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Abstract. We point out a difficulty that arises in extending the group theoretical approach that deductively establishes the quantum theory of a free particle to the case of an interacting particle. Then we develop an approach which overcomes this difficulty. The result is a theory of an interacting particle where the standard theory is characterized by specific covariance properties related to the interaction.

1. Introduction

A very satisfactory approach to the Quantum Theory of a *free* particle can be accomplished by following group theoretical methods, developed in particular by E.P. Wigner and G.W. Mackey. The key starting point is that Galilei's group $\mathcal{G}$, within a non-relativistic theory, is a group of *symmetry* transformations for a free particle; then Wigner's theorem [1],[2] on the representation of symmetries and Mackey's imprimitivity theorem [3],[4],[5] apply. In so doing the specific Quantum Theory of a free particle is explicitly obtained [3],[6],[7],[8],[9] without invoking canonical quantization, but, instead, by means of a mathematical deduction based on symmetry principles. We recall the main mathematical tools that allow for such an approach in section 2, while the necessary formal implications of the concept of Galileian transformations in the Quantum Theory of a particle are outlined in section 3.

But the *Galileian transformations are not symmetries for an interacting particle*, so that an obstacle is found in extending the approach to this case. To overcome this difficulty, the approaches that in the literature extend the cited group theoretical method to the interacting case (e.g., [7],[8],[9]) adopt, among others, the following assumption.

($\mathcal{P}$) *Each Galileian transformation $g \in \mathcal{G}$ is assigned a unitary operator U_g which realizes the corresponding quantum transformations of states and observables according to $\rho \stackrel{g}{\longrightarrow} U_g \rho U_g^{-1}$ and $A \stackrel{g}{\longrightarrow} U_g A U_g^{-1}$; the correspondence $g \to U_g$ is a projective representation.*

Now, in section 4 we show that ($\mathcal{P}$) forces the hamiltonian operator into the form $H = -\frac{1}{2\mu}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right) + \Phi(x_1, x_2, x_3)$. Therefore a Quantum Theory where ($\mathcal{P}$) is assumed cannot describe all empirically known interactions; in particular it excludes electromagnetic interactions. An empirically more adequate approach must abandon assumption ($\mathcal{P}$).

Then in section 5 we undertake the task of establishing the Quantum Theory of an interacting particle through a group theoretical approach, but without assumption ($\mathcal{P}$). The difficulties raised by the loss of symmetry are addressed through a theoretical development based on the *passive interpretation* of the Galileian transformations and on the concept of *σ-conversion*. We find that the form of the dynamical equation is constrained by the covariance properties preserved by the σ-conversion admitted by the interaction; in particular, we exactly identify which covariance properties characterize electromagnetic interaction among all possible ones.

2. Group theoretical key results

Here we recall the main mathematical tools of the cited group theoretical approach.

2.1. Basic achievements

Given the Hilbert space $\mathcal{H}$ of the Quantum Theory of a physical system, we shall make use of the following operator's structures.

- The set $\Omega(\mathcal{H})$ of all self-adjoint operators of $\mathcal{H}$, which represent *quantum observables*.
- The complete, ortho-complemented lattice $\Pi(\mathcal{H})$ of all projections operators of $\mathcal{H}$, i.e. quantum observables with possible outcomes in $\{0,1\}$.
- The set $\Pi_1(\mathcal{H})$ of all rank one orthogonal projections of $\mathcal{H}$.
- The set $\mathcal{S}(\mathcal{H})$ of all density operators of $\mathcal{H}$, which represent *quantum states*.
- The set $\mathcal{U}(\mathcal{H})$ of all unitary operators of the Hilbert space $\mathcal{H}$.

The main mathematical tools will be used in this work are the *imprimitivity theorem* of Mackey and Wigner's *representation theorem*.

Mackey's *imprimitivity* theorem is a representation theorem for *imprimitivity* systems relative to projective representations [4]. The following definition recalls the notion of projective representation.

Definition 2.1. *Let G be a separable, locally compact group with identity element e. A correspondence $U : G \to \mathcal{U}(\mathcal{H})$, $g \to U_g$, with $U_e = \mathbb{1}$, is a projective representation of G if the following conditions hold.*

i) *A complex function $\sigma : G \times G \to \mathbf{C}$ such that $|\sigma(g_1, g_2)| = 1$ for all $g_1, g_2 \in G$, called multiplier, exists such that $U_{g_1 g_2} = \sigma(g_1, g_2) U_{g_1} U_{g_2}$;*

ii) *for all $\phi, \psi \in \mathcal{H}$, the mapping $g \to \langle U_g \phi \mid \psi \rangle$ is a Borel function in g.*

A projective representation with multiplier σ is called σ-representation.

A projective representation is said to be continuous if for any fixed $\psi \in \mathcal{H}$ the mapping $g \to U_g \psi$ from G into $\mathcal{H}$ is continuous with respect to g.

Let $\mathcal{E}$ be the Euclidean group, i.e. the semi-direct product $\mathcal{E} = \mathbf{R}^3 \circledS SO(3)$ between the group of spatial translations $\mathbf{R}^3$ and the group of spatial proper rotations $SO(3)$; each transformation $g \in \mathcal{E}$ bi-univocally corresponds to the pair $(\mathbf{a}, R) \in \mathbf{R}^3 \times SO(3)$ such that $R\mathbf{x} + \mathbf{a} \equiv g(\mathbf{x})$ is the result of the transformation of the spatial point $\mathbf{x} = (x_1, x_2, x_3)$ by g. The general imprimitivity theorem is an advanced mathematical result, but its formulation becomes very simple when adapted to the case of the Euclidean group $\mathcal{E}$, that is the case we are interested to. Then we introduce the concept of *imprimitivity system* and the theorem for this simple case.

Definition 2.2. *Let $\mathcal{H}$ be the Hilbert space of a σ-representation $g \to U_g$ of the Euclidean group $\mathcal{E}$. A projection valued (PV) measure $E : \mathcal{B}(\mathbf{R}^3) \to \Pi(\mathcal{H})$, $\Delta \to E(\Delta)$ is an imprimitivity system for the σ-representation $g \to U_g$ if the relation*

$$U_g E(\Delta) U_g^{-1} = E(g(\Delta)) \equiv E(R(\Delta) + \mathbf{a})$$

holds for all $(\mathbf{a}, R) \in \mathcal{E}$.

Mackey's theorem of imprimitivity. *If a PV measure $E : \mathcal{B}(\mathbf{R}^3) \to \Pi(\mathcal{H})$ is an imprimitivity system for a continuous σ-representation $g \to U_g$ of the Euclidean group $\mathcal{E}$, then a σ-representation $L : SO(3) \to \mathcal{U}(\mathcal{H}_0)$ exists such that, modulo a unitary isomorphism,*

(M.1) $\mathcal{H} = L_2(\mathbf{R}^3, \mathcal{H}_0)$,

(M.2) $(E(\Delta)\psi)(\mathbf{x}) = \chi_{\Delta}(x)\psi(\mathbf{x})$, *where* χ_{Δ} *is the characteristic functional of* Δ,

(M.3) $(U_g\psi)(\mathbf{x}) = L_R\psi(g^{-1}(\mathbf{x})) \equiv L_R\psi(R^{-1}\mathbf{x} - R^{-1}\mathbf{a})$, *for every* $g = (\mathbf{a}, R) \in \mathcal{E}$.

In the literature different equivalent formulations of Wigner's theorem have been proved [2],[11]. Here we formulate those "Wigner's' theorems" we need for our work.

Wigner's theorem 1. *If $S^{(1)} : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H})$ and $S^{(2)} : \Omega(\mathcal{H}) \to \Omega(\mathcal{H})$ are bijective mappings such that*

$$Tr(S^{(1)}[\rho]S^{(2)}[A]) = Tr[\rho A] \tag{1}$$

holds for all $(\rho, A) \in \mathcal{S}(\mathcal{H}) \times \Omega(\mathcal{H})$ for which $Tr(\rho A)$ exists, then either a unitary operator or an anti-unitary operator U of $\mathcal{H}$ exists such that $S^{(1)}[\rho] = U\rho U^*$ and $S^{(2)}[A] = UAU^*$ for all $(\rho, A) \in \mathcal{S}(\mathcal{H}) \times \Omega(\mathcal{H})$, unique up a phase factor.

Wigner's theorem 2. *If $S : \Pi(\mathcal{H}) \to \Pi(\mathcal{H})$ is an automorphism of $\Pi(\mathcal{H})$, i.e. if it is a bijective mapping such that*

$$E_1 \le E_2 \Leftrightarrow S[E_1] \le S[E_2] \quad \text{and} \quad S[E^{\perp}] = (S[E])^{\perp}, \quad \forall E_1, E_2, E \in \Pi(\mathcal{H}),$$

then either a unitary operator or an anti-unitary operator U of $\mathcal{H}$ exists such that $S(E) = UEU^*$ for all $E \in \Pi(\mathcal{H})$, unique up a phase factor.

2.2. Quantum theoretical implications

Wigner's theorem has very important implications in Quantum Theory. We outline those of interest for our aims.

Let $g \to \hat{U}_g$ be every continuous projective representation of Galilei's group $\mathcal{G}$, i.e. the group generated by $\mathcal{E}$ and by Galileian velocity boosts. Let $\mathcal{T}_1 \subseteq \mathcal{G}$ be the abelian sub-group of spatial translations along the x_1 axis. If $\tau_1(a)$ denotes the translation $(x_1, x_2, x_3) \to (x_1 + a, x_2, x_3)$, then we have $\tau_1(a)\tau_1(b) = \tau_1(a+b)$, so that $\hat{U}_{\tau_1(a+b)} = \sigma(\tau_1(a), \tau_1(b))\,\hat{U}_{\tau_1(a)}\hat{U}_{\tau_1(b)}$. The arbitrary phase factor of each $\hat{U}_{\tau_1(a)}$ can be chosen so that, according to Stone's theorem, $\hat{U}_{\tau_1(a)} = e^{-i\hat{P}_1 a}$, where $\hat{P}_1$ is a self-adjoint operator called the *hermitean generator* of the one parameter unitary sub-group $\{e^{-i\hat{P}_1 a}, a \in \mathbf{R}\} \subseteq \mathcal{U}(\mathcal{H})$. Now, the one-parameter abelian sub-groups $\mathcal{T}_\alpha, \mathcal{R}_\alpha$, $\mathcal{B}_\alpha$ of spatial translation, spatial rotations and Galileian boosts, relative to axis x_α, are all additive; then the argument used for $\mathcal{T}_1$ can be repeated to establish the existence of nine hermitean generators $\hat{P}_\alpha$, $\hat{J}_\alpha$, $\hat{G}_\alpha$, $\alpha \in \{1, 2, 3\}$, of the nine one-parameter unitary subgroups $\{e^{-i\hat{P}_\alpha a_\alpha}, a \in \mathbf{R}\}$, $\{e^{-i\hat{J}_\alpha \theta_\alpha}, \theta_\alpha \in \mathbf{R}\}$, $\{e^{-i\hat{G}_\alpha u_\alpha}, u_\alpha \in \mathbf{R}\}$ representing the sub-groups $\mathcal{T}_\alpha, \mathcal{R}_\alpha$, $\mathcal{B}_\alpha$ according to the projective representation $g \to \hat{U}_g$ of the Galilei's group $\mathcal{G}$. The structural properties of $\mathcal{G}$ as a Lie group imply the validity of the following commutation relations [9].

$$\text{(i) } [\hat{P}_\alpha, \hat{P}_\beta] = \mathbf{0}, \quad \text{(ii) } [\hat{J}_\alpha, \hat{P}_\beta] = i\hat{\epsilon}_{\alpha\beta\gamma}\hat{P}_\gamma, \quad \text{(iii) } [\hat{J}_\alpha, \hat{J}_\beta] = i\hat{\epsilon}_{\alpha\beta\gamma}\hat{J}_\gamma,$$
$$\text{(iv) } [\hat{J}_\alpha, \hat{G}_\beta] = i\hat{\epsilon}_{\alpha\beta\gamma}\hat{G}_\gamma, \quad \text{(v) } [\hat{G}_\alpha, \hat{G}_\beta] = \mathbf{0}, \quad \text{(vi) } [\hat{G}_\alpha, \hat{P}_\beta] = i\delta_{\alpha\beta}\mu \mathbb{I}, \tag{2}$$

where $\hat{\epsilon}_{\alpha,\beta,\gamma}$ is the Levi-civita symbol $\epsilon_{\alpha,\beta,\gamma}$ restricted by the condition $\alpha \ne \gamma \ne \beta$, while μ is a non-zero real number which characterizes the projective representation.

Another important achievement implied by Wigner's theorem is the general evolution law of Quantum Theory [5],[6] with respect to a homogeneous time:

$$\rho_t = U_t = e^{-iHt}\rho e^{iHt} \quad \text{and} \quad \frac{d}{dt}A^{(t)} \equiv \dot{A}^{(t)} = i[H, A^{(t)}]. \tag{3}$$

3. Galileian quantum transformations

In this section we recall the concept of *quantum transformation* corresponding to a Galileian transformation $g \in \mathcal{G}$ according to the *active* interpretation.

In correspondence with each Galileian transformation $g \in \mathcal{G}$ we can conceive *two* mappings $R_g^{(1)}$ and $R_g^{(2)}$ which transform quantum states and quantum observables, respectively, according to the active interpretation of the transformation:

$$\rho \to R_g^{(1)}[\rho], \qquad A \to R_g^{(2)}[A]. \tag{4}$$

For instance, if $h \in \mathcal{G}$ is a pure velocity boost characterized by velocity parameters $(u_1, u_2, u_3) \equiv \mathbf{u}$, then $R_h^{(2)}[A]$ is meant to be the observable whose measurement apparatus ${}^h\mathcal{A}$ is identical to an apparatus $\mathcal{A}$ that measures A, also for its location, but it is endowed with a constant velocity $\mathbf{u}$ with respect to $\mathcal{A}$.

If the theory is specializaed to a localizable particle, then three commuting quantum observables $\mathbf{Q} = (Q_1, Q_2, Q_3)$, the *position* observables, exist such that $R_g^{(2)}[\mathbf{Q}] = g^{-1}(\mathbf{Q})$, where the function $g : \mathbf{R}^3 \to \mathbf{R}^3$, $\mathbf{x} \to g(\mathbf{x})$ defines how $g \in \mathcal{G}$ transforms spatial points $\mathbf{x}$.

Remark 3.1. The transformation of the position at time t, i.e. of $\mathbf{Q}^{(t)} = e^{iHt}\mathbf{Q}e^{-iHt}$, requires a function g_t different[1] from g; in general, the following covariance relations hold.

$$(i) \quad R_g^{(2)}[\mathbf{Q}^{(t)}] = g_t^{-1}(\mathbf{Q}^{(t)}); \quad (ii) \text{ in particular, } R_g^{(2)}[\mathbf{Q}] = g^{-1}(\mathbf{Q}). \tag{5}$$

If all transformations $g \in \mathcal{G}$ are *symmetry transformations* for the quantum system, further conditions can be implied in its Quantum Theory. First of all, $R_g^{(1)}$ and $R_g^{(2)}$ have to be bijective on $\mathcal{S}(\mathcal{H})$ and $\Omega(\mathcal{H})$, respectively. Furthermore, the following statement holds.

$$R_{g_1g_2}^{(1)} = R_{g_1}^{(1)} \circ R_{g_2}^{(1)} \text{ and } R_{g_1g_2}^{(2)} = R_{g_1}^{(2)} \circ R_{g_2}^{(2)}, \text{ for all } g_1g_2 \in \mathcal{G}. \tag{6}$$

The symmetry's character of g entails that the expected values of observables do not change if both the quantum state and the observable are transformed by the same symmetry transformation; then, whenever $Tr(\rho A)$ exists, the following statement holds for every $g \in \mathcal{G}$.

$$Tr(R_g^{(1)}[\rho]R_g^{(2)}[A]) = Tr(\rho A). \tag{7}$$

4. The interacting particle problem

For a free particle, the validity of (7) makes possible to apply Wigner's theorem 1, so that for every $g \in \mathcal{G}$ an operator U_g exists such that $R_g^{(1)}[\rho] = U_g \rho U_g^*$ and $R_g^{(2)}[A] = U_g A U_g^*$; together with (6), these relations imply that the restriction to $\mathcal{E}$ of $g \to U_g$ is a projective representation [5]. Then, the relation $U_g \mathbf{Q} U_g^{-1} = g^{-1}(\mathbf{Q})$ follows from (5.ii); it entails that the spectral PV measure of $\mathbf{Q}$ is an imprimitivity system [5]; therefore we can apply Mackey's theorem. In so doing, to each choice of the inducing representation L in Mackey's theorem and of μ in (2), there corresponds a different theory. The simplest choice, i.e. $L : SO(3) \to \mathbf{C}$, $L(R) = 1$, identifies $\mathcal{H} = L_2(\mathbf{R}^3)$ as the Hilbert space of the theory, and the position operators as $(Q_\alpha \psi) = x_\alpha \psi(\mathbf{x})$; moreover, by making use of Galileian invariance, valid for a free particle, it can be proved that the form of the hamiltonian operator must be $H = -\frac{1}{2\mu}\sum_{\alpha=1}^3 \frac{\partial^2}{\partial x_\alpha^2}$. Hence, the simplest theory

[1] For instance, the pure boost $h \in \mathcal{G}$ characterized by a velocity $\mathbf{u} = (u, 0, 0)$, does not change the position at all; hence $h(\mathbf{x}) = \mathbf{x}$, i.e. $R_h^{(2)}(\mathbf{Q}) = h^{-1}(\mathbf{Q}) = \mathbf{Q}$. But $\mathbf{Q}^{(t)}$ represents the position measured with a delay t, therefore $R_h^{(2)}[\mathbf{Q}^{(t)}] = (Q_1^{(t)} - ut, Q_2, Q_3) \equiv h_t(\mathbf{Q}^{(t)})$, where $h_t(\mathbf{x}) = (x_1 - ut, x_2, x_3)$.

deduced from symmetry principles by the group theoretical approach is the standard Quantum Theory of a spin-0 free particle.

If the system is an *interacting* particle, then the *group of Galileian transformations is not a group of symmetry transformations*, so that conditions (6) and (7) fail to hold and we find an obstacle in extending the group theoretical approach to the interacting particle. However, in the literature several proposals can be found [5],[7],[9] where the group theoretical methods are extended to the interacting case. The aforesaid obstacles are overcome by assuming, more or less implicitly, that the following statement holds (e.g., see [5], page 201 and [7], page 236).

($\mathcal{P}$) *Each Galileian transformation g is represented in the formalism of the Quantum Theory by a unitary or anti-unitary operator U_g in such a way that*

i) $R^{(2)}_g[A] = U_g A U^*_g$ *is the result of the active transformation of the observable A by g;*

ii) $R^{(2)}_{g_1 g_2} = R^{(2)}_{g_1} \circ R^{(2)}_{g_2}$, *for all $g_1, g_2 \in \mathcal{G}$.*

By making use of these assumptions, the cited approaches deduce that in the Quantum Theory of a spin-0 particle undergoing an interaction homogeneous in time the hamiltonian operator H must have the following form, able to describe also interactions with electromagnetic fields.

$$H = \frac{1}{2\mu}\sum_{\alpha=1}^{3}\left(-i\frac{\partial}{\partial x_\alpha} - a_\alpha(\mathbf{Q})\right)^2 + \Phi(\mathbf{x}). \tag{8}$$

Now we prove, instead, the following statement.

($\mathcal{R}$) *Assumption ($\mathcal{P}$) forces the hamiltonian of the Quantum Theory of a spin-0 particle undergoing an interaction homogeneous in time into the form*

$$H = \frac{1}{2\mu}\sum_{\alpha=1}^{3}\left(-i\frac{\partial}{\partial x_\alpha}\right)^2 + \Phi(\mathbf{x}).$$

To prove ($\mathcal{R}$), first we imply from ($\mathcal{P}$) that $g \to U_g$ is a projective representation of the Galileian group $\mathcal{G}$. Then, according to section 2.2, the sub-groups $\mathcal{T}_\alpha$, $\mathcal{R}_\alpha$, $\mathcal{B}_\alpha$ can be represented by the one parameter unitary sub-groups $\{e^{-iP_\alpha a}\}_{a\in\mathbf{R}}$, $\{e^{-iJ_\alpha \theta}\}_{\theta\in\mathbf{R}}$, $\{e^{iG_\alpha u}\}_{u\in\mathbf{R}}$, in such a way that the hermitean generators $P_\alpha, J_\alpha, G_\alpha$ satisfy (2). Once defined the self-adjoint operators $F_\alpha = \frac{G_\alpha}{\mu}$, it can be proved that relations (2) imply that

$$U_g \mathbf{F} U_g^{-1} = g^{-1}(\mathbf{F}). \tag{9}$$

Since the F_α's commute with each other, according to spectral theory, there is a unique PV measure $P : \mathcal{B}(\mathbf{R}^3) \to \Pi(\mathcal{H})$ such that $F_\alpha = \int \lambda dE^{(\alpha)}_\lambda$, where $E^{(1)}_\lambda = P((-\infty, \lambda] \times \mathbf{R}^2)$, $E^{(2)}_\lambda = P(\mathbf{R}\times(-\infty,\lambda]\times\mathbf{R})$, $E^{(3)}_\lambda = P(\mathbf{R}^2\times(-\infty,\lambda])$. Then (9) easily implies that $\Delta \to P(\Delta)$ is an imprimitivity system for the restriction to $\mathcal{E}$ of $g \to U_g$; therefore Mackey's theorem applies. In so doing, the simplest choice for $\mathcal{H}_0$, i.e. $\mathcal{H}_0 = \mathbf{C}$, leads to identify $\mathcal{H}$, F_α, P_α, $\hat{U}_g$ as

$$\mathcal{H} = L_2(\mathbf{R}^3), \quad (F_\alpha\psi)(\mathbf{x}) = x_\alpha\psi(\mathbf{x}), \quad P_\alpha = -i\frac{\partial}{\partial x_\alpha}, \quad (U_g\psi)(\mathbf{x}) = \psi\left(g^{-1}(\mathbf{x})\right). \qquad (10).$$

Both $\mathbf{F}$ and $\mathbf{P}$ in (10) are *complete* systems of operators in $L_2(\mathbf{R}^3)$, and $(\mathbf{F},\mathbf{P})$ is an *irreducible* system of operators [5]. Then we can easily prove the following proposition.

Proposition 4.1. *If ($\mathcal{P}$) holds, then in the simplest Quantum Theory of a localizable interacting particle the equality $\mathbf{F} = \mathbf{Q}$ holds.*

Proof. If $g \in \mathcal{T}_1$ and $(\mathcal{P})$ holds, then $U_g = e^{-iP_\beta a} = 1\!\!I - iP_\beta a + o_1(a)$, where $o_1(a)$ is an infinitesimal operator of order greater than 1 with respect to a; relation (5.ii) implies $[Q_\alpha, P_\beta] = i\delta_{\alpha\beta}1\!\!I$. So, by using (2.vi) we obtain $[F_\alpha - Q_\alpha, P_\beta] = \mathbf{0}$; but (5.ii) for $U_g = e^{iG_\beta u} = 1\!\!I + iG_\beta u + o_2(u)$ implies also $[F_\alpha - Q_\alpha, F_\beta] = \mathbf{0}$, i.e. $F_\alpha - Q_\alpha = c_\alpha 1\!\!I$ ≡constant for the irreducibility of $(\mathbf{F}, \mathbf{P})$; on the other hand, $(\mathcal{P}.\mathrm{i})$ together with (2.iv) and (5.ii) for $U_g = e^{-iJ_\alpha\theta}$ imply $[J_\alpha, F_\beta - Q_\beta] = i\hat{\epsilon}_{\alpha,\beta,\gamma}(F_\gamma - Q_\gamma) = [J_\alpha, c_\beta 1\!\!I] = \mathbf{0}$; thus, $F_\alpha - Q_\alpha = \mathbf{0}$. •

Prop. 4.1 together with (5.i) is sufficient to determine the form of the hamiltonian operator H consistent with $(\mathcal{P})$. First, we determine $[G_\alpha, \dot{Q}_\beta]$. Let us start with

$$e^{iG_\alpha u}\dot{Q}_\beta e^{-iG_\alpha u} = \dot{Q}_\beta + i[G_\alpha, \dot{Q}_\beta]u + o(u). \tag{11}$$

By making use of $\dot{Q}_\beta = i[H, Q_\beta] = \lim_{t\to 0}\frac{Q_\beta^{(t)} - Q_\beta}{t}$ and of $e^{iG_\alpha u}Q_\beta^{(t)}e^{-iG_\alpha u} = Q_\beta^{(t)} - \delta_{\alpha\beta}ut1\!\!I$, implied by (5.i), we also find

$$e^{iG_\alpha u}\dot{Q}_\beta e^{-iG_\alpha u} = \lim_{t\to 0} e^{iG_\alpha u}\frac{Q_\beta^{(t)} - Q_\beta}{t}e^{-iG_\alpha u} = \lim_{t\to 0}\frac{Q_\beta^{(t)} - \delta_{\alpha\beta}ut1\!\!I - Q_\beta}{t} = \dot{Q}_\beta - \delta_{\alpha\beta}u1\!\!I. \tag{12}$$

The comparison between (11) and (12) yields

$$[G_\alpha, \dot{Q}_\beta] = [Q_\alpha, \mu\dot{Q}_\beta] = i\delta_{\alpha\beta}1\!\!I, \text{ which implies } [F_\alpha, \mu\dot{Q}_\beta - P_\beta] = \mathbf{0}. \qquad (13).$$

By using $U_g = e^{-iP_\alpha a}$ and $U_g = e^{-iJ_\alpha\theta}$ instead of $e^{iG_\alpha u}$ in this last argument[2], we obtain $[P_\alpha, \mu\dot{Q}_\beta - P_\beta] = \mathbf{0}$ and $[J_\alpha, \mu\dot{Q}_\beta] = i\hat{\epsilon}_{\alpha,\beta,\gamma}\mu\dot{Q}_\gamma$. These two relations, together with (13), imply $\dot{Q}_\beta = (1/\mu)P_\beta$. To determine H, by making use of (2.vi) we obtain

$$i[H, Q_\beta] = \dot{Q}_\beta = \frac{1}{\mu}P_\beta = i\left[(1/2\mu)(P_1^2 + P_2^2 + P_3^2), \frac{G_\alpha}{\mu}\right] \equiv i[(1/2\mu)(P_1^2 + P_2^2 + P_3^2), Q_\alpha]. \tag{14}$$

Then the completeness of $\mathbf{Q}$ implies that $H - (1/2\mu)(P_1^2 + P_2^2 + P_3^2)$ is a function of $\mathbf{Q}$. Thus

$$H = -\frac{1}{2\mu}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right) + \Phi(\mathbf{Q}). \tag{15}$$

Thus, assumption $(\mathcal{P})$ forbids the description of electro-magnetic interactions.

5. Quantum Theory of an interacting particle

According to the conclusion of the last section, to develop a Quantum Theory of a particle, able to describe also electromagnetic interactions, assumption $(\mathcal{P})$ must be abandoned besides the symmetry's implications (6) and (7). In this section we undertake such a development.

First, in sect. 5.1, we replace the active transformation $A \to R_g^{(2)}[A]$ of observables by the *passive* transformation $A \to S_g[A]$. This allows us to prove, without making use of (7), that a unitary or an anti-unitary operator U_g exists such that $S_g[A] = U_g A U_g^*$, for every $g \in \mathcal{G}$.

Another problematic task is to exclude anti-unitary operators U_g without the condition that $g \to U_g$ is a projective representation, condition necessary in the known arguments that obtain such an exclusion [3],[5], [10]. Such a further problem is addressed and solved in sect. 5.2.

However, our correspondence $U : \mathcal{G} \to \mathcal{U}(\mathcal{H})$, $S_g[A] = U_g A U_g^{-1}$ is yet not a projective representation, so that Mackey's theorem cannot apply. In section 5.3 we argue how $g \to U_g$

[2] It is just by extending the argument to translations ($U_g = e^{-iP_\alpha a}$) and rotations ($U_g = e^{-iJ_\alpha\theta}$) that we can prove $(\mathcal{R})$. If the argument is limited to boosts ($U_g = e^{iG_\alpha a}$) [5],[7],[9] only the loser form (8) of H is obtained.

can be changed into a projective representation by means of a *σ-conversion*. We show that the operators representing position are identifiable with the multiplication operators in the case that the interaction admits a σ-conversion that leaves unaltered the covariance properties of the position with respect to $\mathcal{G}$. Therefore our approach explains such an identification by the existence of a determined covariance property related to the interaction.

In section 5.4 we determine the dynamical law for this class of interactions, whose hamiltonian operator turns out to be more general than the known hamiltonian of a particle in an electromagnetic field. However, in section 5.5 we completely characterize the electromagnetic interaction, among all possible ones, in terms of covariance properties preserved by the σ-conversion.

5.1. Passive interpretation of quantum transformations

To recall the meaning of the passive interpretation we make use of the Quantum Theory of a free particle, where $\mathcal{H} = L_2(\mathbf{R}^3)$ and the position operators Q_α are the multiplication operators defined by $(Q_\alpha\psi)(\mathbf{x}) = x_\alpha\psi(\mathbf{x})$. In fact, these operators Q_α represent the position observables of the particle just as referred to a pre-specified *spatial reference frame* Σ_0; if Σ_g is the frame related to such a pre-fixed Σ_0 just by g, then the position operators with respect to Σ_g must be $g^{-1}(\mathbf{Q})$, in general different from $\mathbf{Q}$. Such a dependence on the reference frame is not an exclusive feature of the position observables; for each $g \in \mathcal{G}$, we can introduce a mapping

$$S_g : \Omega(\mathcal{H}) \to \Omega(\mathcal{H}), \quad A \to S_g[A] \tag{16}$$

with the following interpretation.

(PI) *If the operator A represents a given observable with respect to the pre-fixed reference frame Σ_0, the operator $S_g[A]$ represents that observable with respect to Σ_g.*

Statement (PI) expresses the *passive* interpretation of the transformation g. For instance, since remark 3.1 extends to the mapping S_g, then the operators $\mathbf{Q}^{(t)}$ transform according to

$$S_g[\mathbf{Q}^{(t)}] = g_t^{-1}(\mathbf{Q}^{(t)}), \quad \text{and, as a particular case, } S_g[\mathbf{Q}] = g^{-1}(\mathbf{Q}). \tag{17}$$

The following statements are conceptually implied by the meaning of the passive quantum transformations.

(S.1) For every $g \in \mathcal{G}$, the mapping S_g is bijective.

(S.2) Let f be any fixed real Borel function such that if A is a self-adjoint operator, then $B = f(A)$ is a self-adjoint operator too. Since in Quantum Theory a measurement of the quantum observable $f(A)$ can be performed by measuring A and then transforming the obtained outcome a by the purely mathematical transformation f into the outcome $b = f(a)$ of $f(A)$, the following equality has to hold:

$$f(S_g[A]) = S_g[f(A)]. \tag{18}$$

Conditions (S.1) and (S.2) are sufficient to show further properties of the mappings S_g, according to the following propositions.

Proposition 5.1. *Let $S : \Omega(\mathcal{H}) \to \Omega(\mathcal{H})$ be a bijective mapping such that $S[f(A)] = f(S[A])$, for every function as in (S2). Then the following statements hold.*

i) *If $E \in \Pi(\mathcal{H})$ then $S[E] \in \Pi[\mathcal{H}]$, i.e., the mapping S is an extension of a bijection of $\Pi(\mathcal{H}]$.*

ii) *If $A, B \in \Omega(\mathcal{H})$ and $A + B \in \Omega(\mathcal{H})$, then $[A, B] = \mathbf{0}$ implies $S[A + B] = S[A] + S[B]$.*
This partial additivity immediately implies $S[A] = \mathbf{0}$ if and only if $A = \mathbf{0}$.

iii) *For all* $E, F \in \Pi(\mathcal{H})$, $EF = \mathbf{0}$ *implies* $S[E+F] = S[E] + S[F] \in \Pi(\mathcal{H})$*; as a consequence,* $E \le F$ *if and only if* $S[E] \le S[F]$.

iv) $S[P] \in \Pi_1(\mathcal{H})$ *if and only if* $P \in \Pi_1(\mathcal{H})$.

Proof. (i) If $E \in \Pi(\mathcal{H})$ and $f(\lambda) = \lambda^2$ then $f(E) = E$ holds; so $S[f(E)] = f(S[E])$ implies $(S_g[E])^2 \equiv f(S[E]) = S[E^2] \equiv S[E]$, i.e. $S^2[E] = S[E]$.

(ii) If $[A, B] = \mathbf{0}$ then $C \in \Omega(\mathcal{H})$ and two functions f_a, f_b exist so that $A = f_a(C)$ and $B = f_b(C)$; once defined the function $f = f_a + f_b$, we have $S[A+B] \equiv S[f(C)] = f(S[C]) = f_a(S[C]) + f_b(S[C]) = S[f_a(C)] + S[f_b(C)] \equiv S[A] + S[C]$.

(iii) If $EF = \mathbf{0}$, then $[E, F] = \mathbf{0}$ and $(E + F) \in \Pi(\mathcal{H})$ hold. Statements (i) and (ii) imply $S[E+F] = S[E] + S[F] \in \Pi(\mathcal{H}]$.

(iv) If $P \in \Pi_1(\mathcal{H})$ then $S[P] \in \Pi(\mathcal{H})$ by (i). If $Q \in \Pi_1(\mathcal{H})$ and $Q \le S([P]$ then $P_0 \equiv S^{-1}[Q] \le P$ by (iii); but P is rank 1, therefore $P_0 = P$ and $Q = S[P]$. •

Corollary 5.1. *From prop. 5.1 immediately follows that the restriction of* S *to* $\Pi(\mathcal{H})$ *is a bijection that also satisfies* $S[\mathbf{0}] = \mathbf{0}$, $S[\mathbb{1}] = \mathbb{1}$, $E \le F$ *iff* $S[E] \le S[F]$, $S[E^\perp] = (S[E])^\perp$.

In virtue of corollary 5.1, by Wigner's Theorem 2 the following proposition is easily proved.

Proposition 5.2. *If a mapping* S *satisfies the hypothesis of Prop. 5.1, then a unitary or an anti-unitary operator exists such that* $S[A] = UAU^*$ *for every* $A \in \Omega(\mathcal{H})$*; if another unitary or anti-unitary operator* V *satisfies* $S[A] = VAV^*$ *for every* $A \in \Omega$*, then* $V = e^{i\theta}U$ *with* $\theta \in \mathbf{R}$.

5.2. Further implications

Propositions 5.1 and 5.2 imply that for every transformation $g \in \mathcal{G}$, the corresponding *passive* quantum transformation is given by $A \to S_g[A] = U_g A U_g^*$. If the correspondence $g \to S_g$ satisfied $S_{g_1 g_2} = S_{g_1} \circ S_{g_2}$, so that $g \to U_g$ would be a projective representation, then it could be easily proved, according to [3],[5],[10],[12], that every U_g must be unitary. But in presence of interaction $g \to U_g$ in general *is not a projective representation*. Now we prove that anti-unitary U_g can be excluded under the only hypothesis that the correspondence $g \to S_g$ is continuous according to the following continuity notion given by Bargmann [12].

Definition 5.1. *Given* $P_1, P_2 \in \Pi_1(\mathcal{H})$*, the distance* $d(P_1, P_2)$ *is defined as the minimal distance* $\|\psi_1 - \psi_2\|$ *between vectors* ψ_1, ψ_2 *such that* $P_1 = |\psi_1\rangle\langle\psi_1|$ *and* $P_2 = |\psi_2\rangle\langle\psi_2|$*, i.e.,* $d(P_1, P_2) = [2(1 - |\langle\psi_1 \mid \psi_2\rangle|]^{1/2}$.

A correspondence $g \to T_g$ *from* $\mathcal{G}$ *into the set of all automorphims of* $\Pi(\mathcal{H})$ *is continuous if for any fixed* $P \in \Pi_1(\mathcal{H})$ *the mapping from* $\mathcal{G}$ *into* $\Pi_1(\mathcal{H})$, $g \to T_g[P]$ *is continuous in* g *with respect the distance* d *above defined on* $\Pi_1(\mathcal{H})$.

Bargmann proved that if a correspondence $g \to T_g$ from $\mathcal{G}$ into the set of all automorphisms of $\Pi(\mathcal{H})$ is continuous then the arbitrary phase factor of the operator U_g for which $T_g[A] = U_g A U_g^*$ can be chosen so that $U_g\psi$ is continuous with respect to g in the topology of $\mathcal{H}$ [12].

Proposition 5.3. *If the correspondence* $g \to S_g$ *in (16) is continuous according to def. 5.1, then every operator* U_g *such that* $S_g[A] = U_g A U_g^*$ *for all* $A \in \Omega(\mathcal{H})$ *is unitary.*

Proof. Since $S_e[A] = A = \mathbb{1}A\mathbb{1}^*$, we can choose $U_e = \mathbb{1}$ which is unitary. Hence, because of the continuity of $g \to U_g$, a maximal neighborhood K_e of e must exist so that U_g is unitary for all $g \in K_e$; otherwise a sequence $g_n \to e$ would exist with U_{g_n} anti-unitary and so $\langle\psi \mid \varphi\rangle = \langle U_{g_n}\varphi \mid U_{g_n}\psi\rangle$, and then $\langle\psi \mid \varphi\rangle = \lim_{n\to\infty}\langle U_{g_n}\varphi \mid U_{g_n}\psi\rangle = \langle U_e\varphi \mid U_e\psi\rangle = \langle\varphi \mid \psi\rangle$ which cannot hold for all ψ, φ unless $\mathcal{H}$ is real. Now, such a neighborhood K_e has no boundary, and since $\mathcal{G}$ is a connected group, $K_e = \mathcal{G}$; indeed, if $g_0 \in \partial K_e$, two sequences $g_n \to g_0$ and $h_n \to g_0$ would exist with U_{g_n} unitary and U_{h_n} anti-unitary; therefore, the continuity of U_g would imply that U_{g_0} should simultaneously be unitary and anti-unitary. •

5.3. σ-conversions

By switching to the passive interpretation, under a continuity condition we were able to establish the existence of a continuous correspondence $U : \mathcal{G} \to \mathcal{U}(\mathcal{H})$, $g \to U_g$, such that $S_g[A] = U_g A U_g^{-1}$. The argument of section 4, which shows that assumption $(\mathcal{P})$ is empirically inadequate, can be easily adapted to show that such a correspondence is not a σ-representation, in general, even if the passive interpretation is adopted. Without such a representation property we cannot apply Mackey's theorem to proceed with our approach. Now we address this obstacle.

The correspondence $g \to U_g$, can be *converted* into a σ-representation if we multiply each operator U_g by a suitable unitary operator V_g of $\mathcal{H}$; namely, V_g is a unitary operator such that the correspondence $g \to \hat{U}_g = V_g U_g$ turns out to be a σ-representation. The transition from $g \to U_g$ to $g \to \hat{U}_g = V_g U_g$ will be called *σ-conversion*; the mapping $V : \mathcal{G} \to \mathcal{U}(\mathcal{H})$, $g \to V_g$ that realizes the σ-conversion will be called *σ-conversion mapping*. If $g \to V_g$ is a σ-conversion mapping for $g \to U_g$ and $\theta : \mathcal{G} \to \mathbf{R}$ is a real function, then also $g \to e^{i\theta(g)} V_g$ is a σ-conversion mapping, provided that $e^{i\theta(e)} = 1$. In any case, $V_e = 1\!\!I$ must hold.

If $g \to V_g$ is a σ-conversion mapping for U_g then, according to sect. 2.2, the σ-representation $g \to \hat{U}_g = V_g U_g$ has nine hermitean generators $\hat{P}_\alpha$, $\hat{J}_\alpha$, $\hat{G}_\alpha$ for which (2) hold. Moreover, the common spectral measure of the triple $\mathbf{F} = \hat{\mathbf{G}}/\mu$ is an imprimitivity system for the restriction of $g \to \hat{U}_g$ to $\mathcal{E}$. Then Mackey's theorem allows us to identify $\mathcal{H}$ with $L_2(\mathbf{R}^3, \mathcal{H}_0)$ and explicitly indicates the concrete form of $\mathbf{F}$ and of $\hat{P}_\alpha$, $\hat{J}_\alpha$, $\hat{G}_\alpha$. In the following, we restrict ourselves to the case $\mathcal{H}_0 = \mathbf{C}$; is so doing we can take $\mathcal{H} = L_2(\mathbf{R}^3)$, $(F_\alpha\psi)(\mathbf{x}) = x_\alpha\psi(\mathbf{x})$, $\hat{P}_\alpha = -i\frac{\partial}{\partial x_\alpha}$, $\hat{J}_\alpha = F_\beta \hat{P}_\gamma - F_\gamma \hat{P}_\beta$ where (α, β, γ) is a cyclic permutation of $(1, 2, 3)$, and $\hat{G}_\alpha = \mu F_\alpha$.

But $\hat{U}_g$ *is not* the unitary operator which realizes the *passive* transformation: $S_g[A] \neq \hat{U}_g A \hat{U}_g^{-1}$. Moreover, $\mathbf{F}$ *is not* the triple $\mathbf{Q}$ representing the position. So, our explicit realization of the mathematical formalism of the theory is, in general, devoid of physical significance. But the approach can go on if we restrict our investigation to those interactions which admit σ-conversions $U_g \to \hat{U}_g = V_g U_g$ which are *Q-covariant*, i.e. σ-conversions that *leave unaltered the covariance properties of the position operators* $\mathbf{Q}$, i.e. such that

$$\hat{U}_g \mathbf{Q} \hat{U}_g^{-1} = g^{-1}(\mathbf{Q}) \;\; \forall g \in \mathcal{G}. \tag{19}$$

The proof of Prop. 4.1 can be easily adapted to show that if the interaction admits a Q-covariant σ-conversion then $\mathbf{F} = \mathbf{Q}$ in the present case too. Hence, the operator which represents the position in the mathematical formalism of the theory is the multiplication operator: $Q_\alpha\psi(\mathbf{x}) = x_\alpha\psi(\mathbf{x})$. So, in the present approach the identification of the multiplication operator with the position operator is implied by the possibility that the interaction admits a σ-conversion which preserves the covariance properties of the position operators. The existence of interaction which are not Q-covariant is not excluded, of course.

However, the operators $\hat{U}_g$ continue *to be not* the representative of the transformations of $\mathcal{G}$, i.e., $S_g[A] = \hat{U}_g A \hat{U}_g^{-1}$ does not hold. The following proposition specify how $\hat{U}_g$ relates to the unitary operator U_g that realizes the transformation g according to the passive interpretation.

Proposition 5.4. *The σ-conversion mapping of a Q-covariant σ-conversion has the form* $g \to V_g = e^{i\theta(g,\mathbf{Q})}$, *for some function* θ.

Proof. The relation $V_g U_g \mathbf{Q} U_g^{-1} V_g^{-1} = g^{-1}(\mathbf{Q})$ implied by (19) and (17) imply $V_g(g^{-1}(\mathbf{Q}))V_g^{-1} = g^{-1}(\mathbf{Q})$, i.e. $[V_g, g^{-1}(\mathbf{Q})] = \mathbf{0}$. Then $[V_g, f(g^{-1}(\mathbf{Q}))] = \mathbf{0}$ for every sufficiently regular function f; by taking $f = g$ we have $[V_g, \mathbf{Q}] = \mathbf{0}$. The thesis follows from the completeness of $\mathbf{Q}$ and the unitary character of V_g. •

5.4. Dynamical equation for Q-covariant interactions
Let us consider a velocity boost $g \in \mathcal{G}$ such that $\hat{U}_g = e^{i\hat{G}_\alpha u}$, for an interaction with Q-covariant σ-conversion. Since $\hat{G}_\alpha = \mu F_\alpha = \mu Q_\alpha$, we can write $\hat{U}_g = e^{i\mu Q_\alpha u}$; therefore

$$\hat{U}_g \dot{Q}_\beta \hat{U}_g^{-1} = \dot{Q}_\beta + i\mu[Q_\alpha, \dot{Q}_\beta]u + o_1(u). \tag{20}$$

On the other hand

$$\hat{U}_g \dot{Q}_\beta \hat{U}_g^{-1} = \lim_{t\to 0} V_g U_g \frac{(Q_\beta^{(t)} - Q_\beta)}{t} U_g^{-1} V_g^{-1}. \tag{21}$$

By making use of $U_g Q_\beta^{(t)} U_g^{-1} = Q_\beta^{(t)} - \delta_{\alpha\beta} ut \mathbb{I}$, implied by (17), in (21) and then comparing with (20) we obtain

$$\hat{U}_g \dot{Q}_\beta \hat{U}_g^{-1} = V_g \dot{Q}_\beta V_g^{-1} - \delta_{\alpha\beta} u \mathbb{I} = \dot{Q}_\beta + i\mu[Q_\alpha, \dot{Q}_\beta]u + o_1(u). \tag{22}$$

But Prop. 5.4 implies that $V_g = e^{i\varsigma_\alpha(u,\mathbf{Q})}$, for some fuction ς_α; replacing in (22) we obtain

$$\dot{Q}_\beta + i[\varsigma_\alpha(u,\mathbf{Q}), \dot{Q}_\beta] + o_2(u) - \delta_{\alpha\beta} u \mathbb{I} = \dot{Q}_\beta + i\mu[Q_\alpha, \dot{Q}_\beta]u + o_1(u). \tag{23}$$

Since $e^{i\varsigma_\alpha(0,\mathbf{Q})} = \mathbb{I}$, the expansion of ς_α with respect to u yields $\varsigma_\alpha(u,\mathbf{Q}) = \frac{\partial \varsigma_\alpha}{\partial u}(0,\mathbf{Q})u + o_3(u)$; by replacing this last relation in (23) we obtain $\mu[Q_\alpha, \dot{Q}_\beta] = [\eta_\alpha(\mathbf{Q}), \dot{Q}_\beta] + i\delta_{\alpha\beta}\mathbb{I}$, where $\eta_\alpha(\mathbf{Q}) = \frac{\partial \varsigma_\alpha}{\partial u}(0,\mathbf{Q})$. By replacing $\dot{Q}_\beta = i[H, Q_\beta]$ in this equation we can apply Jacobi's identity, and in so doing we obtain $[Q_\beta, \mu\dot{Q}_\alpha] = [Q_\beta, \dot{\eta}_\alpha(\mathbf{Q})] + i\delta_{\alpha\beta}\mathbb{I}$, i.e.

$$[Q_\beta, \dot{\eta}_\alpha(\mathbf{Q}) - \mu\dot{Q}_\alpha] = -i\delta_{\alpha\beta}\mathbb{I} = [Q_\beta, -\hat{P}_\alpha]. \tag{24}$$

By the completeness of $\mathbf{Q}$, from (24) we imply that an operator $f_\alpha(\mathbf{Q})$, function of $\mathbf{Q}$, must exist such that the equation $\dot{\eta}(\mathbf{Q}) - \mu\dot{Q}_\alpha + \hat{P}_\alpha = f_\alpha(\mathbf{Q})$ holds; then we can rewrite (24) as

$$i[H, \mu Q_\alpha - \eta_\alpha(\mathbf{Q})] = \hat{P}_\alpha - f_\alpha(\mathbf{Q}). \tag{25}$$

This is a general dynamical equation for a localizable particle whose interaction admits Q-covariant σ-conversions.

5.5. Characterization of the electromagnetic interaction
Once derived the general dynamical law (25) for a localizable particle with Q-covariant and homogeneous in time interaction, it is worth to re-discover the equation currently assumed in quantum physics as a particular case of the general equation (25). The nowadays adopted Schroedinger equation for a spin-0 particle has the form

$$i\frac{d}{dt}\psi_t = \left\{ \frac{1}{2m} \sum_{\alpha=1}^{3} [\hat{P}_\alpha - a_\alpha(\mathbf{Q})]^2 + \Phi(\mathbf{Q}) \right\} \psi_t,$$

i.e. the Hamiltonian operator is $H = (1/2\mu)\sum_{\alpha=1}^{3}\{\hat{P}_\alpha - a_\alpha(\mathbf{x})\}^2 + \Phi(\mathbf{Q})$. The consequent Quantum Theory corresponds to the case that the function η_α in the general law (25) is a constant function. Let us show this result.

Proposition 5.5. *The hamiltonian operator H of an interacting particle which admits Q-covariant σ-conversion has the form $H = (1/2\mu)\sum_{\alpha=1}^{3}\{\hat{P}_\alpha - a_\alpha(\mathbf{x})\}^2 + \Phi(\mathbf{Q})$ if and only if the functions η_α in (25) are constant functions. In this case $a_\alpha = f_\alpha$.*

Proof. If η_α is a constant function, then (25) transforms into $i[H,\mu Q_\alpha] = \hat{P}_\alpha - f_\alpha(\mathbf{Q})$ which holds if $H_0 = \frac{1}{2\mu}\sum_{\alpha=1}^{3}[\hat{P}_\alpha - f_\alpha(\mathbf{Q})]^2$ replaces H. Hence the operator $H - H_0$ must be a function Φ of $\mathbf{Q}$ because of the completeness of $\mathbf{Q}$. Then $\eta_\alpha(\mathbf{Q}) = c_\alpha 1\!\!1$ implies $H = \frac{1}{2\mu}\sum_{\alpha=1}^{3}[\hat{P}_\alpha - f_\alpha(\mathbf{Q})]^2 + \Phi(\mathbf{Q})$.

Now we prove the converse. Let us suppose that $H = \frac{1}{2\mu}\sum_{\alpha=1}^{3}\{\hat{P}_\alpha - a_\alpha(\mathbf{Q})\}^2 + \Phi(\mathbf{Q})$; by replacing this H in (25) to obtain

$$\begin{aligned}
i[H,\mu Q_\alpha - \eta_\alpha(\mathbf{Q})] &= \hat{P}_\alpha - f_\alpha(\mathbf{Q}) = \\
&= \frac{i}{2\mu}\sum_\beta[P_\beta^2,\mu Q_\alpha] - \frac{i}{2\mu}\sum_\beta[a_\beta\hat{P}_\beta,\mu Q_\alpha] - \frac{i}{2\mu}\sum_\beta[\hat{P}_\beta a_\beta,\mu Q_\alpha] + \frac{i}{2\mu}\sum_\beta[a_\beta^2,\mu Q_\alpha] + \\
&\quad - \frac{i}{2\mu}\sum_\beta[P_\beta^2,\eta_\alpha] + \frac{i}{2\mu}\sum_\beta[a_\beta\hat{P}_\beta,\eta_\alpha] + \frac{i}{2\mu}\sum_\beta[\hat{P}_\beta a_\beta,\eta_\alpha] - \frac{i}{2\mu}\sum_\beta[a_\beta^2,\eta_\alpha] + \\
&\quad + i[\Phi(\mathbf{Q}),\mu Q_\alpha - \eta_\alpha].
\end{aligned}$$

The last term and the fourth terms of the second and third lines are zero. Then we have

$$\begin{aligned}
i[H,\mu Q_\alpha - \eta_\alpha(\mathbf{Q})] &= \hat{P}_\alpha - f_\alpha(\mathbf{Q}) \qquad (26)\\
&= \hat{P}_\alpha - \frac{i}{2}\sum_\beta(a_\beta\hat{P}_\beta Q_\alpha - Q_\alpha a_\beta\hat{P}_\beta + \hat{P}_\beta a_\beta Q_\alpha - Q_\alpha\hat{P}_\beta a_\beta) + \\
&\quad - \frac{i}{2\mu}\sum_\beta[P_\beta^2,\eta_\alpha] + \frac{i}{2\mu}\sum_\beta(a_\beta\hat{P}_\beta\eta_\alpha - \eta_\alpha a_\beta\hat{P}_\beta + \hat{P}_\beta a_\beta\eta_\alpha - \eta_\alpha\hat{P}_\beta a_\beta) \\
&= \hat{P}_\alpha - \frac{i}{2}\sum_\beta(a_\beta[\hat{P}_\beta,Q_\alpha] + [\hat{P}_\beta,Q_\alpha]a_\beta) - \frac{i}{2\mu}\sum_\beta[P_\beta^2,\eta_\alpha] + \frac{i}{2\mu}\sum_\beta(a_\beta[\hat{P}_\beta,\eta_\alpha] + [\hat{P}_\beta,\eta_\alpha]a_\beta) \\
&= \hat{P}_\alpha - \frac{i}{2}(-2ia_\alpha) - \frac{i}{2\mu}\sum_\beta[P_\beta^2,\eta_\alpha] + \frac{i}{2\mu}\sum_\beta\left(-2i\,a_\beta\frac{\partial\eta_\alpha}{\partial q_\beta}\right) \\
&= \hat{P}_\alpha - a_\alpha + \frac{1}{\mu}\sum_\beta a_\beta\frac{\partial\eta_\alpha}{\partial q_\beta} - \frac{i}{2\mu}\sum[P_\beta^2,\eta_\alpha].
\end{aligned}$$

From the second and last members of this equations' chain we obtain $f_\alpha(\mathbf{Q}) = a_\alpha - \frac{1}{\mu}\sum_\beta a_\beta\frac{\partial\eta_\alpha}{\partial q_\beta} + \frac{i}{2\mu}\sum[P_\beta^2,\eta_\alpha]$, which implies that $\sum_\beta[P_\beta^2,\eta_\alpha]$ is a function of $\mathbf{Q}$. Therefore we have

$$\begin{aligned}
\sum_\beta[P_\beta^2,\eta_\alpha] = \phi_\alpha(\mathbf{Q}) &= \sum_\beta(\hat{P}_\beta[\hat{P}_\beta,\eta_\alpha] + [\hat{P}_\beta,\eta_\alpha]\hat{P}_\beta) = (-i)\sum_\beta\left(\hat{P}_\beta\frac{\partial\eta_\alpha}{\partial q_\beta} + \frac{\partial\eta_\alpha}{\partial q_\beta}\hat{P}_\beta\right) \\
&= (-i)\sum_\beta\left(\left[\hat{P}_\beta,\frac{\partial\eta_\alpha}{\partial q_\beta}\right] + 2\frac{\partial\eta_\alpha}{\partial q_\beta}\hat{P}_\beta\right) = (-i)\sum_\beta\left((-i)\frac{\partial^2\eta_\alpha}{\partial q_\beta^2} + 2\frac{\partial\eta_\alpha}{\partial q_\beta}\hat{P}_\beta\right).
\end{aligned}$$

Then $\sum_\beta\frac{\partial\eta_\alpha}{\partial q_\beta}\hat{P}_\beta$ is a function of $\mathbf{Q}$, and this implies $\sum_\beta\left[Q_\gamma,\frac{\partial\eta_\alpha}{\partial q_\beta}\hat{P}_\beta\right] = \mathbf{0} = \frac{\partial\eta_\alpha}{\partial q_\gamma}[Q_\gamma,\hat{P}_\gamma] = i\frac{\partial\eta_\alpha}{\partial q_\gamma}$ for every γ; therefore $\frac{\partial\eta_\alpha}{\partial q_\gamma} = \mathbf{0}$; thus η_α is a constant function. By using this result in the equality between the second and the last members of (26) we obtain $a_\alpha = f_\alpha$. •

Now we show that the case $\eta_\alpha =$constant is *completely* characterized in terms of covariance properties tied to the interaction.

Proposition 5.6. *The Q-covariant σ-conversions for which $\eta_\alpha(\mathbf{Q}) =$constant are those and only those which leave unaltered the covariant properties of $\mathbf{Q}^{(t)}$ with respect to the Galileian boosts $g \in \mathcal{G}$, at the first order in the boost's velocity, i.e. the σ-conversions such that*

$$\hat{U}_g \mathbf{Q}^{(t)} \hat{U}_g^{-1} = g_t^{-1}(\mathbf{Q}^{(t)}) + o(u), \quad \textit{where } \hat{U}_g = e^{i\hat{G}_\alpha u} \tag{27}$$

at the first order in u.

Proof. Let $\hat{U}_g = e^{i\hat{G}_\alpha u} = V_g U_g$ be the σ-converted unitary operator associated with the Galileian boost g, where $V_g = e^{i\varsigma_\alpha(u,\mathbf{Q})}$ according to Prop.2. By starting from (27) and by expanding $e^{\pm i\varsigma_\alpha(u,\mathbf{Q})}$ with respect to u we obtain

$$e^{i\hat{G}_\alpha u} Q_\beta^{(t)} e^{-i\hat{G}_\alpha u} = V_g U_g Q_\beta^{(t)} U_g^{-1} V_g^{-1} = Q_\beta^{(t)} + i[\eta_\alpha(\mathbf{Q}), Q_\beta^{(t)}]u - \delta_{\alpha\beta} ut\mathbb{I} + o_1(u). \tag{28}$$

The covariance properties of $\mathbf{Q}^{(t)}$ with respect to our Galileian boost g are expressed by $S_g[Q_\beta^{(t)}] = Q_\beta^{(t)} - \delta_{\alpha\beta} ut\mathbb{I}$. Hence the equation

$$e^{i\hat{G}_\alpha u} Q_\beta^{(t)} e^{-i\hat{G}_\alpha u} = Q_\beta^{(t)} - \delta_{\alpha\beta} ut\mathbb{I} + o_2(u) \tag{29}$$

is the condition in order that the σ-conversion leave unaltered the covariance properties of $\mathbf{Q}^{(t)}$ with respect to the Galileian boosts, at the first order in u; the comparison between (28) and (29) implies that such a condition holds if and only if

$$[\eta_\alpha(\mathbf{Q}), Q_\beta^{(t)}] = \mathbf{0}. \tag{30}$$

If η_α =constant then (30) holds, of course. Therefore, in order to prove the proposition, it is sufficient to prove the inverse implication. Hence we suppose that (30) holds. By expanding $e^{\pm i\hat{G}_\alpha u}$ with respect to u we find $e^{i\hat{G}_\alpha u} Q_\beta^{(t)} e^{-\hat{G}_\alpha u} = Q_\beta^{(t)} + i[\hat{G}_\alpha, Q_\beta^{(t)}]u + o_3(u)$, so that (29) holds if and only if $i[\hat{G}_\alpha, Q_\beta^{(t)}] = -\delta_{\alpha\beta} t\mathbb{I}$, i.e. if and only if $[Q_\alpha, (\mu Q_\beta^{(t)})/t] = i\delta_{\alpha\beta} \equiv [Q_\alpha, \hat{P}_\beta]$; then the completeness of $\mathbf{Q}$ implies that a function φ_β must exist such that $(\mu Q_\beta^{(t)})/t - \hat{P}_\beta = \varphi_\beta(\mathbf{Q})$ and hence the following equation holds.

$$Q_\beta^{(t)} = \frac{t}{\mu}(\varphi_\beta(\mathbf{Q}) + \hat{P}_\beta). \tag{31}$$

Now, the condition $[\eta_\alpha(\mathbf{Q}), Q_\alpha] = \mathbf{0}$ is obviously satisfied for all α; but it and the necessary and sufficient condition (30), where (31) is used, imply also $[\eta_\alpha(\mathbf{Q}), \hat{P}_\beta] = \mathbf{0}$; therefore $\eta_\alpha(\mathbf{Q})$ is a constant operator $c_\alpha\mathbb{I}$. •

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