Teorie Relativistiche: Dispensa 1

Four steps for Deriving Lorentz' Transformations

Based on:
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1. Introduction

Let Σ and Σ' be two inertial frames. If a material point P moves according to a law $(t, \vec{x}(t))$ with respect to Σ, with respect to Σ' its motion will be described by a different law $(t', \vec{x'}(t'))$. To obtain the law $(t', \vec{x'}(t'))$ which corresponds to $(t, \vec{x}(t))$ a transformation is needed. The transformations which are coherent with Electromagnetism and the Principle of Relativity, asserting that the physical laws must be the same in all inertial frames, are the Lorentz' transformations.

The usual derivations of Lorentz' Transformations assume that the light travels with the same velocity $c = 1/\sqrt{\epsilon_0\mu_0} \approx 3 \cdot 10^8 m/s$ in all inertial reference frames; this assumption can be motivated by arguing that the solutions of Maxwell equations in empty space are electro-magnetic waves which propagate just with a velocity $c = 1/\sqrt{\epsilon_0\mu_0}$, hence depending only on universal constants. The full comprehension of this motivation requires to solve partial differential equations. To achieve the derivation of Lorentz' transformations, further assumptions of mathematical regularity of the searched transformations, as linearity and continuity, must be required.

Here we present an alternative derivation of Lorentz' Transformations, which makes use of physical principles, without assuming as a postulate the invariance of velocity of light and without any assumption on the regularity of the transformations. In particular, an empirical implication of a well known Electro-magneto-statics law is deduced, and then we impose that this law holds in all inertial frames, according to the Principle of Relativity.

Moreover, we use the principle that electrical charge does not change with velocity, empirically justified by a quite familiar experience: if a solid body, with no (net) charge is heated, then its (net) charge remains zero. Since the increase of the temperature corresponds to an increase of the average kinetic energy of the particles constituting the body, the velocity of electrons becomes much larger than the velocity of more massive nuclei. If the charge was dependent on the velocity, the change of charge due to electrons should overcome the change due to nuclei, and the body would acquire a (net) electrical charge. But this phenomenon has never been observed.

The entire derivation is carried out by means of conceptual reasoning based on symmetry and reciprocity rather than on mathematical technicalities. It consists of the following steps:

(i) to formulate an empirical law as a consequence of electro-magneto-statics;
(ii) to show the invariance of distances transversal to the relative motion;
(iii) to show how lengths parallel to the relative motion transform (Lorentz Contraction);
(iv) to derive Lorentz's Transformations.

2. An empirical consequence of electro-magneto-statics

We consider two parallel rectilinear wires in an inertial frame Σ, which carry a constant electrical current i and an uniform charge density λ, placed at a distance r from each other. The Electro-magnetic fields produced by charges and currents would provoke an attraction or repulsion between the wires. To equilibrate this interaction we introduce a device $\mathcal{I}$ consisting of suitable distributions of springs in the plane of the wires, which act on both wires (Figure 1), and they are chosen in such a way that their action establishes equilibrium. If we change the values of λ or i, the equilibrium is broken, in general. However, if these values are changed into suitable ones $\hat{\lambda}$, $\hat{i}$ equilibrium is kept.

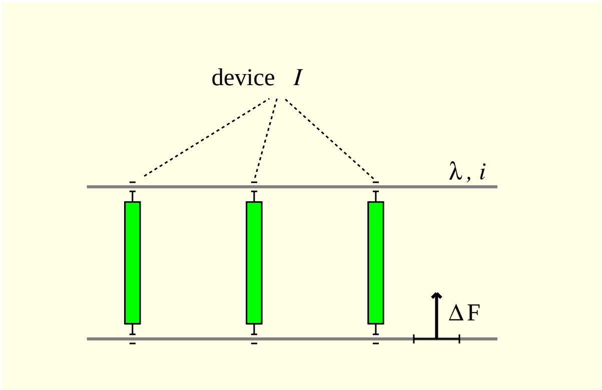

Figure 1. Device $\mathcal{I}$ yields equilibrium

The problem we want to address here is that of finding the pairs $\hat{\lambda}, \hat{i}$ which do not perturb the equilibrium established by $\mathcal{I}$. According to Electro-magneto-statics, the force $\mathbf{\Delta F}$ acting on a piece of wire of length L has modulus $\Delta F = \left|\frac{\lambda^2}{2\pi\epsilon_0 r} - \frac{\mu_0 i^2}{2\pi r}\right| L$ (Fig. 1). Now we argue that if a different pair $(\hat{\lambda}, \hat{i})$ of charge density and current does not perturb the equilibrium established by the action of $\mathcal{I}$, then this implies that

$$\hat{\Delta F} \equiv \left|\frac{\hat{\lambda}^2}{2\pi\epsilon_0 r} - \frac{\mu_0 \hat{i}^2}{2\pi r}\right| L = \Delta F \equiv \left|\frac{\lambda^2}{2\pi\epsilon_0 r} - \frac{\mu_0 i^2}{2\pi r}\right| L.$$

Then, once introduced the function

$$\phi(\lambda, i) = \frac{\lambda^2}{2\pi\epsilon_0 r} - \frac{\mu_0 i^2}{2\pi r}, \tag{1}$$

we formulate the following law.

($\mathcal{L}$) If the action of device $\mathcal{I}$ yields equilibrium for both the two pairs of values (λ, i) and $(\hat{\lambda}, \hat{i})$, then

$$\phi(\hat{\lambda}, \hat{i}) = \phi(\lambda, i).$$

While the argument leading to ($\mathcal{L}$) makes use of the concept of force, it is a remarkable fact that the interpretation of law ($\mathcal{L}$) does not require to attribute the meaning of density of force to function ϕ. In the formulation of ($\mathcal{L}$) function ϕ plays the role of mere empirical tool for ruling over equilibrium in this particular experimental situation. The empirical validity of ($\mathcal{L}$) can be experimentally verified without making reference to any underlying mechanical theory [1]. Thus we consider ($\mathcal{L}$) an empirical law which, according to the Principle of Relativity, must hold in all inertial frames.

3. Invariance of distances transversal to the relative motion

Let Σ' be another frame which moves with a constant velocity v in the direction parallel to the wires, with respect to Σ. If device $\mathcal{I}$ establishes equilibrium between the wires with respect to Σ, then the equilibrium holds also with respect to Σ'; indeed, we shall show now that, as a consequence of the Principle of Relativity, the distance r' between the wires with respect to Σ' must have the same value r of the distance between the wires with respect to Σ. Equilibrium in Σ means that r does not change; then also r' must not change, i.e. equilibrium holds also in Σ'.

Let us consider two inertial frames Σ_1 and Σ_2, which move with respect to each other with a constant velocity v. Let us suppose that two material rectilinear wires are parallel to the direction of the relative motion, and that one wire is at rest with respect to Σ_1, while the other is at rest with respect to Σ_2. By r_1 and r_2 we denote the distances between the two wires with respect to Σ_1 and Σ_2, respectively. Between these distances the inequality $r_1 \leq r_2$ or $r_2 \leq r_1$ must hold. Let us suppose that the first case is realized.

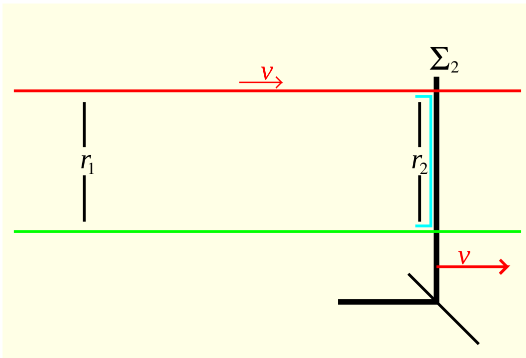

Figure 2. The view of Σ_1

An observer at rest in Σ_2 sees a physical situation identical to that seen by an observer at rest in Σ_1, with the roles of r_1 and r_2 exchanged (Figures 2, 3).

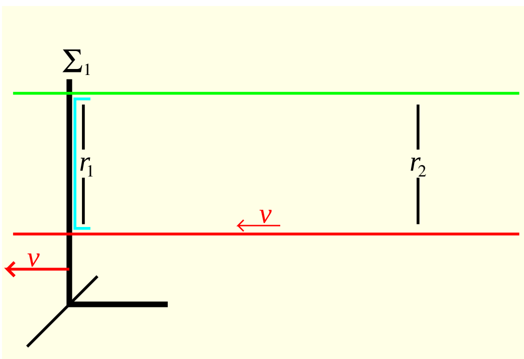

Figure 3. The view of Σ_2

Therefore, as a consequence of the Principle of Relativity $r_2 \leq r_1$ should hold, i.e. $r_1 = r_2$; hence the distances transversal to the relative motions are invariant.

4. Deriving Lorentz contractions

While tranversal distances must be invariant, we are going to show how the lenghts along the direction of the relative motion may undergo modifications.

4.1. Invariant device

Device $\mathcal{I}$ in $(\mathcal{L})$ can be conceived so that it appears to Σ physically indistinguishable from how it appears to Σ'.

Such an "invariant" $\mathcal{I}$ consists of two uniform distributions, $\mathcal{D}_1$ and $\mathcal{D}_2$, of springs, the spring of each distribution being identical with each other. The distibution $\mathcal{D}_1$ is at rest with respect to frame Σ. The second distribution, $\mathcal{D}_2$, is at rest with respect to Σ'. Let ρ_1, ρ_2 be the densities of the springs of $\mathcal{D}_1$ and $\mathcal{D}_2$ with respect to Σ, while ρ_1' and ρ_2' denote the values of these densities with respect to Σ'. Hence, with respect to Σ device $\mathcal{I}$ consists of a distribution at rest with density ρ_1 and another distribution with density ρ_2 which moves with velocity v (Figure 4).

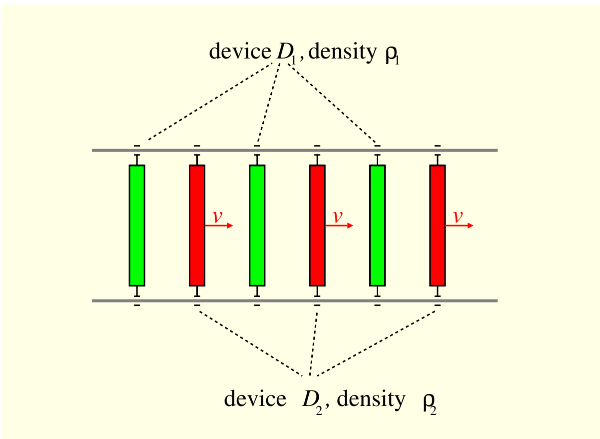

Figure 4. Device $\mathcal{I} = \mathcal{D}_1 + \mathcal{D}_2$ with respect to Σ

With respect to Σ', device $\mathcal{I}$ consists of a distribution at rest with density ρ_2' and a distribution with density ρ_1' which moves with velocity v (Figure 5).

Let the density ρ_2 of $\mathcal{D}_2$ with respect to Σ be chosen in such a way that $\rho_2' = \rho_1$, this implies that if a distribution which moves with velocity v has denstiy ρ_2, in a frame where it is at rest its density must be ρ_1; reciprocally, $\rho_1' = \rho_2$ must hold. Thus, as regards to the densities, device $\mathcal{I}$ appears to Σ identical to that seen by Σ', apart from an exchange of the roles of $\mathcal{D}_1$ and $\mathcal{D}_2$. In the same way, the value of any magnitude characterizing $\mathcal{D}_2$, an which determines its action on the wires – as for instance the angle between wires and springs – is chosen in such a way that device $\mathcal{I}$ appears to

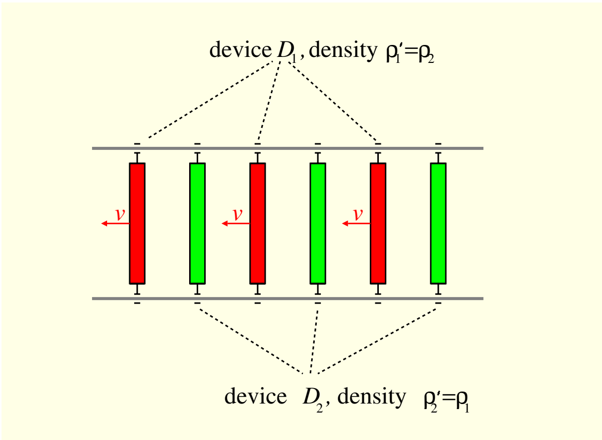

Figure 5. Device $\mathcal{I} = \mathcal{D}_1 + \mathcal{D}_2$ with respect to Σ'

Σ' identical to that seen by Σ. The invariance of $\mathcal{I}$ is completed by the fact that the distance r between the wires is invariant, as proved in the second step.

4.2. Lorentz's Contraction

Now we specialize to the case that all charges together with the two wires of second step are at rest in Σ, so that $\lambda \neq 0$ ed $i = 0$. Let $\mathcal{I}$ be the device so far devised which establishes equilibrium in Σ. As argued in section 3, such an equilibrium must hold also in Σ'. But the device which determines equilibrium in Σ', where charge density and current are λ' and i', is the same (indistinguishable from) that which yields equilibrium in Σ. Therefore, according to law $(\mathcal{L})$ $\phi(\lambda', i') = \phi(\lambda, i = 0)$. Now, in Σ' the current is produced by the motion of the wires, therefore $i' = \lambda' v$. Thus $\phi(\lambda', \lambda' v) = \phi(\lambda, 0)$ must hold; more precisely, by (1)

$$\frac{\lambda^2}{2\pi\epsilon_0 r} = \left| \frac{\lambda'^2}{2\pi\epsilon_0 r} - \frac{\mu_0}{2\pi}\frac{(\lambda' v)^2}{r} \right| \tag{2}$$

which implies $\lambda^2 = \lambda'^2(1 - \epsilon_0\mu_0 v^2)$; therefore, if we set $\epsilon_0\mu_0 = \frac{1}{c^2}$, we have

$$\lambda = \lambda' \sqrt{1 - \frac{v^2}{c^2}}. \tag{3}$$

This result says that the charge density is not invariant.

Now we consider a piece of wire of length L carrying a charge δQ with respect to Σ. With respect to Σ', this same piece of wire has a length L' and carries a charge $\delta Q' = \delta Q$.

Then

$$\lambda = \frac{\delta Q}{L} \qquad \text{and} \qquad \lambda' = \frac{\delta Q'}{L'} = \frac{\delta Q}{L'}.$$

Therefore, (3) becomes

$$\frac{\delta Q}{L} = \frac{\delta Q}{L'}\sqrt{1 - \frac{v^2}{c^2}}$$

which leads to

$$L' = L\sqrt{1 - \frac{v^2}{c^2}}. \tag{4}$$

This relation is known as *Lorentz's Contraction*.

5. Lorentz's Transformations

Let Σ' be an inertial frame which moves with a constant velocity v with respect to frame Σ in the direction of the x axis. The x axes of Σ' and Σ overlap, while the y axis of Σ' lies in the plane xy of Σ and the z axis of Σ' lies in the plane xz of Σ, and such that at time $t = 0$ the two origins of the axes of the two frames coincide. The fourth step consists of three sub-steps:

(I) first, we consider the case in which particle P is at rest in Σ on the x axis and we use Lorentz's Contraction to derive its law of motion in Σ';

(II) we extend (I) to a particle at rest in any spatial point of Σ;

(III) by means of the results of (II), we derive the law of motion in Σ' when the particle moves in Σ according to any known law, i.e. the Lorentz's Transformations.

Sub-step (I). If particle P is at rest in the point of coordinate x of the x axis with respect to Σ, its motion with respect to Σ' is described by the "world line" $(t', x'(t'))$, where $x'(t')$ is the x coordinate of the particle at time t' with respect to Σ'. The particle moves with respect to Σ' with a velocity $-v$.

The value x represents the length l of the segment $[0, x]$ on the spatial x axis of Σ. This length l is related to the length l' of this same segment with respect to Σ' just by Lorentz's Contraction

$$l' = l\sqrt{1 - \frac{v^2}{c^2}} \equiv x\sqrt{1 - \frac{v^2}{c^2}}. \tag{5}$$

But l' is also the difference between the coordinate of P and of the origin of Σ, with respect to Σ', which are $x'(t')$ and $-vt'$:

$$l' = x'(t') - (-vt') = x'(t') + vt'. \tag{6}$$

Therefore, by equating (5) and (6) we get

$$x'(t') = x\sqrt{1 - \frac{v^2}{c^2}} - vt', \quad \text{for all } t'. \tag{7}$$

This relation is the law of motion with respect to Σ' in the case (I). The same argument can be used to show that if a particle is at rest in point x' with respect to Σ', then its world line $(t, x(t))$ with respect to Σ is given by

$$x(t) = x'\sqrt{1 - \frac{v^2}{c^2}} + vt, \quad \text{for all } t. \tag{8}$$

Sub-step (II). Now we consider a particle P at rest in the point of spatial coordinates (x, y, z) with respect to Σ. Our aim is to find the world line $(t', x'(t'), y'(t'), z'(t'))$

with respect to Σ'. Let us imagine a parallelepiped at rest in Σ with a vertex in the origin of Σ, three edges lying along the axes x, y, z, and particle P in the vertex with the greatest distance from the origin (Figure 6). With respect to Σ' the coordinates $(x'(t'), y'(t'), z'(t'))$ of P at time t' coincide with those of this last vertex. The coordinates $y'(t')$ e $z'(t')$ are distances between edges of the parallelepiped parallel to the relative motion, i.e. they are distances transversal to the relative motion and therefore are invariant. For the x coordinate, we can repeat the argument of sub-step (I); thus, with respect to Σ', particle P moves according to

$$\begin{cases} x'(t') = x\sqrt{1-\frac{v^2}{c^2}} - vt' \\ y'(t') = y \\ z'(t) = z \end{cases} \tag{9}$$

for all t', while x, y, z are constant.

Reciprocally if a particle is at rest in the point (x', y', z') with respect to Σ', then its law of motion with respect to Σ is

$$\begin{cases} x(t) = x'\sqrt{1-\frac{v^2}{c^2}} + vt \\ y(t) = y' \\ z(t) = z' \end{cases} \tag{10}$$

for all t, while x', y', z' are constant.

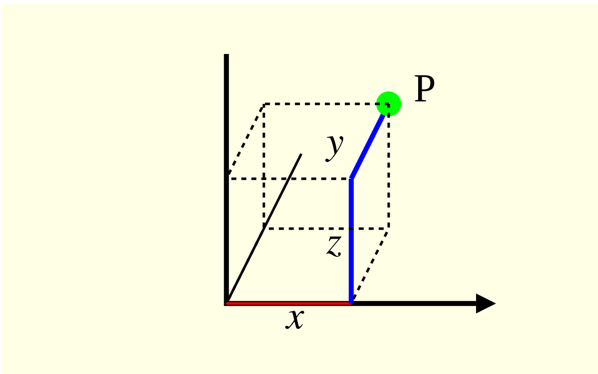

Figure 6. A particle at rest in Σ

Sub-step (III). Now we let particle P move in an arbitrary way with respect to Σ; suppose that at time t_0 it is in the point (x_0, y_0, z_0). Let us imagine a particle A at rest with respect to Σ and another particle B at rest in Σ', such that both A and B collide with particle P just in the space-time point (t_0, x_0, y_0, z_0). Our assumption is simply that this threefold collision occurs also in Σ' in a space-time point denoted by (t'_0, x'_0, y'_0, z'_0) (Figure 7). The space-time point of the collision must belong to the world line of particle A in Σ'. Therefore by (9) we have

$$\begin{cases} x'_0 = x_0\sqrt{1-\frac{v^2}{c^2}} - vt'_0 \\ y'_0 = y_0 \\ z'_0 = z_0 \end{cases} \tag{11}$$

where (x_0, y_0, z_0) are the constant coordinates of A with respect to Σ.

Reciprocally, the space-time point of the impact with respect to Σ must belong to the world line of B in Σ. By (10)

$$\begin{cases} x_0 = x'_0\sqrt{1-\frac{v^2}{c^2}} + vt_0 \\ y_0 = y'_0 \\ z_0 = z'_0 \end{cases} \tag{12}$$

where (x'_0, y'_0, z'_0) are the constant coordinates of B with respect to Σ'.

By rewriting (11) and (12) in explicit form, we get the usual form of Lorentz's Transformations:

$$\begin{cases} x'_0 = \frac{x_0 - vt_0}{\sqrt{1-\frac{v^2}{c^2}}} \\ y'_0 = y_0 \\ z'_0 = z_0 \\ t'_0 = \frac{t_0 - \frac{v}{c^2}x_0}{\sqrt{1-\frac{v^2}{c^2}}}. \end{cases} \tag{13}$$

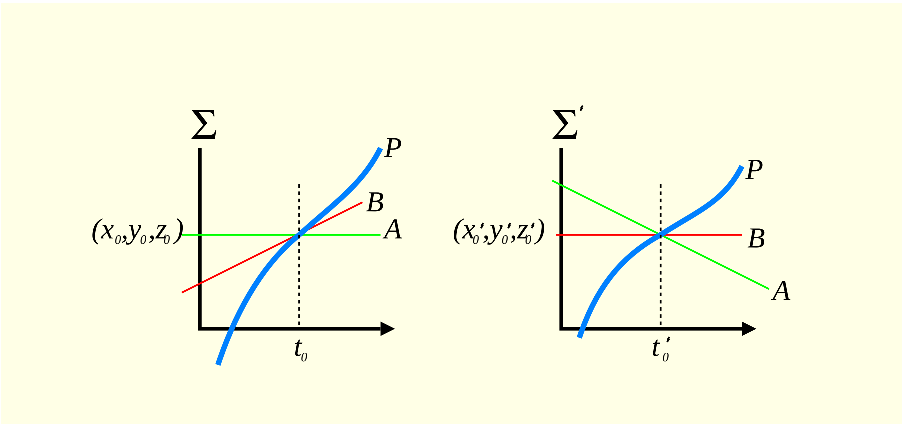

Figure 7. World lines of P, A and B with respect to Σ and to Σ'

References

[1] In so doing, we follow Ampére's attitude: "*The main advantage of the formulas so established [...] is that of remaining independent of the hypotheses, both from those used by their authors in the research of the formulas, and from those that replace the formers in the future. [...] Whatever the physical cause one wants to attribute to the phenomena produced by such an [electro-dynamical] action, the formula obtained by me will be always the expression of real facts. [...] The adopted [method] which led me to the desired results [...] consists in verifying, by means of experience, that an electrical conductor remains in equilibrium under equal forces [...]*" Translated by the authors from Ampère A M 1826 *Théorie des phénomènes électro-dynamiques, uniquement déduite de l'expérience* (Méquignon-Marvis, Paris).

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