if $g \to U_g$ were a projective representation and (3) held, then only interactions with linear 'potentials' Φ could be described. Thus, in order to develop an empirically more comprehensive Quantum Theory of an interacting particle, we have to remove the 'projectivity' condition or (3). Since (3) expresses just the notion of position, in this work we give up the projectivity condition for $g \to U_g$; without it, however, Mackey's imprimitivity theorem does not apply and the approach finds an obstacle. Now we address this obstacle.

The continuous correspondence $g \to U_g$, can be *converted* into a continuous projective representation if we multiply each operator U_g by a suitable unitary operator V_g of $\mathcal{H}$; namely, V_g is a unitary operator such that the correspondence $g \to \hat{U}_g = V_g U_g$ turns out to be a projective representation; the transition from $\{g \to U_g\}$ to $\{g \to \hat{U}_g \equiv V_g U_g\}$ will be called *σ-conversion*. If $\{U_g \to V_g U_g\}$ is a σ-conversion and $\theta : \mathcal{G} \to \mathbb{R}$ is a real function such that $e^{i\theta(e)} = 1$, then also $\{U_g \to [e^{i\theta(g)} V_g] U_g\}$ is a σ-conversion.

Remark 3.1 Since non-trivial continuous projective representations $\tilde{U} : \mathcal{G} \to \mathcal{U}(\tilde{\mathcal{H}})$ exist, we can state that a σ-conversion to non-trivial projective representations $g \to \hat{U}_g$ always exists. Indeed, both $\tilde{\mathcal{H}}$ and the Hilbert space $\mathcal{H}$ of the Quantum Theory of our interacting particle have the same (countably infinite) dimension. Therefore a unitary mapping $W : \mathcal{H} \to \tilde{\mathcal{H}}$ exists. Hence $\hat{U} : \mathcal{G} \to \mathcal{U}(\mathcal{H})$, $g \to \hat{U}_g = W\tilde{U}_g W^{-1}$ is a projective representation of $\mathcal{G}$ in $\mathcal{H}$. If we define $V_g = \hat{U}_g U_g^{-1}$, then $V_g U_g = \hat{U}_g$; therefore $\{U_g \to V_g U_g\}$ is a σ-conversion. This argument leaves completely undetermined the σ-conversion; actually, there is a σ-conversion for each projective representation of $\mathcal{G}$. This 'degeneracy' will be eliminated by physically meaningful further conditions, as argued in Remark 4.1.

3.1 Mathematical Formalism of the Theory

If $\{U_g \to V_g U_g\}$ is a σ-conversion then, according to Section 2, the continuous projective representation $g \to \hat{U}_g = V_g U_g$ has nine self-adjoint generators $\hat{P}_\alpha$, $\hat{J}_\alpha$, $\hat{G}_\alpha$ for which (1) hold. These commutation relations imply

$$\hat{U}_g \mathbf{F} \hat{U}_g^{-1} = \mathsf{g}(\mathbf{F}), \ \forall g \in \mathcal{G}, \tag{7}$$

for the triple $\mathbf{F} = \hat{\mathbf{G}}/\mu$, and then its common spectral measure is an imprimitivity system for $\hat{U}\ |_{\mathcal{E}}$. So, according to the imprimitivity theorem we can explicitly identify $\mathcal{H}$ modulo a unitary isomorphism, but also F_α, $\hat{P}_\alpha$, $\hat{J}_\alpha$ and $\hat{G}_\alpha$ by means of (M.3) in the theorem:

$$\mathcal{H} = L_2(\mathbb{R}^3, \mathcal{H}_0), \qquad (F_\alpha\psi)(\mathbf{x}) = x_\alpha\psi(\mathbf{x}), \qquad \hat{P}_\alpha = -i\frac{\partial}{\partial x_\alpha}, \tag{8}$$

$$\hat{J}_\alpha = F_\beta\hat{P}_\gamma - F_\gamma\hat{P}_\beta + S_\alpha, \qquad \hat{G}_\alpha = \mu F_\alpha.$$

Here (α, β, γ) is any cyclic permutation of (1,2,3); the S_α are operators that act on $\mathcal{H}_0$ only, i.e. their action is $(S_\alpha\psi)(\mathbf{x}) = \hat{s}_\alpha\psi(\mathbf{x})$ where the $\hat{s}_\alpha$ are self-adjoint operators of $\mathcal{H}_0$ which form a representation of the commutation rules $[\hat{s}_\alpha, \hat{s}_\beta] = i\hat{\epsilon}_{\alpha\beta\gamma}\hat{s}_\gamma$. Since the reducibility of the inducing representation $L : SO(3) \to \mathcal{U}(\mathcal{H}_0)$ implies the reducibility of $\hat{U} : \mathcal{G} \to \mathcal{U}(\mathcal{H})$, if $\hat{U}$ is irreducible then also $(\hat{s}_1, \hat{s}_2, \hat{s}_3)$ must be an irreducible representation of $[\hat{s}_\alpha, \hat{s}_\beta] = i\hat{\epsilon}_{\alpha\beta\gamma}\hat{s}_\gamma$; in this case, modulo unitary isomorphisms, $\mathcal{H}_0$ is one of the finite-dimensional Hilbert spaces $\mathbb{C}^{2s+1}$, with $s \in \frac{1}{2}\mathbb{N}$: the $\hat{s}_\alpha$ are the familiar spin operators of $\mathbb{C}^{2s+1}$.

Hence, the mathematical formalism of the Quantum Theory of an interacting particle has been explicitly identified. However, the operators $\mathbf{Q} = (Q_1, Q_2, Q_3)$ representing the position are not identified. So, the mathematical formalism (8) turns out to be devoid of physical significance.

In order that the formalism become that of an effective Quantum Theory of an interacting particle at least two tasks shoud be accomplished.

First, the operators $\mathbf{Q}$ of $\mathcal{H} = L_2(\mathbb{R}^3, \mathcal{H}_0)$ in (8), that represent the position of the particle, should be explicitly determined. We address this task in Section 3.2.

Second, the wave equation ruling over the time evolution should be determined. In Section 4, once derived a general dynamical law, *specific* wave equations corresponding to *specific* features of the interaction are determined.

3.2 Q-Covariant σ-Conversions

The position operators $\mathbf{Q}$ can be determined within the formalism (8) for interactions that have the particular feature of admitting a σ-conversion $U_g \to \hat{U}_g = V_g U_g$ that *leaves unaltered the covariance properties of the position operators* $\mathbf{Q}$, i.e. such that

$$\hat{U}_g \mathbf{Q} \hat{U}_g^{-1} = \mathsf{g}(\mathbf{Q}), \quad \forall g \in \mathcal{G}. \tag{9}$$

A σ-conversion satisfying (9) is said to be *Q-covariant*.

The relevance of the concept of Q-covariance stands in the fact that it is a necessary and sufficient condition for representing position of a particle by the multiplication operators; indeed, the following proposition holds.

Proposition 3.1 *If a σ-conversion for a particle yields an irreducible projective representation $\hat{U}$, then it is a Q-covariant σ-conversion if and only if the position operators $\mathbf{Q}$ coincide with $\mathbf{F}$.*

Proof If $\mathbf{Q} = \mathbf{F} = \hat{\mathbf{G}}/\mu$, then (9) and (1) imply $\hat{U}_g \mathbf{Q} \hat{U}_g^{-1} = \mathsf{g}(\mathbf{Q}) \equiv \hat{U}_g \mathbf{F} \hat{U}_g^{-1} = \mathsf{g}(\mathbf{F})$.

Conversely, if $\hat{U} : \mathcal{G} \to \mathcal{U}(\mathcal{H})$ is an irreducible projective representation obtained from $U : \mathcal{G} \to \mathcal{U}(\mathcal{H})$ through a Q-covariant σ-conversion, then (9) for $\hat{U}_g = e^{i\hat{G}_\beta u} = e^{i\mu F_\beta u}$ and (1.v) imply $[Q_\alpha - F_\alpha, F_\beta] = [Q_\alpha, F_\beta] - [F_\alpha, F_\beta] = \mathbb{O} - \mathbb{O} = \mathbb{O}$; therefore $(Q_\alpha - F_\alpha)\psi(\mathbf{x}) = (f_\alpha(\mathbf{Q})\psi)(\mathbf{x}) = f_\alpha(\mathbf{x})\psi(\mathbf{x})$, where $f_\alpha(\mathbf{x})$ is a self-adjoint operator of $\mathcal{H}_0$. However, the Q-covariance and (1.vi) imply also $[Q_\alpha - F_\alpha, \hat{P}_\beta] = [Q_\alpha, \hat{P}_\beta] - [Q_\alpha, \hat{P}_\beta] = i\delta_{\alpha\beta}\mathbb{I} - i\delta_{\alpha\beta}\mathbb{I} = \mathbb{O}$, i.e. $[f_\alpha(\mathbf{Q}), \hat{P}_\beta] = \mathbb{O}$ for all $\mathbf{x}$; this relation, since $\hat{P} = -i\frac{\partial}{\partial x_\alpha}$, implies that $\frac{\partial f_\alpha}{\partial x_\alpha}(\mathbf{x}) = 0$, for all α, β; therefore $f_\alpha(\mathbf{x})$ is an operator $\hat{f}_\alpha$ of $\mathcal{H}_0$ which *does not depend* on $\mathbf{x}$. Now, since $\hat{f}_\alpha = Q_\alpha - F_\alpha$, also $[\hat{f}_\alpha, \hat{f}_\beta] = \mathbb{O}$ holds; moreover, from (3.i) for a pure spatial rotation g about x_α and from (9.iv) we obtain $[\hat{J}_\alpha, Q_\beta - F_\beta] = i\hat{\epsilon}_{\alpha\beta\gamma}(Q_\gamma - F_\gamma) = i\hat{\epsilon}_{\alpha\beta\gamma}\hat{f}_\gamma$; but the irreducibility of $\hat{U}$ implies the irreducibility of the inducing projective representation $L : SO(3) \to \mathcal{U}(\mathcal{H}_0)$, so that $\mathcal{H}_0$ is finite dimensional; then $[\hat{f}_\alpha, \hat{f}_\beta] = \mathbb{O}$ and $[\hat{J}_\alpha, \hat{f}_\beta] = i\hat{\epsilon}_{\alpha\beta\gamma}\hat{f}_\gamma$ can hold only if $\hat{f}_\alpha = \mathbb{O}$, i.e. $F_\alpha = Q_\alpha$. $\square$

Following a customary habit, we say that a particle, whose interaction admits Q-covariant σ-conversion, is *elementary* if $\hat{U}$ is irreducible. The following proposition specify how in the Quantum Theory of an elementary particle each $\hat{U}_g$ is related to the unitary operator U_g that realizes the quantum transformation corresponding to g.

Proposition 3.2 *The action of each operator V_g of a Q-covariant σ-conversion is of the kind $(V_g\psi)(\mathbf{x}) = \left(e^{i\theta(g,\mathbf{Q})}\psi\right)(\mathbf{x}) = e^{i\theta(g,\mathbf{x})}\psi(\mathbf{x})$, where $\theta(g,\mathbf{x})$ is a self-adjoint operator of $\mathcal{H}_0$ which depends on $\mathbf{x}$ and on g.*

Proof Relations (9) and (3) imply $V_g U_g \mathbf{Q} U_g^{-1} V_g^{-1} = \mathsf{g}(\mathbf{Q})$, i.e. $V_g(\mathsf{g}(\mathbf{Q}))V_g^{-1} = \mathsf{g}(\mathbf{Q})$, which implies $[V_g, \mathsf{g}(\mathbf{Q})] = \mathbb{O}$. Then $[V_g, \mathsf{f}(\mathsf{g}(\mathbf{Q}))] = \mathbb{O}$ for every sufficiently regular function f; by taking $\mathsf{f} = \mathsf{g}^{-1}$ we have $[V_g, \mathbf{Q}] = \mathbb{O}$. Then $(V_g\psi)(\mathbf{x}) = \mathsf{h}_g(\mathbf{x})\psi(\mathbf{x})$, where $\mathsf{h}_g(\mathbf{x})$ is an operator of $\mathcal{H}_0$. Finally, the unitary character of V_g imposes that $h_g(\mathbf{x})$ must be unitary as an operator of $\mathcal{H}_0$; thus a self-adjoint operator $\theta(g,\mathbf{x})$ of $\mathcal{H}_0$ exists such that $\mathsf{h}_g(\mathbf{x}) = e^{i\theta(g,\mathbf{x})}$. □

4 Wave Equations

Now we derive a general dynamical equation ruling over the time evolution of an elementary particle whose interaction admits Q-covariant σ-conversion. In so doing we shall suppose that the mapping $g \to V_g$ is differentiable with respect to the parameters $a_\alpha, \theta_\alpha, u_\alpha$ of the group $\mathcal{G}$.

Let us consider the pure velocity boost $g \in \mathcal{G}$ such that, in the formalism (8), $\hat{U}_g = e^{i\hat{G}_\alpha u}$. Since $\hat{G}_\alpha = \mu F_\alpha = \mu Q_\alpha$, we can write $\hat{U}_g = e^{i\mu Q_\alpha u}$; therefore

$$\hat{U}_g \dot{Q}_\beta \hat{U}_g^{-1} \equiv \lim_{t\to 0} V_g U_g \frac{(Q_\beta^{(t)} - Q_\beta)}{t} U_g^{-1} V_g^{-1} = \dot{Q}_\beta + i\mu[Q_\alpha, \dot{Q}_\beta]u + o_1(u). \quad (10)$$

By making use in (10) of $U_g Q_\beta^{(t)} U_g^{-1} = Q_\beta^{(t)} - \delta_{\alpha\beta} u t\, \mathbb{I}$, implied by (3), we obtain

$$\hat{U}_g \dot{Q}_\beta \hat{U}_g^{-1} = V_g \dot{Q}_\beta V_g^{-1} - \delta_{\alpha\beta} u\, \mathbb{I} = \dot{Q}_\beta + i\mu[Q_\alpha, \dot{Q}_\beta]u + o_1(u). \quad (11)$$

But Prop. 3.2 implies that $V_g = e^{i\varsigma_\alpha(u,\mathbf{Q})}$, where $\varsigma_\alpha(u,\mathbf{x})$ is a self-adjoint operator of $\mathcal{H}_0$; replacing in (11) we obtain

$$\dot{Q}_\beta + i[\varsigma_\alpha(u,\mathbf{Q}), \dot{Q}_\beta] + o_2(u) - \delta_{\alpha\beta} u\, \mathbb{I} = \dot{Q}_\beta + i\mu[Q_\alpha, \dot{Q}_\beta]u + o_1(u). \quad (12)$$

Since $e^{i\varsigma_\alpha(0,\mathbf{Q})} = \mathbb{I}$, the expansion of ς_α with respect to u yields $\varsigma_\alpha(u,\mathbf{Q}) = \frac{\partial\varsigma_\alpha}{\partial u}(0,\mathbf{Q})u + o_3(u)$; by replacing this last relation in (12) we obtain

$$\mu[Q_\alpha, \dot{Q}_\beta] = [\eta_\alpha(\mathbf{Q}), \dot{Q}_\beta] + i\delta_{\alpha\beta}\, \mathbb{I},$$

where $\eta_\alpha(\mathbf{Q}) = \frac{\partial\varsigma_\alpha}{\partial u}(0,\mathbf{Q})$. By replacing $\dot{Q}_\beta = i[H, Q_\beta]$ in this last equation we can apply Jacobi's identity, and in so doing we obtain $[Q_\beta, \mu\dot{Q}_\alpha] = [Q_\beta, \dot{\eta}_\alpha(\mathbf{Q})] + i\delta_{\alpha\beta}\, \mathbb{I}$, i.e.

$$[Q_\beta, \dot{\eta}_\alpha(\mathbf{Q}) - \mu\dot{Q}_\alpha] = -i\delta_{\alpha\beta}\, \mathbb{I} = [Q_\beta, -\hat{P}_\alpha].$$

Hence $[\dot{\eta}_\alpha(\mathbf{Q}) - \mu\dot{Q}_\alpha - \hat{P}_\alpha, Q_\beta] = \mathbb{O}$, from which we imply that for every $\mathbf{x} \in \mathbb{R}^3$ an operator $f_\alpha(\mathbf{x})$ of $\mathcal{H}_0$, must exist such that the equation $\{\dot{\eta}(\mathbf{Q}) - \mu\dot{Q}_\alpha + \hat{P}_\alpha\}\psi(\mathbf{x}) = f_\alpha(\mathbf{x})\psi(\mathbf{x})$ holds, that we can rewrite as

$$i[H, \mu Q_\alpha - \eta_\alpha(\mathbf{Q})] = \hat{P}_\alpha - f_\alpha(\mathbf{Q}). \tag{13}$$

This is a general dynamical equation for a localizable particle whose interaction admits Q-covariant σ-conversions; according to such a law, the effects of the interaction on the dynamics are encoded in the six "fields" η_α, f_α.

The general law (13) does not specify the concrete form of the hamiltonian operator H. In fact, equation (13) is satisfied by different inequivalent concrete operators H. This equation has been determined by requiring that the covariance properties of $\mathbf{Q}$ with respect to the whole group $\mathcal{G}$ are invariant under the σ-conversion; in Sections 4.1–4.3 we show how specific forms of H, i.e. of the wave equation, are implied by more stringent invariance of this kind.

4.1 Boosts Sub-Group and Electromagnetic Interaction

By $o^{(t)}(u)$ we denote an operator infinitesimal of order greater than 1 with respect to $\sqrt{t^2+u^2}$, that satisfies $\lim_{t\to 0}\frac{o^{(t)}(u)}{t} = o(u)$ and $\lim_{u\to 0}\frac{o^{(t)}(u)}{u} = \tilde{o}(t)$, where the operators $o(u)$ and $\tilde{o}(t)$ are infinitesimal of order greater than 1. We say that the σ-conversion leaves unaltered the covariance properties of $\mathbf{Q}^{(t)}$ at the first order with respect to Galileian boosts if the following equalities hold for all α and β.

$$e^{i\hat{G}_\alpha u}Q_\beta^{(t)}e^{-i\hat{G}_\alpha u} = S_g[Q_\beta^{(t)}] + o^{(t)}(u) = Q_\beta^{(t)} - \delta_{\alpha\beta}ut\,\mathbb{I} + o^{(t)}(u), \tag{14}$$

Proposition 4.1 *If a Q-covariant σ-conversion leaves unaltered the covariance properties of $\mathbf{Q}^{(t)}$ under galileian boosts at the first order, then*

$$[\eta_\alpha(\mathbf{Q}), Q_\beta^{(t)}] = \hat{o}(t)\,, \ \text{where } \lim_{t\to 0}\frac{\hat{o}(t)}{t} = \mathbb{O}; \tag{15}$$

$$(i) \quad [\hat{G}_\alpha, Q_\beta^{(t)}] = i\delta_{\alpha\beta}t + \hat{o}(t)\,, \qquad (ii) \quad [\hat{G}_\alpha, \dot{Q}_\beta] = i\delta_{\alpha\beta}\,; \tag{16}$$

$$\dot{Q}_\beta = \frac{1}{\mu}\left(\hat{P}_\beta + a_\beta(\mathbf{Q})\right), \tag{17}$$

where $a_\beta(\mathbf{x})$ is a self-adjoint operator of $\mathcal{H}_0$, for any $\mathbf{x} \in \mathbb{R}^3$.

Proof Let $\hat{U}_g = e^{i\hat{G}_\alpha u} = V_g U_g$ be the σ-converted unitary operator associated with the galileian boost g, where $V_g = e^{i\varsigma_\alpha(u,\mathbf{Q})}$ as in (12). By making use of these equalities, of $U_g Q_\beta^{(t)} U_g^{-1} = Q_\beta^{(t)} - \delta_{\alpha\beta}ut\,\mathbb{I}$ and by expanding $e^{\pm i\varsigma_\alpha(u,\mathbf{Q})}$ in u we obtain

$$e^{i\hat{G}_\alpha u}Q_\beta^{(t)}e^{-i\hat{G}_\alpha u} = Q_\beta^{(t)} + i[\eta_\alpha(\mathbf{Q}), Q_\beta^{(t)}]u - \delta_{\alpha\beta}ut\,\mathbb{I} + \omega_1(u, Q_\beta^{(t)})\,, \tag{18}$$

where $\lim_{u\to 0}\frac{\omega_1(u,Q_\beta^{(t)})}{u} = \mathbb{O}$. Replacing (18) in (14), we imply (15).

By expanding $e^{\pm i\hat{G}_\alpha u}$ with respect to u we find

$e^{i\hat{G}_\alpha u}Q_\beta^{(t)}e^{-i\hat{G}_\alpha u} = Q_\beta^{(t)} + i[\hat{G}_\alpha, Q_\beta^{(t)}]u + \omega_2(u, Q_\beta^{(t)})$, where $\lim_{u\to 0}\frac{\omega_2(u,Q_\beta^{(t)})}{u} = \mathbb{O}$, so that if (14) holds then $i[\hat{G}_\alpha, Q_\beta^{(t)}] = -\delta_{\alpha\beta}t\,\mathbb{I} + \hat{o}(t)$ follows; therefore (16) hold.

Finally, $\hat{G}_\alpha = \mu Q_\alpha$, (16.i) imply $[\mu Q_\alpha, \dot{Q}_\beta] = i\delta_{\alpha\beta} = [\hat{G}_\alpha, \dot{Q}_\beta] = [Q_\alpha, \hat{P}_\beta]$, and then a self-adjoint operator $a_\beta(\mathbf{x})$ of $\mathcal{H}_0$ must exists for every $\mathbf{x}$ such that (17) hold. □

If we put $H_0 = \frac{1}{2\mu}\sum_\gamma \left(\hat{P}_\gamma + a_\gamma(\mathbf{Q})\right)^2$, then $i[H_0, Q_\beta] = \frac{1}{\mu}\left(\hat{P}_\beta + a_\beta(\mathbf{Q})\right)$ straightforwardly follows. Whenever (14) holds, Prop. 4.1 implies $i[H_0, Q_\beta] = \dot{Q}_\beta = i[H, Q_\beta]$; therefore

$$H = H_0 + \Phi(\mathbf{Q}) = \frac{1}{2\mu}\sum_\gamma \left(\hat{P}_\gamma + a_\gamma(\mathbf{Q})\right)^2 + \Phi(\mathbf{Q}),$$

where $\Phi(\mathbf{x})$ is a self-adjoint operator of $\mathcal{H}_0$. Then the wave equation is

$$i\frac{\partial}{\partial t}\psi_t = H\psi_t = \left\{\frac{1}{2\mu}\sum_\gamma \left(\hat{P}_\gamma + a_\gamma(\mathbf{Q})\right)^2 + \Phi(\mathbf{Q})\right\}\psi_t. \tag{19}$$

Thus, the usual wave equation of a particle interacting with an electromagnetic field is implied by the fact that it admits σ-conversions that leave unaltered the covariance properties of $\mathbf{Q}^{(t)}$ with respect to boosts at the first order.

Remark 4.1 The condition for the σ-conversion of being Q-covariant and of leaving unaltered the covariance properties of $\mathbf{Q}^{(t)}$ with respect to boosts at first order eliminates the indeterminateness of the σ-conversion, i.e. of the irreducible projective representation $\hat{U} : \mathcal{G} \to \mathcal{U}(\mathcal{H})$ that results from it, highlighted in Remark 3.1.

Indeed, each such projective representation is completely identified by the pair (μ, s). *Now, the mentioned conditions imply* $\hat{G}_\alpha = \mu Q_\alpha$ *and, by Prop. 4.1, that* $\hat{P}_\alpha = \mu\dot{Q}_\alpha - a_\alpha$, *then* $i\mu \equiv [\hat{G}_\alpha, \hat{P}_\alpha] = \mu^2[Q_\alpha, \dot{Q}_\alpha]$. *Therefore* μ *is determined by the physical observables* Q_α *and* $\dot{Q}_\alpha$: *the physics of the system imposes the value of* μ. *On the other hand, the operator* $S^2 = S_1^2 + S_2^2 + S_3^2$ *in an irreducible projective representation is the constant operator* $S^2 = s(s+1)\mathbb{I}$. *So the value* $s \in \frac{1}{2}\mathbb{N}$ *identifying* $\hat{U}$, *and hence the* σ*-conversion, is the unique possible outcome of a measurement of the observable represented by* S^2.

4.2 Invariance Under Spatial Translations

Let us now suppose that the interaction admits a Q-covariant σ-conversion that leaves unaltered the covariance properties of $\mathbf{Q}^{(t)}$ with respect to spatial translations at the first order, i.e. we are supposing that

$$e^{-i\hat{P}_\alpha a} Q_\beta^{(t)} e^{i\hat{P}_\alpha a} = Q_\beta^{(t)} - \delta_{\alpha\beta} a\,\mathbb{I} + o^{(t)}(a), \tag{20}$$

This relation implies

$$(i)\quad [Q_\beta^{(t)}, \hat{P}_\alpha] = i\delta_{\alpha\beta} + \hat{o}(t) \qquad \text{and hence} \qquad (ii)\quad [\dot{Q}_\beta, \hat{P}_\alpha] = \mathbb{O}. \tag{21}$$

Therefore we can state that $\dot{Q}_\beta = v_\beta(\hat{\mathbf{P}})$, where $v_\beta(\mathbf{p})$ is a self-adjoint operator of $\mathcal{H}_0$, for all $\mathbf{p} \in \mathbb{R}^3$. Since $[Q_\alpha, v_\beta(\hat{\mathbf{P}})] = i\frac{\partial v_\beta}{\partial p_\alpha}(\hat{\mathbf{P}})$, Jacobi identity for $\left[Q_\alpha, [H, Q_\beta]\right]$ implies

$$i\frac{\partial v_\beta}{\partial p_\alpha}(\hat{\mathbf{P}}) = [Q_\alpha, \dot{Q}_\beta] = i\left[Q_\alpha, [H, Q_\beta]\right] = [Q_\beta, \dot{Q}_\alpha] = i\frac{\partial v_\alpha}{\partial p_\beta}(\hat{\mathbf{P}}).$$

Hence $\mathbf{v}(\mathbf{p}) = (v_1(\mathbf{p}), v_2(\mathbf{p}), v_3(\mathbf{p}))$ is an irrotational field, so that $v_\alpha(\mathbf{p}) = \frac{\partial F}{\partial p_\alpha}(\mathbf{p})$, for some function F, where $F(\mathbf{p})$ is a self-adjoint operator of $\mathcal{H}_0$. Therefore we are led to the following equalities.

$$\dot{Q}_\alpha = v_\alpha(\hat{\mathbf{P}}) = \frac{\partial F}{\partial p_\alpha}(\hat{\mathbf{P}}) = i[F(\hat{\mathbf{P}}), Q_\alpha] = i[H, Q_\alpha].$$

The last equality implies that a function $\Psi : \mathbb{R}^3 \to \Omega(\mathcal{H}_0)$ exists such that $H = F(\hat{\mathbf{P}}) + \Psi(\mathbf{Q})$. Then the wave equation is

$$i\frac{\partial}{\partial t}\psi_t = H\psi_t = \left\{F(\hat{\mathbf{P}}) + \Psi(\mathbf{Q})\right\}\psi_t. \tag{22}$$

4.3 Invariance Under Both Boosts and Translations

If the covariance properties of $\mathbf{Q}^{(t)}$ with respect to boosts and to spatial translations are invariant at the first order under the σ-conversion, then (19) and (22) imply

$$\begin{aligned} H &= F(\hat{\mathbf{P}}) + \Psi(\mathbf{Q}) = \frac{1}{2\mu}\sum_\gamma \left(\hat{P}_\gamma + a_\gamma(\mathbf{Q})\right)^2 + \Phi(\mathbf{Q}) \\ &= \frac{1}{2\mu}\sum_\gamma \left(\hat{P}_\gamma^2 + a_\gamma(\mathbf{Q})\hat{P}_\gamma + \hat{P}_\gamma a_\gamma(\mathbf{Q}) + a_\gamma^2(\mathbf{Q})\right) + \Phi(\mathbf{Q}). \end{aligned}$$

These equalities, since $a_\gamma(\mathbf{Q})\hat{P}_\gamma + \hat{P}_\gamma a_\gamma(\mathbf{Q}) = [a_\gamma(\mathbf{Q}), \hat{P}_\gamma] + 2\hat{P}_\gamma a_\gamma(\mathbf{Q}) = i\frac{\partial a_\gamma}{\partial x_\gamma}(\mathbf{Q}) + 2\hat{P}_\gamma a_\gamma(\mathbf{Q})$, imply

$$\frac{1}{\mu}\sum_\gamma \hat{P}_\gamma a_\gamma(\mathbf{Q}) = \left(F(\hat{\mathbf{P}}) - \frac{1}{2\mu}\sum_\gamma \hat{P}_\gamma^2\right) + \Psi(\mathbf{Q}) - \frac{i}{2\mu}\sum_\gamma \frac{\partial a_\gamma}{\partial x_\gamma}(\mathbf{Q}) - \Phi(\mathbf{Q}) - \sum_\gamma a_\gamma^2(\mathbf{Q}),$$

i.e.

$$\frac{1}{\mu}\sum_\beta \hat{P}_\beta a_\beta(\mathbf{Q}) = F_1(\hat{\mathbf{P}}) + F_2(\mathbf{Q}), \tag{23}$$

where $F_1(\hat{\mathbf{P}}) = \left(F(\hat{\mathbf{P}}) - \frac{1}{2\mu}\sum_\beta \hat{P}_\beta^2\right)$ and $F_2(\mathbf{Q}) = \Psi(\mathbf{Q}) - \frac{i}{2\mu}\sum_\gamma \frac{\partial a_\gamma}{\partial x_\gamma}(\mathbf{Q}) - \Phi(\mathbf{Q}) - \sum_\gamma a_\gamma^2(\mathbf{Q})$. Then

$$\left[Q_\gamma, \frac{1}{\mu}\sum_\beta \hat{P}_\beta a_\beta(\mathbf{Q})\right] = \frac{i}{\mu}a_\gamma(\mathbf{Q}) = \frac{\partial F_1}{\partial x_\gamma}(\hat{\mathbf{P}}).$$

The commutator of this equation with $\hat{P}_\alpha$ yields

$$[\hat{P}_\alpha, a_\gamma(\mathbf{Q})] = -i\frac{\partial a_\gamma}{\partial x_\alpha}(\mathbf{Q}) = -i\mu\left[\hat{P}_\alpha, \frac{\partial F_1}{\partial x_\gamma}(\hat{\mathbf{P}})\right] = \mathbb{O}.$$

Therefore, $a_\gamma(\mathbf{Q})$ is an operator that acts as follows

$$\left(a_\gamma(\mathbf{Q})\psi\right)(\mathbf{x}) = \hat{a}_\gamma\psi(\mathbf{x}),$$

where $\hat{a}_\gamma$ is an operator of $\mathcal{H}_0$ which does not depend on $\mathbf{x}$.

Thus, $H = \frac{1}{2\mu}\sum_\gamma (\hat{P}_\gamma + \hat{a}_\gamma)^2 + \Phi(\mathbf{Q})$, and the wave equation is

$$i\frac{\partial}{\partial t}\psi_t = H\psi_t = \left\{\frac{1}{2\mu}\sum_\gamma (\hat{P}_\gamma + \hat{a}_\gamma)^2 + \Phi(\mathbf{Q})\right\}\psi_t\,. \tag{24}$$

In the spin-0 case, if the σ-conversion leaves unaltered also the covariance properties of $\mathbf{Q}^{(t)}$ at first order with respect to the rotations subgroup, then $a_\gamma = 0$ [9]. In other words, this twofold invariance characterizes non-magnetic interactions.

4.4 Conclusive Remarks

We have implied the known forms (19) and (24) of the wave equation as consequences of peculiar properties of the interaction. According to our approach, if the σ-conversion admitted by the interaction undergone by the particle does not leave unaltered the covariance properties of $\mathbf{Q}^{(t)}$ with respect to the subgroup of boosts, then the wave equation can be different from the known ones. Therefore our development opens to the possibility for interactions different form those described by (19) and (24). The present work has a general character, and these possibilities have not been specifically explored; however, the present theoretical framework allows for such an investigation. This possibility is precluded by other practiced methods for quantizing the interaction; for instance the methods based on the gauge principle – without entering Quantum Field Theories – immediately lead to wave equations of the form (19). On the other hand, the method of canonical quantization is constitutionally constrained to the wave equations implied by the classical equations.

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