Univerität Basel

Herbsemester 2012

Master course A. Surroca - L. Paladino

Some topics on modular functions, elliptic functions and transcendence theory

Sheet of exercises n.4

4.1. Let $\gamma \in SL_2(\mathbb{Z})$ and let $L' = \mathbb{Z}^2 \setminus \{(0,0)\}$. Show that right multiplication by γ defines a bijection from L' to L'.

4.2. Let $k \geq 3$. Use Exercise **3.7** to show that G_k is bounded on

$$\Omega = \{\tau \in \mathcal{H} : |\mathrm{Re}(\tau)| \leq A, \mathrm{Re}(\tau) \geq B\},$$

where A and B are fixed positive numbers.

Deduce from Exercise **4.1** that G_k is weakly modular and in particular that $G_k(\tau + 1) = G_k(\tau)$. Show that therefore G_k is bounded as $\mathrm{Im}(\tau) \to \infty$.

4.3. Let $\tau \in \mathcal{H}$. Prove that

(a) $\pi \cot \pi\tau = \frac{1}{\tau} + \sum_{d=1}^{\infty}(\frac{1}{\tau - d} + \frac{1}{\tau + d})$;

(b) $\pi \cot \pi\tau = \pi i - 2\pi i \sum_{m=0}^{\infty} e^{2m\tau\pi i}$, with $m \in \mathbb{N}$.

4.4. As usual let ζ denote the Riemann zeta function and B_k denote the Bernoulli numbers, with $k \in \mathbb{N}$. Use the expressions of Exercise **4.3** (a) and Exercise **4.3** (b) for $\pi \cot \pi\tau$ to show

(a) $\pi\tau \cot \pi\tau = 1 - 2\sum_{k=1}^{\infty} \zeta(2k)\tau^{2k}$;

(b) $\pi\tau \cot \pi\tau = \tau\pi i + \sum_{k=0}^{\infty} B_k \frac{(2\pi\tau i)^k}{k!}$.

Use the two formulas to prove that for $k \geq 2$ even, the Riemann zeta function satisfies

$$2\zeta(k) = -\frac{(2\pi i)^k}{k!} B_k,$$

and in particular $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$, $\zeta(6) = \pi^6/945$. Deduce that the normalized Eisenstein series of weight k

$$E_k(\tau) = \frac{G_k(\tau)}{2\zeta(k)} = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,$$

has rational coefficents with a common denominator.