Univerität Basel

Herbsemester 2012

Master course A. Surroca - L. Paladino

Some topics on modular functions, elliptic functions and transcendence theory

Sheet of exercises n.3

3.1. Let

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Show that

$$\frac{d\gamma(\tau)}{d\tau} = \frac{1}{(c\tau + d)^2}, \qquad \text{for } \tau \in \mathcal{H}.$$

3.2. Show that the set $\mathcal{M}_k(\mathrm{SL}_2(\mathbb{Z})$ of the modular forms of weight k forms a vector space over $\mathbb{C}$.

3.3. Let f be a modular form of weight k and let g be a modular form of weight l. Show that the product fg is a modular form of weight $k+l$.

3.4. Show that $\mathcal{S}_k(\mathrm{SL}_2(\mathbb{Z}))$ is a vector subspace of $\mathcal{M}_k(\mathrm{SL}_2(\mathbb{Z}))$ and show that $\mathcal{S}(\mathrm{SL}_2(\mathbb{Z}))$ is an ideal in $\mathcal{M}(\mathrm{SL}_2(\mathbb{Z}))$.

3.5. Let $k \geq 3$ be an integer and let $L' = \mathbb{Z}^2 \setminus \{(0,0)\}$. Show that the series

$$\sum_{(c,d) \in L'} (\sup\{|c|, |d|\})^{-k}$$

converges, by considering the partial sums over expanding squares.

3.6. Let $k \geq 3$ be an integer and let $L' = \mathbb{Z}^2 \setminus \{(0,0)\}$. Fix positive numbers A and B and let

$$\Omega = \{\tau \in \mathcal{H} : |\mathrm{Re}(\tau)| \leq A, \mathrm{Re}(\tau) \geq B\}.$$

Prove that there exists a constant $C > 0$ such that

$$|\tau + \delta| > C\sup\{1, |\delta|\}, \quad \text{for all } \tau \in \Omega \text{ and } \delta \in \mathbb{R}.$$

3.7. By using exercises **3.5** and **3.6**, prove that the series defining $G_k(\tau)$ converges absolutely and uniformly, for $\tau \in \Omega$. Conclude that $G_k(\tau)$ is holomorphic on $\mathcal{H}$.