Some topics on modular functions, elliptic functions and transcendence theory

Sheet of exercises n.6

6.1. Let $\Lambda \in \mathcal{L}$. Prove that the series

$$\sum_{\gamma \in \Lambda \setminus \{0\}} \frac{1}{|\gamma|^k}$$

converges for $k > 2$. (Hint: use Exercise **3.7**).

6.2. Let $k \in \mathbb{Z}$, $k > 1$. For $\Lambda \in \mathcal{L}$, define

$$\mathcal{F}_k(\Lambda) := \sum_{\gamma \in \Lambda \setminus \{0\}} \frac{1}{\gamma^k}.$$

Prove that $\mathcal{F}_k$ is a lattice function of weight k.

6.3. Prove that for $k \geq 4$,

$$\dim(\mathcal{M}_k) = \dim(\mathcal{S}_k) + 1.$$

(Hint: consider the linear form $\varphi : \mathcal{M}_k \to \mathbb{C}$, defined by $\varphi(f) := f(\infty)$).

6.4. Prove that for $k \geq 0$,

$$\dim(\mathcal{M}_k) = \begin{cases} \left[\frac{k}{12}\right] & \text{if } k \equiv 1 \,(\mathrm{mod}\,12) \\ \left[\frac{k}{12}\right] + 1 & \text{if } k \not\equiv 1 \,(\mathrm{mod}\,12). \end{cases}$$