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A STRONG COMPARISON PRINCIPLE FOR THE p-LAPLACIAN

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ABSTRACT. We consider weak solutions of the differential inequality of p-Laplacian type

$$-\Delta_p u - f(u) \leq -\Delta_p v - f(v)$$

such that $u \leq v$ on a smooth bounded domain in $\mathbb{R}^N$ and either u or v is a weak solution of the corresponding Dirichlet problem with zero boundary condition. Assuming that $u < v$ on the boundary of the domain we prove that $u < v$, and assuming that $u \equiv v \equiv 0$ on the boundary of the domain we prove $u < v$ unless $u \equiv v$. The novelty is that the nonlinearity f is allowed to change sign. In particular, the result holds for the model nonlinearity $f(s) = s^q - \lambda s^{p-1}$ with $q > p-1$.

1. Introduction and statement of the results

Throughout this article Ω will be a bounded smooth domain of $\mathbb{R}^N$ with $N \geq 2$. A function $w \in C^{1,\alpha}(\overline{\Omega})$ (see [6, 8, 12]) solves the equation

$$-\Delta_p w = f(w) \text{ weakly on } \Omega$$

(where $p > 1$ and f is a continuous real function that is locally Lipschitz on its domain) if and only if

$$\int_\Omega |\nabla w|^{p-2}\nabla w \cdot \nabla\phi \, dx = \int_\Omega f(w)\phi \, dx \qquad \forall\ \phi \in W_0^{1,p}(\Omega)\ . \tag{1.1}$$

In this paper we consider the following problem:

$$\begin{cases} -\Delta_p w &= f(w) \quad \text{weakly on } \Omega, \\ \quad w &> 0 \quad\quad\ \text{on } \Omega, \\ \quad w &= 0 \quad\quad\ \text{on } \partial\Omega\ . \end{cases} \tag{1.2}$$

We restrict our attention to the case of positive solutions, and we recall that by the strong maximum principle for the p-Laplacian under quite general hypotheses on f (see [10, 13]) any nonnegative solution is in fact strictly positive.

Two functions $u, v \in C^{1,\alpha}(\overline{\Omega})$ satisfy the inequality

$$-\Delta_p u - f(u) \leq -\Delta_p v - f(v) \quad \text{weakly on } \Omega$$

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if and only if

$$\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\psi\ dx - \int_\Omega f(u)\psi\ dx \le \int_\Omega |\nabla v|^{p-2}\nabla v\cdot\nabla\psi\ dx - \int_\Omega f(v)\psi\ dx$$

for every $\psi \in W_0^{1,p}(\Omega)$ such that $\psi \ge 0$ a.e. Throughout this paper we will assume

$$(A)_p \begin{cases} \text{both } u \text{ and } v \text{ are nonnegative on } \Omega, \\ \text{either } u \text{ or } v \text{ solves problem (1.2)}, \\ -\Delta_p u - f(u) \le -\Delta_p v - f(v) \quad \text{weakly on } \Omega\ . \end{cases}$$

We say that a Strong Comparison Principle (SCP for short) holds for two functions $u, v \in C^{1,\alpha}(\overline{\Omega})$ satisfying $(A)_p$ if from the inequalities

$$u \le v \quad \text{on } \Omega$$

we can infer the alternative

$$u < v \text{ on } \Omega \ \text{ unless } \quad u \equiv v \text{ on } \Omega.$$

We want to prove that, under suitable boundary conditions, such an SCP holds.

The novelty of the paper is that f can be a sign changing nonlinearity. For example, our assumptions allow us to consider nonlinearities such as

$$f(s) = s^q - \lambda s^{p-1} \qquad\qquad (\text{ with } q > p-1).$$

Even when f has definite sign, it is well known that this is a hard task due to the nonlinear degenerate nature of the p-Laplace operator. In fact, comparison principles are not equivalent in this case to maximum principles as for the case of linear operators. We refer the readers to [10] and the references therein for an interesting overview on this topic, and we recall here some known results.

In [3] it is proved that, if f is locally Lipschitz, a Strong Comparison Principle holds in any connected component of $\Omega \setminus Z_{u,v}$ where $Z_{u,v} \equiv \{x \in \Omega \,|\, \nabla u(x) = 0 = \nabla v(x)\}$. In [7] it is proved that, if f is positive and nondecreasing, a Strong Comparison Principle holds assuming that u, v are both solutions of problem (1.2) or assuming as the boundary condition in (1.2) that $u < v$ on $\partial\Omega$. The results in [7] have been recently extended to a more general class of operators in [9], where also some interesting estimates on the set of possible touching points are proved. The assumptions of Theorem 1.3 in [9] are equivalent, in our context, to assuming that f is positive and nondecreasing. Also, we point out some interesting results in [1, 2] where the case of solutions of (1.2) is considered and a Strong Comparison Principle is proved for a particular class of problems involving nonlinearities that do not change sign.[1]

Some details of our proofs are similar to the ones in [1, 2]. In particular, we point out that we will use a Divergence Theorem stated and proved in [2], together with some regularity results from [4]. The crucial tool anyway is a general result recently obtained in [5] where the case of positive nonlinearities is considered. Here we adapt Theorem 1.4 in [5] for future use.

[1]The nonlinearities considered in [1, 2] could change sign if the solutions u, v change sign. Anyway this does not occur since the authors show that the solutions are nonnegative.

Lemma 1.1. *Assume* $\frac{2N+2}{N+2} < p \leq 2$ *or* $p \geq 2$*. Let* $u,\ v \in C^{1,\alpha}(\overline{\Omega})$ *satisfy* $(A)_p$ *and* f *satisfy the following hypothesis:*

$$\begin{aligned} &(f_1) \quad f \text{ is continuous on } [0, +\infty)\ , \\ &(f_2) \quad f \text{ is locally Lipschitz continuous on } (0, +\infty)\ . \end{aligned}$$

Assume that u *is a solution of* (1.2) *in* Ω *and assume that* $f(u)$ *has a definite sign on a domain* $\Omega' \subseteq \Omega$ *(let us say* $f(u) > 0$*); if* $u \leq v$ *and* $u \neq v$ *in* Ω'*, then* $u < v$ *in* Ω'*.*

The same result follows assuming that v *is a solution of* (1.2) *in* Ω *and* $f(v)$ *has a definite sign on* Ω'*.*

Remark 1.2. The restriction $p > \frac{2N+2}{N+2}$ allows $|\nabla u|^{p-2}$ to be in $L^1(\Omega)$ (in [4] see Theorem 2.3). Lemma 1.1 follows from Theorem 1.4 in [5] by simple considerations. In Theorem 1.4 of [5] only the assumption $f(u) > 0$ is considered, however it is clear from its proof that the assumption $f(u) < 0$ is equivalent to the assumption $f(u) > 0$. The statement of Lemma 1.1 is a local version of Theorem 1.4 in [5] since it holds in any domain $\Omega' \subseteq \Omega$. Looking at the proof of Theorem 1.4 in [5] this causes only that a local version of Theorem 2.1 in [5] (see also Theorem 1.1 in [4]) is needed. The latter can be found in [11].

The aim of this paper is to deal with sign changing nonlinearities. More precisely, we keep hypothesis $(A)_p$, (f_1), (f_2) without assuming that $f(u)$ or $f(v)$ has definite sign. We simply assume

$$(f_3) \quad f(t) \begin{cases} = 0 & \text{if } t = 0 \text{ or } t = k > 0, \\ < 0 & \text{if } t \in (0, k), \\ > 0 & \text{if } t \in (k, +\infty), \end{cases}$$

$$(f_4) \quad f \text{ is nondecreasing on some open interval } I_k \text{ containing } k\ .$$

We prove the following

Theorem 1.3. *Assume* $\frac{2N+2}{N+2} < p \leq 2$ *or* $p \geq 2$*. Let* $u,\ v \in C^{1,\alpha}$ *satisfy* $(A)_p$ *with* f *fulfilling* $(f_1), (f_2), (f_3), (f_4)$*, and assume that* $u \leq v$ *in* Ω*. Then, if* $u < v$ *on* $\partial\Omega$*, it follows*

$$u < v \ \text{ in } \Omega.$$

Theorem 1.4. *Assume* $\frac{2N+2}{N+2} < p \leq 2$ *or* $p \geq 2$*. Let* $u,\ v \in C^{1,\alpha}$ *both satisfy* (1.2) *with* f *fulfilling* $(f_1), (f_2), (f_3), (f_4)$*, and assume that* $u \leq v$ *in* Ω*. Then, if* $u \equiv v \equiv 0$ *on* $\partial\Omega$*, the following alternative holds:*

$$u < v \ \text{ in } \Omega \quad \text{or} \quad u \equiv v \ \text{ in } \Omega.$$

2. Proof of Theorem 1.3

Let us consider the set where u and v possibly coincide:

$$C_{u,v} = \left\{ x \in \Omega \ : \ u(x) = v(x) \right\}\ .$$

We want to show that $C_{u,v} = \emptyset$. By contradiction, we assume that the closed set $C_{u,v}$ is not empty. This, under our hypothesis, equals $\partial C_{u,v} \neq \emptyset$.

2.1. **Step 1.** We claim that at each $x \in \partial C_{u,v}$ we have $u(x) = k$. We already know that $u \equiv v > 0$ on $C_{u,v} \supset \partial C_{u,v}$ since either u or v is a solution of problem (1.2). Assume by contradiction that there exists some $\bar{x} \in \partial C_{u,v}$ such that $u(\bar{x}) \neq k$. By hypothesis (f_3), we have $f\big(u(\bar{x})\big) \neq 0$. Without loss of generality we can consider $f\big(u(\bar{x})\big) > 0$, and u as a solution of problem (1.2); in this case we can find an open ball $B(\bar{x}, r_{\bar{x}})$ centered at $\bar{x}$ such that $f(u) > 0$ on $B(\bar{x}, r_{\bar{x}})$. Since $\bar{x} \in \partial C_{u,v}$, u can not coincide with v on the whole $B(\bar{x}, r_{\bar{x}})$, thus we can apply Lemma 1.1 getting $u < v$ on $B(\bar{x}, r_{\bar{x}})$, and this contradicts the hypothesis $u(\bar{x}) = v(\bar{x})$.

2.2. **Step 2.** By assuming $C_{u,v} \neq \emptyset$, the function $dist(x, C_{u,v})$ is well defined at each $x \in \Omega$ and we can consider the open set

$$C^{\epsilon}_{u,v} = \big\{x \in \Omega \ : \ dist(x, C_{u,v}) < \epsilon\big\} \qquad \text{(where } \epsilon > 0\text{)}.$$

Since $u \equiv v \equiv k$ on $\partial C_{u,v}$, we can claim that there exists a $\bar{\epsilon} > 0$ such that

$$\forall x \in C^{\bar{\epsilon}}_{u,v} \setminus C_{u,v} \qquad u(x) \in I_k \quad \text{and} \quad v(x) \in I_k. \tag{2.1}$$

On the contrary we would have that

$$\forall \epsilon > 0 \ \exists x_\epsilon \in C^{\epsilon}_{u,v} \setminus C_{u,v} \qquad u(x_\epsilon) \notin I_k \quad \text{or} \quad v(x_\epsilon) \notin I_k.$$

By choosing $\epsilon = \frac{1}{n}$ there would exist a sequence (x_n) such that

$$x_n \in C^{\frac{1}{n}}_{u,v} \setminus C_{u,v} \qquad u(x_n) \notin I_k \quad \text{or} \quad v(x_n) \notin I_k.$$

From this sequence we could extract a subsequence $(x_{n'})$ such that

$$x_{n'} \in C^{\frac{1}{n'}}_{u,v} \setminus C_{u,v} \qquad w(x_{n'}) \notin I_k$$

where w would be either u or v. As Ω is bounded we could extract from $(x_{n'})$ a subsequence $(x_{n''})$ that would necessarily converge to some point $z \in \partial C_{u,v}$ where $w(z) = k$. But this would end the contradiction $w(x_{n''}) \to k$ and $w(x_{n''}) \notin I_k$.

2.3. **Step 3 [Contradiction].** By construction we have that $u < v$ on $\partial C^{\bar{\epsilon}}_{u,v}$. As $\partial C^{\bar{\epsilon}}_{u,v}$ is compact, there exists some $\rho > 0$ such that $u + \rho < v$ on $\partial C^{\bar{\epsilon}}_{u,v}$. Let us consider the function $w_\rho : \bar{\Omega} \to [0, +\infty)$ defined as follows:

$$w_\rho = \begin{cases} \big(u + \rho - v\big)^+ & \text{on } C^{\bar{\epsilon}}_{u,v}, \\ \\ 0 & \text{on } \bar{\Omega} \setminus C^{\bar{\epsilon}}_{u,v}\ . \end{cases}$$

Since $u + \rho < v$ on $\partial C^{\bar{\epsilon}}_{u,v}$, we have that $w_\rho \in W^{1,p}_0\big(\Omega\big)$ and

$$\nabla w_\rho = \begin{cases} \nabla u - \nabla v & \text{where } w_\rho > 0, \\ \\ 0 & \text{elsewhere.} \end{cases}$$

As w_ρ is a test function, we can use it in (1.1) obtaining[2]

$$\begin{aligned}
\int_\Omega \left|\nabla u\right|^{p-2} \nabla u \cdot \nabla w_\rho \quad &= \int_\Omega f(u) w_\rho \\
&\qquad\qquad \left\langle \textit{since } w_\rho = 0 \textit{ on } \Omega \setminus C^{\bar{\epsilon}}_{u,v} \right\rangle \\
&= \int_{C^{\bar{\epsilon}}_{u,v}} f(u) w_\rho \\
&= \int_{C^{\bar{\epsilon}}_{u,v} \setminus C_{u,v}} f(u) w_\rho + \int_{C_{u,v}} f(u) w_\rho \\
&= \int_{C^{\bar{\epsilon}}_{u,v} \setminus C_{u,v}} f(u) w_\rho + \int_{C_{u,v}} f(v) w_\rho \\
&\qquad\qquad \left\langle \textit{ by } (2.1) \textit{ and } (f_4) \right\rangle \\
&\leq \int_{C^{\bar{\epsilon}}_{u,v} \setminus C_{u,v}} f(v) w_\rho + \int_{C_{u,v}} f(v) w_\rho \\
&= \int_{C^{\bar{\epsilon}}_{u,v}} f(v) w_\rho = \int_\Omega f(v) w_\rho \\
&\left\langle \textit{ recall } (A)_p \textit{ and } u \textit{ is a solution of } (1.2) \right\rangle \\
&\leq \int_\Omega \left|\nabla v\right|^{p-2} \nabla v \cdot \nabla w_\rho
\end{aligned}$$

that is,

$$\begin{aligned}
\int_\Omega &\left(\left|\nabla u\right|^{p-2} \quad \nabla u - \left|\nabla v\right|^{p-2} \nabla v\right) \cdot \nabla w_\rho \\
&= \int_{\{w_\rho > 0\}} \left(\left|\nabla u\right|^{p-2} \nabla u - \left|\nabla v\right|^{p-2} \nabla v\right) \cdot (\nabla u - \nabla v) \leq 0 \ .
\end{aligned}$$

By recalling (see for example [3]) that there exists some positive constant C_p such that for each $\eta, \eta' \in \mathbb{R}^N$

$$\left(|\eta|^{p-2}\eta - |\eta'|^{p-2}\eta'\right) \cdot (\eta - \eta') \geq C_p\left(|\eta| + |\eta'|\right)^{p-2} \ |\eta - \eta'|^2 \ ,$$

we get

$$C_p \int_{\{w_\rho > 0\}} \left(\left|\nabla u\right| + \left|\nabla v\right|\right)^{p-2} \left|\nabla u - \nabla v\right|^2 \leq 0 \ .$$

This implies that $u - v$ equals some constant on $\{w_\rho > 0\}$, that is, w_ρ is a constant on $\{w_\rho > 0\}$. By continuity of w_ρ this constant must be zero since $w_\rho = 0$ on $\partial C^{\bar{\epsilon}}_{u,v}$. Thus, we have that $w_\rho \equiv 0$ in $C^{\bar{\epsilon}}_{u,v}$, that is,

$$u + \rho \leq v \quad \text{ on } C^{\bar{\epsilon}}_{u,v} \quad \left(\text{i.e. } u < v \quad \text{ on } C^{\bar{\epsilon}}_{u,v}\right),$$

and this contradicts the fact that $C^{\bar{\epsilon}}_{u,v} \supset C_{u,v} \neq \emptyset$.

[2]We put comments between $\langle \ \rangle$ brackets.

3. Proof of Theorem 1.4

Since $u = 0$ on $\partial\Omega$ and $u \in C^{1,\alpha}(\overline{\Omega})$, there exists an open neighborhood U of $\partial\Omega$ such that $0 < u < k$ on $V = U \cap \Omega$. Since $f(u) < 0$ on V, there the SCP holds by Lemma 1.1; therefore $u \equiv v$ on V or $u < v$ on V. In the latter case we can find a set $\Gamma^\epsilon = \{x \in \Omega \,:\, dist(x, \partial\Omega) \geq \epsilon\}$ for a suitable $\epsilon > 0$ such that $u < v$ on $\partial\Gamma^\epsilon$; exploiting Theorem 1.3, we get $u < v$ on Γ^ϵ, and therefore $u < v$ on Ω. Thus, in the sequel we will consider the former case $\big(u \equiv v$ on $V\big)$ and prove that u must coincide with v on Ω. As in Theorem 1.3, we define $C_{u,v} = \{x \in \Omega \,:\, u(x) = v(x)\}$ and $C^\epsilon_{u,v} = \{x \in \Omega \,:\, dist(x, C_{u,v}) < \epsilon\}$. Let us assume by contradiction that there exists some $x_0 \in \Omega$ such that $u(x_0) < v(x_0)$. Arguing as in Section 2.2 of the proof of Theorem 1.3, we can always find an ϵ such that $0 < \epsilon < dist\big(x_0, \partial C_{u,v}\big)$ and

$$\forall x \in C^\epsilon_{u,v} \setminus C_{u,v} \qquad u(x) \in I_k \quad \text{and} \quad v(x) \in I_k \ .$$

Let us observe that $\big(\Omega \setminus C_{u,v}\big) \cap C^\epsilon_{u,v}$ is a nonempty open set and $\partial C^\epsilon_{u,v} \setminus \partial\Omega \neq \emptyset$ by the assumption $0 < \epsilon < dist\big(x_0, \partial C_{u,v}\big)$. Moreover at each $x \in \partial C^\epsilon_{u,v} \setminus \partial\Omega$ we have $u(x) < v(x)$. By compactness of $\partial C^\epsilon_{u,v} \setminus \partial\Omega$ and continuity of u and v, there exists $\rho > 0$ such that $u + \rho < v$ on $\partial C^\epsilon_{u,v} \setminus \partial\Omega$. Let us define

$$w_\rho = \begin{cases} \big(u + \rho - v\big)^+ & \text{on } C^\epsilon_{u,v}, \\ \\ 0 & \text{on } \Omega \setminus C^\epsilon_{u,v} \ . \end{cases}$$

We have that $w_\rho \in W^{1,p}\big(\Omega\big)$ and

$$\nabla w_\rho = \begin{cases} \nabla u - \nabla v & \text{where } w_\rho > 0, \\ \\ 0 & \text{elsewhere.} \end{cases}$$

Let us observe that $\nabla w_\rho = \nabla u - \nabla v = 0$ on $\overline{V}$. This allows us to use w_ρ "as a test function" even if $w_\rho \notin W^{1,p}_0(\Omega)$; in fact, we will see that the boundary terms appearing in the Divergence Theorem for u and v coincide.

As pointed out in [5], a C^1 solution of (1.2), with f as in our hypothesis, belongs to the class $C^2(\Omega \setminus Z)$, where $Z = \{x \in \Omega \,:\, \nabla u(x) = 0\}$; therefore the generalized derivatives of $|\nabla u|^{p-2} u_{x_i}$ coincide with the classical ones on $\Omega \setminus Z$. Moreover in [5] it was proved that $|\nabla u|^{p-2} u_{x_i} \in W^{1,2}(\Omega)$. Since $w_\rho \in W^{1,2}(\Omega)$ we have that $div\big(w_\rho |\nabla u|^{p-2} \nabla u\big) \in L^1$. The vector field $w_\rho |\nabla u|^{p-2} \nabla u$ belongs to $\big[C^0(\Omega)\big]^N$, so we can apply the Divergence Theorem as stated in [2] pag.742, obtaining

$$\int_\Omega div\big(w_\rho |\nabla u|^{p-2} \nabla u\big)\ dx = \int_{\partial\Omega} w_\rho |\nabla u|^{p-2} \frac{\partial u}{\partial \nu}\ d\sigma.$$

Since $div\big(w_\rho|\nabla u|^{p-2}\nabla u\big) = w_\rho div\big(|\nabla u|^{p-2}\nabla u\big) + |\nabla u|^{p-2}\nabla u \cdot \nabla w_\rho$ and also $-div\big(|\nabla u|^{p-2}\nabla u\big) = f(u)$ almost everywhere, we obtain (exploiting as in Theorem 1.3)

$$\begin{aligned}
\int_\Omega \quad |\nabla u|^{p-2}\nabla u \cdot \nabla w_\rho \, dx &= \int_\Omega f(u)w_\rho \, dx + \int_{\partial\Omega} w_\rho |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} \, d\sigma \\
&= \int_\Omega f(u)w_\rho \, dx + \int_{\partial\Omega} w_\rho |\nabla v|^{p-2}\frac{\partial v}{\partial \nu} \, d\sigma \\
&= \int_{C^\epsilon_{u,v}} f(u)w_\rho \, dx + \int_{\partial\Omega} w_\rho |\nabla v|^{p-2}\frac{\partial v}{\partial \nu} \, d\sigma \\
&= \int_{C^\epsilon_{u,v}\cap C_{u,v}} f(u)w_\rho \, dx + \int_{C^\epsilon_{u,v}\setminus C_{u,v}} f(u)w_\rho \, dx + \int_{\partial\Omega} w_\rho |\nabla v|^{p-2}\frac{\partial v}{\partial \nu} \, d\sigma \\
&= \int_{C^\epsilon_{u,v}\cap C_{u,v}} f(v)w_\rho \, dx + \int_{C^\epsilon_{u,v}\setminus C_{u,v}} f(u)w_\rho \, dx + \int_{\partial\Omega} w_\rho |\nabla v|^{p-2}\frac{\partial v}{\partial \nu} \, d\sigma \\
&\leq \int_{C^\epsilon_{u,v}\cap C_{u,v}} f(v)w_\rho \, dx + \int_{C^\epsilon_{u,v}\setminus C_{u,v}} f(v)w_\rho \, dx + \int_{\partial\Omega} w_\rho |\nabla v|^{p-2}\frac{\partial v}{\partial \nu} \, d\sigma \\
&= \int_\Omega f(v)w_\rho \, dx + \int_{\partial\Omega} w_\rho |\nabla v|^{p-2}\frac{\partial v}{\partial \nu} \, d\sigma \\
&=_{(*)} \int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla w_\rho \, dx \; .
\end{aligned}$$

Arguing as in Theorem 1.3 we conclude the contradiction $w_\rho = 0$ (that is, $u+\rho \leq v$) in $C^\epsilon_{u,v} \supset C_{u,v} \neq \emptyset$.

Remark 3.1. Further extensions are possible. For example, one may guess that in Theorem 1.4 the thesis is still valid by assuming that $u,\ v \in C^{1,\alpha}$ simply satisfy $(A)_p$, instead of both being solutions of (1.2). This is actually true if the function that is not a solution of (1.2) (let us say v) shares the same regularity as the solution u. In such a case the Divergence Theorem can still be applied to v giving, with $(A)_p$, the inequality $\leq$ instead of the equality at the final step $(*)$. However, we skipped such a statement because here shortness and simplicity is our aim.

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