A STRONG COMPARISON PRINCIPLE FOR THE $p$-LAPLACIAN

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Abstract. We consider weak solutions of the differential inequality of $p$-Laplacian type

$$-\Delta_p u - f(u) \leq -\Delta_p v - f(v)$$

such that $u \leq v$ on a smooth bounded domain in $\mathbb{R}^N$ and either $u$ or $v$ is a weak solution of the corresponding Dirichlet problem with zero boundary condition.

Assuming that $u < v$ on the boundary of the domain we prove that $u < v$, and assuming that $u \equiv v \equiv 0$ on the boundary of the domain we prove $u < v$ unless $u \equiv v$. The novelty is that the nonlinearity $f$ is allowed to change sign. In particular, the result holds for the model nonlinearity $f(s) = s^q - \lambda s^{p-1}$ with $q > p - 1$.

1. Introduction and statement of the results

Throughout this article $\Omega$ will be a bounded smooth domain of $\mathbb{R}^N$ with $N \geq 2$. A function $w \in C^{1,\alpha}(\Omega)$ (see [6, 8, 12]) solves the equation

$$-\Delta_p w = f(w) \text{ weakly on } \Omega$$

(where $p > 1$ and $f$ is a continuous real function that is locally Lipschitz on its domain) if and only if

$$\int_\Omega |\nabla w|^{p-2} \nabla w \cdot \nabla \phi \, dx = \int_\Omega f(w)\phi \, dx \quad \forall \ \phi \in W_0^{1,p}(\Omega).$$

In this paper we consider the following problem:

$$\begin{cases}
-\Delta_p w &= f(w) \text{ weakly on } \Omega, \\
w &> 0 \text{ on } \Omega, \\
w &= 0 \text{ on } \partial \Omega.
\end{cases}$$

We restrict our attention to the case of positive solutions, and we recall that by the strong maximum principle for the $p$-Laplacian under quite general hypotheses on $f$ (see [10, 13]) any nonnegative solution is in fact strictly positive.

Two functions $u, v \in C^{1,\alpha}(\Omega)$ satisfy the inequality

$$-\Delta_p u - f(u) \leq -\Delta_p v - f(v) \text{ weakly on } \Omega$$

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if and only if

\[ \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \psi \, dx - \int_{\Omega} f(u) \psi \, dx \leq \int_{\Omega} |\nabla v|^{p-2} \nabla v \cdot \nabla \psi \, dx - \int_{\Omega} f(v) \psi \, dx \]

for every \( \psi \in W^{1,p}_0(\Omega) \) such that \( \psi \geq 0 \text{ a.e.} \). Throughout this paper we will assume

\[
(A)_p \left\{ \begin{array}{ll}
\text{both } u \text{ and } v \text{ are nonnegative on } \Omega, \\
\text{either } u \text{ or } v \text{ solves problem } (1.2), \\
-\Delta_p u - f(u) \leq -\Delta_p v - f(v) \text{ weakly on } \Omega.
\end{array} \right.
\]

We say that a Strong Comparison Principle (SCP for short) holds for two functions \( u, v \in C^{1,\alpha}(\Omega) \) satisfying \((A)_p\) if from the inequalities

\[ u \leq v \text{ on } \Omega \]

we can infer the alternative

\[ u < v \text{ on } \Omega \text{ unless } u \equiv v \text{ on } \Omega. \]

We want to prove that, under suitable boundary conditions, such an SCP holds. The novelty of the paper is that \( f \) can be a sign changing nonlinearity. For example, our assumptions allow us to consider nonlinearities such as

\[ f(s) = s^q - \lambda s^{p-1} \quad (\text{with } q > p - 1). \]

Even when \( f \) has definite sign, it is well known that this is a hard task due to the nonlinear degenerate nature of the \( p \)-Laplace operator. In fact, comparison principles are not equivalent in this case to maximum principles as for the case of linear operators. We refer the readers to [10] and the references therein for an interesting overview on this topic, and we recall here some known results.

In [3] it is proved that, if \( f \) is locally Lipschitz, a Strong Comparison Principle holds in any connected component of \( \Omega \setminus Z_{u,v} \) where \( Z_{u,v} = \{ x \in \Omega | \nabla u(x) = 0 = \nabla v(x) \} \). In [7] it is proved that, if \( f \) is positive and nondecreasing, a Strong Comparison Principle holds assuming that \( u, v \) are both solutions of problem (1.2) or assuming as the boundary condition in (1.2) that \( u < v \) on \( \partial \Omega \). The results in [7] have been recently extended to a more general class of operators in [9], where also some interesting estimates on the set of possible touching points are proved. The assumptions of Theorem 1.3 in [9] are equivalent, in our context, to assuming that \( f \) is positive and nondecreasing. Also, we point out some interesting results in [1, 2] where the case of solutions of (1.2) is considered and a Strong Comparison Principle is proved for a particular class of problems involving nonlinearities that do not change sign.

Some details of our proofs are similar to the ones in [1, 2]. In particular, we point out that we will use a Divergence Theorem stated and proved in [2], together with some regularity results from [4]. The crucial tool anyway is a general result recently obtained in [5] where the case of positive nonlinearities is considered. Here we adapt Theorem 1.4 in [5] for future use.

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1The nonlinearities considered in [1, 2] could change sign if the solutions \( u, v \) change sign. Anyway this does not occur since the authors show that the solutions are nonnegative.
Lemma 1.1. Assume \( \frac{2N+2}{N+2} < p \leq 2 \) or \( p \geq 2 \). Let \( u, v \in C^{1,\alpha}(\Omega) \) satisfy \((A)_p\) and \( f\) satisfy the following hypothesis:

\[
\begin{align*}
(f_1) & \quad f \text{ is continuous on } [0, +\infty), \\
(f_2) & \quad f \text{ is locally Lipschitz continuous on } (0, +\infty).
\end{align*}
\]

Assume that \( u \) is a solution of \((1.2)\) in \( \Omega \) and assume that \( f(u) \) has a definite sign on a domain \( \Omega' \subseteq \Omega \) (let us say \( f(u) > 0 \)); if \( u \leq v \) and \( u \neq v \) in \( \Omega' \), then \( u < v \) in \( \Omega' \).

The same result follows assuming that \( v \) is a solution of \((1.2)\) in \( \Omega \) and \( f(v) \) has a definite sign on \( \Omega' \).

Remark 1.2. The restriction \( p > \frac{2N+2}{N+2} \) allows \(|\nabla u|^{p-2}\) to be in \( L^1(\Omega) \) (in [4] see Theorem 2.3). Lemma [1.1] follows from Theorem 1.4 in [5] by simple considerations. In Theorem 1.4 of [5] only the assumption \( f(u) > 0 \) is considered, however it is clear from its proof that the assumption \( f(u) < 0 \) is equivalent to the assumption \( f(u) > 0 \). The statement of Lemma [1.1] is a local version of Theorem 1.4 in [5] since it holds in any domain \( \Omega' \subseteq \Omega \). Looking at the proof of Theorem 1.4 in [5] this causes only that a local version of Theorem 2.1 in [4] (see also Theorem 1.1 in [4]) is needed. The latter can be found in [11].

The aim of this paper is to deal with sign changing nonlinearities. More precisely, we keep hypothesis \((A)_p\), \((f_1)\), \((f_2)\) without assuming that \( f(u) \) or \( f(v) \) has definite sign. We simply assume

\[
(f_3) \quad f(t) \begin{cases} = 0 & \text{if } t = 0 \text{ or } t = k > 0, \\ < 0 & \text{if } t \in (0, k), \\ > 0 & \text{if } t \in (k, +\infty), \end{cases}
\]

\[
(f_4) \quad f \text{ is nondecreasing on some open interval } I_k \text{ containing } k.
\]

We prove the following

Theorem 1.3. Assume \( \frac{2N+2}{N+2} < p \leq 2 \) or \( p \geq 2 \). Let \( u, v \in C^{1,\alpha} \) satisfy \((A)_p\) with \( f \) fulfilling \((f_1)\), \((f_2)\), \((f_3)\), \((f_4)\), and assume that \( u \leq v \) in \( \Omega \). Then, if \( u < v \) on \( \partial \Omega \), it follows

\[ u < v \text{ in } \Omega. \]

Theorem 1.4. Assume \( \frac{2N+2}{N+2} < p \leq 2 \) or \( p \geq 2 \). Let \( u, v \in C^{1,\alpha} \) both satisfy \((A)_p\) with \( f \) fulfilling \((f_1)\), \((f_2)\), \((f_3)\), \((f_4)\), and assume that \( u \leq v \) in \( \Omega \). Then, if \( u \equiv v \equiv 0 \) on \( \partial \Omega \), the following alternative holds:

\[ u < v \text{ in } \Omega \quad \text{or} \quad u \equiv v \text{ in } \Omega. \]

2. Proof of Theorem 1.3

Let us consider the set where \( u \) and \( v \) possibly coincide:

\[ C_{u,v} = \{ x \in \Omega : u(x) = v(x) \}. \]

We want to show that \( C_{u,v} = \emptyset \). By contradiction, we assume that the closed set \( C_{u,v} \) is not empty. This, under our hypothesis, equals \( \partial C_{u,v} \neq \emptyset \).
2.1. **Step 1.** We claim that at each \( x \in \partial C_{u,v} \) we have \( u(x) = k \). We already know that \( u \equiv v > 0 \) on \( C_{u,v} \supset \partial C_{u,v} \) since either \( u \) or \( v \) is a solution of problem (1.2). Assume by contradiction that there exists some \( \bar{x} \in \partial C_{u,v} \) such that \( u(\bar{x}) \neq k \). By hypothesis (f3), we have \( f(u(\bar{x})) \neq 0 \). Without loss of generality we can consider \( f(u(\bar{x})) > 0 \), and \( u \) as a solution of problem (1.2); in this case we can find an open ball \( B(\bar{x}, r) \) centered at \( \bar{x} \) such that \( f(u) > 0 \) on \( B(\bar{x}, r) \). Since \( \bar{x} \in \partial C_{u,v} \), \( u \) can not coincide with \( v \) on the whole \( B(\bar{x}, r) \), thus we can apply Lemma 1.1 getting \( u < v \) on \( B(\bar{x}, r) \), and this contradicts the hypothesis \( u(\bar{x}) = v(\bar{x}) \).

2.2. **Step 2.** By assuming \( u \equiv v \neq 0 \) on \( \partial C_{u,v} \), the function \( \text{dist}(x, C_{u,v}) \) is well defined at each \( x \in \Omega \) and we can consider the open set

\[
C_{u,v}^\epsilon = \{ x \in \Omega : \text{dist}(x, C_{u,v}) < \epsilon \}
\text{ (where } \epsilon > 0 \).
\]

Since \( u \equiv v \equiv k \) on \( \partial C_{u,v} \), we can claim that there exists a \( \bar{\epsilon} > 0 \) such that

\[
\forall x \in C_{u,v}^\epsilon \setminus C_{u,v} \quad u(x) \in I_k \quad \text{and} \quad v(x) \in I_k.
\]

On the contrary we would have that

\[
\forall \epsilon > 0 \exists x_\epsilon \in C_{u,v}^\epsilon \setminus C_{u,v} \quad u(x_\epsilon) \notin I_k \quad \text{or} \quad v(x_\epsilon) \notin I_k.
\]

By choosing \( \epsilon = \frac{1}{n} \) there would exist a sequence \( (x_n) \) such that

\[
x_n \in C_{u,v}^\frac{1}{n} \setminus C_{u,v} \quad u(x_n) \notin I_k \quad \text{or} \quad v(x_n) \notin I_k.
\]

From this sequence we could extract a subsequence \( (x_{n'}) \) such that

\[
x_{n'} \in C_{u,v}^\frac{1}{n'} \setminus C_{u,v} \quad w(x_{n'}) \notin I_k
\]

where \( w \) would be either \( u \) or \( v \). As \( \Omega \) is bounded we could extract from \( (x_{n'}) \) a subsequence \( (x_{n''}) \) that would necessarily converge to some point \( z \in \partial C_{u,v} \) where \( w(z) = k \). But this would end the contradiction \( w(x_{n''}) \to k \) and \( w(x_{n''}) \notin I_k \).

2.3. **Step 3 [Contradiction].** By construction we have that \( u < v \) on \( \partial C_{u,v}^\epsilon \). As \( \partial C_{u,v}^\epsilon \) is compact, there exists some \( \rho > 0 \) such that \( u + \rho < v \) on \( \partial C_{u,v}^\epsilon \). Let us consider the function \( w_\rho : \Omega \to [0, +\infty) \) defined as follows:

\[
w_\rho = \begin{cases} 
(u + \rho - v)^+ & \text{on } C_{u,v}^\epsilon, \\
0 & \text{on } \bar{\Omega} \setminus C_{u,v}^\epsilon.
\end{cases}
\]

Since \( u + \rho < v \) on \( \partial C_{u,v}^\epsilon \), we have that \( w_\rho \in W^{1,\varphi}_0(\Omega) \) and

\[
\nabla w_\rho = \begin{cases} 
\nabla u - \nabla v & \text{where } w_\rho > 0, \\
0 & \text{elsewhere}.
\end{cases}
\]
As \( w_\rho \) is a test function, we can use it in \( (\text{1.1}) \) obtaining\(^2\)
\[
\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla w_\rho = \int_\Omega f(u) w_\rho \\
= \int_{C^\epsilon_{u,v}} f(u) w_\rho \\
= \int_{C^\epsilon_{u,v}} f(u) w_\rho + \int_{C_{u,v}} f(u) w_\rho \\
= \int_{C^\epsilon_{u,v}} f(u) w_\rho + \int_{C_{u,v}} f(v) w_\rho \\
\quad \text{by (2.1) and (f4)} \\
\leq \int_{C^\epsilon_{u,v}} f(v) w_\rho + \int_{C_{u,v}} f(v) w_\rho \\
= \int_{C^\epsilon_{u,v}} f(v) w_\rho = \int_{\Omega} f(v) w_\rho \\
\quad \text{recall (A)p and u is a solution of (1.2)} \\
\leq \int_{\Omega} |\nabla v|^{p-2} \nabla v \cdot \nabla w_\rho
\]
that is,
\[
\int_\Omega \left( |\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right) \cdot \nabla w_\rho \\
= \int_{\{w_\rho > 0\}} \left( |\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right) \cdot (\nabla u - \nabla v) \leq 0 .
\]

By recalling (see for example [3]) that there exists some positive constant \( C_p \) such that for each \( \eta, \eta' \in \mathbb{R}^N \)
\[
(|\eta|^{p-2} \eta - |\eta'|^{p-2} \eta') \cdot (\eta - \eta') \geq C_p (|\eta| + |\eta'|)^{p-2} |\eta - \eta'|^2 ,
\]
we get
\[
C_p \int_{\{w_\rho > 0\}} \left( |\nabla u| + |\nabla v| \right)^{p-2} |\nabla u - \nabla v|^2 \leq 0 .
\]

This implies that \( u - v \) equals some constant on \( \{w_\rho > 0\} \), that is, \( w_\rho \) is a constant on \( \{w_\rho > 0\} \). By continuity of \( w_\rho \) this constant must be zero since \( w_\rho = 0 \) on \( \partial C^\epsilon_{u,v} \). Thus, we have that \( w_\rho \equiv 0 \) in \( C^\epsilon_{u,v} \), that is,
\[
u + \rho \leq v \quad \text{on } C^\epsilon_{u,v} \quad \text{(i.e. } u < v \text{ on } C^\epsilon_{u,v}),
\]
and this contradicts the fact that \( C^\epsilon_{u,v} \supset C_{u,v} \neq \emptyset \).

\(^2\)We put comments between \( ( ) \) brackets.
3. Proof of Theorem 1.3

Since \( u = 0 \) on \( \partial \Omega \) and \( u \in C^{1,\alpha}(\overline{\Omega}) \), there exists an open neighborhood \( U \) of \( \partial \Omega \) such that \( 0 < u < k \) on \( V = U \cap \Omega \). Since \( f(u) < 0 \) on \( V \), there the SCP holds by Lemma 1.1 therefore \( u \equiv v \) on \( V \) or \( u < v \) on \( V \). In the latter case we can find a set \( \Gamma^c = \{ x \in \Omega : \text{dist}(x, \partial \Omega) \geq \epsilon \} \) for a suitable \( \epsilon > 0 \) such that \( u < v \) on \( \Gamma^c \); exploiting Theorem 1.3, we get \( u < v \) on \( \Gamma^c \), and therefore \( u < v \) on \( \Omega \). Thus, in the sequel we will consider the former case \( (u \equiv v \) on \( V \)) and prove that \( u \) must coincide with \( v \) on \( \Omega \). As in Theorem 1.3, we define \( C_{u,v} = \{ x \in \Omega : u(x) = v(x) \} \) and \( C^c_{u,v} = \{ x \in \Omega : \text{dist}(x, C_{u,v}) < \epsilon \} \). Let us assume by contradiction that there exists some \( x_0 \in \Omega \) such that \( u(x_0) < v(x_0) \). Arguing as in Section 2.2 of the proof of Theorem 1.3, we can always find an \( \epsilon \) such that \( 0 < \epsilon < \text{dist}(x_0, \partial C_{u,v}) \) and

\[
\forall x \in C^c_{u,v} \setminus C_{u,v} \quad u(x) \in I_k \quad \text{and} \quad v(x) \in I_k.
\]

Let us observe that \((\Omega \setminus C_{u,v}) \cap C^c_{u,v} \) is a nonempty open set and \( \partial C^c_{u,v} \setminus \partial \Omega \neq \emptyset \) by the assumption \( 0 < \epsilon < \text{dist}(x_0, \partial C_{u,v}) \). Moreover at each \( x \in \partial C^c_{u,v} \setminus \partial \Omega \) we have \( u(x) < v(x) \). By compactness of \( \partial C^c_{u,v} \setminus \partial \Omega \) and continuity of \( u \) and \( v \), there exists \( \rho > 0 \) such that \( u + \rho < v \) on \( \partial C^c_{u,v} \setminus \partial \Omega \). Let us define

\[
w_\rho = \begin{cases} 
(u + \rho - v)^+ & \text{on } C^c_{u,v}, \\
0 & \text{on } \Omega \setminus C^c_{u,v}.
\end{cases}
\]

We have that \( w_\rho \in W^{1,p}(\Omega) \) and

\[
\nabla w_\rho = \begin{cases} 
\nabla u - \nabla v & \text{where } w_\rho > 0, \\
0 & \text{elsewhere}.
\end{cases}
\]

Let us observe that \( \nabla w_\rho = \nabla u - \nabla v = 0 \) on \( \nabla \). This allows us to use \( w_\rho \) “as a test function” even if \( w_\rho \notin W^{1,p}_0(\Omega) \); in fact, we will see that the boundary terms appearing in the Divergence Theorem for \( u \) and \( v \) coincide.

As pointed out in [5], a \( C^1 \) solution of (1.2), with \( f \) as in our hypothesis, belongs to the class \( C^2(\Omega \setminus Z) \), where \( Z = \{ x \in \Omega : \nabla u(x) = 0 \} \); therefore the generalized derivatives of \( |\nabla u|^{p-2}u_x \) coincide with the classical ones on \( \Omega \setminus Z \). Moreover in [5] it was proved that \( |\nabla u|^{p-2}u_x \in W^{1,2}(\Omega) \). Since \( w_\rho \in W^{1,2}(\Omega) \) we have that \( \text{div}(w_\rho |\nabla u|^{p-2} \nabla u) \in L^1 \). The vector field \( w_\rho |\nabla u|^{p-2} \nabla u \) belongs to \( [C^0(\Omega)]^N \), so we can apply the Divergence Theorem as stated in [2] pag.742, obtaining

\[
\int_\Omega \text{div}(w_\rho |\nabla u|^{p-2} \nabla u) \, dx = \int_{\partial \Omega} w_\rho |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} \, d\sigma.
\]
Since $div(w_{\rho}|\nabla u|^{p-2}\nabla u) = w_{\rho}div(|\nabla u|^{p-2}\nabla u \cdot \nabla w_{\rho})$ and also $-div(|\nabla u|^{p-2}\nabla u) = f(u)$ almost everywhere, we obtain (exploiting as in Theorem 1.3)

$$
\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla w_{\rho} \, dx = \int_\Omega f(u)w_{\rho} \, dx + \int_{\partial\Omega} w_{\rho}|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} \, d\sigma
$$

$$
= \int_\Omega f(u)w_{\rho} \, dx + \int_{\partial\Omega} w_{\rho}|\nabla v|^{p-2}\frac{\partial v}{\partial \nu} \, d\sigma
$$

$$
= \int_{C_{u,v}} f(u)w_{\rho} \, dx + \int_{C_{u,v}\setminus C_{u,v}} f(u)w_{\rho} \, dx + \int_{\partial\Omega} w_{\rho}|\nabla v|^{p-2}\frac{\partial v}{\partial \nu} \, d\sigma
$$

$$
\leq \int_{C_{u,v}\cap C_{u,v}} f(v)w_{\rho} \, dx + \int_{C_{u,v}\setminus C_{u,v}} f(u)w_{\rho} \, dx + \int_{\partial\Omega} w_{\rho}|\nabla v|^{p-2}\frac{\partial v}{\partial \nu} \, d\sigma
$$

$$
= \int_\Omega f(v)w_{\rho} \, dx + \int_{\partial\Omega} w_{\rho}|\nabla v|^{p-2}\frac{\partial v}{\partial \nu} \, d\sigma
$$

$$
= (\ast) \int_\Omega |\nabla u|^{p-2}\nabla v \cdot \nabla w_{\rho} \, dx .
$$

Arguing as in Theorem 1.3 we conclude the contradiction $w_{\rho} = 0$ (that is, $u + \rho \leq v$) in $C_{u,v} \supset C_{u,v} \neq \emptyset$.

Remark 3.1. Further extensions are possible. For example, one may guess that in Theorem 1.4 the thesis is still valid by assuming that $u, v \in C^{1,\alpha}$ simply satisfy $(A)_{\rho}$ instead of both being solutions of $(1.2)$. This is actually true if the function that is not a solution of $(1.2)$ (let us say $v$) shares the same regularity as the solution $u$. In such a case the Divergence Theorem can still be applied to $v$ giving, with $(A)_{\rho}$, the inequality $\leq$ instead of the equality at the final step $(\ast)$. However, we skipped such a statement because here shortness and simplicity is our aim.

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