Univerität Basel

Herbsemester 2012

Master course A. Surroca - L. Paladino

Some topics on modular functions, elliptic functions and transcendence theory

Sheet of exercises n.10

For all the sheet, let Λ be a complex lattice.

10.1. For $\omega \in \Lambda$, $\omega \neq 0$, let

$$g_\omega(z) = \left(1 - \frac{z}{\omega}\right) e^{\frac{z}{\omega} + \frac{1}{2}\left(\frac{z}{\omega}\right)^2}.$$

Prove that the infinite product

$$\prod_{\omega \in \Lambda \setminus \{0\}} g_\omega(z)$$

converges absolutely and uniformly on every bounded subset of the plane and then defines an entire function.

10.2. Let $\sigma_\Lambda(z) = \sigma(z) = z \prod_{\omega \in \Lambda \setminus \{0\}} g_\omega(z)$ be the Weierstrass σ-function associated to the lattice Λ. Prove that

$$\frac{\sigma'(z)}{\sigma(z)} - \frac{1}{z} = \sum_{\omega \in \Lambda \setminus \{0\}} \left(\frac{1}{z - \omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right).$$

10.3. Define the Weierstrass ζ-function associated to the lattice Λ by

$$\zeta(z) = \zeta_\Lambda(z) := \frac{\sigma'(z)}{\sigma(z)}.$$

Prove that the functions $\sigma_\Lambda(z)$ and $\zeta_\Lambda(z)$ have the following homogeneity properties with respect to z and Λ. For $z \in \mathbb{C}$ and $\lambda \in \mathbb{C}^*$,

$$\sigma_{\lambda\Lambda}(\lambda z) = \lambda\sigma_\lambda(z) \quad \text{and} \quad \zeta_{\lambda\Lambda}(\lambda z) = \lambda^{-1}\zeta_\lambda(z).$$

10.4. Prove that the functions $\sigma(z)$ and $\zeta(z)$ are odd.

10.5. Prove that the functions $\wp_\Lambda(z)$ and $\wp'_\Lambda(z)$ have the following homogeneity properties with respect to z and Λ. For $z \in \mathbb{C}$ and $\lambda \in \mathbb{C}^*$,

$$\wp_{\lambda\Lambda}(\lambda z) = \lambda^{-2}\wp_\Lambda(z) \quad \text{and} \quad \wp'_{\lambda\Lambda}(\lambda z) = \lambda^{-3}\wp_\Lambda(z).$$

10.6. Let $\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ and let $\eta_\Lambda(\omega)$ be the function defined by $\zeta_\Lambda(z + \omega) = \zeta_\Lambda(z) + \eta_\Lambda(\omega)$, for all $z \in \mathbb{C}$ and $\omega \in \Lambda$. Denote $\eta_1 := \eta_\Lambda(\omega_1)$ and $\eta_2 := \eta_\Lambda(\omega_2)$. Prove the Legendre relation

$$-\eta_2\omega_1 + \eta_1\omega_2 = 2\pi i.$$