Univerität Basel

Herbsemester 2012

Master course A. Surroca - L. Paladino

Some topics on modular functions, elliptic functions and transcendence theory

Sheet of exercises n.11

For all the sheet, let Λ be a complex lattice, ζ_Λ be the Riemann zeta function and σ_Λ be the Riemann sigma function.

11.1. Let $k > 2$ be an even integer. Recall that we defined the Eisenstein series of Λ by

$$G_k(\Lambda) = \sum_{\omega \in \Lambda \setminus \{0\}} \frac{1}{\omega^k},$$

which is a lattice function. Prove that in a neighborhood of $z = 0$, the Laurent expansion of $\zeta_\Lambda(z)$ is

$$\zeta_\Lambda(z) = \frac{1}{z} - \sum_{k=1}^{\infty} G_{2k+2}(\Lambda) z^{2k+1}.$$

11.2. For $i \in \{1, 2, ..., n\}$, let $\{a_i\}$ and $\{b_i\}$ be points of the complex plane satisfying $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$. Prove that the function

$$f(z) = \frac{\sigma_\Lambda(z - a_1)...\sigma_\Lambda(z - a_n)}{\sigma_\Lambda(z - b_1)...\sigma_\Lambda(z - b_n)}$$

is Λ-periodic.

11.3. a) Prove that for all $u, v \in \mathbb{C} \setminus \Lambda$,

$$\wp(u) - \wp(v) = -\frac{\sigma(u + v) \cdot \sigma(u - v)}{\sigma^2(u) \cdot \sigma^2(v)}.$$

b) From **a)** deduce that

$$\sigma(2u) = -\wp'(u)\sigma(u)^4.$$