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A WEAK MAXIMUM PRINCIPLE FOR THE LINEARIZED OPERATOR OF m-LAPLACE EQUATIONS WITH APPLICATIONS TO A NONDEGENERACY RESULT

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Abstract. We consider the Dirichlet problem for positive solutions of the equation $-\Delta_m(u) = f(u)$ in a bounded, smooth domain Ω, with f positive and locally Lipschitz continuous. We prove a weak maximum principle in small domains for the linearized operator that we exploit to prove a weak maximum principle for the linearized operator. We then consider the case $f(s) = s^q$ and prove a nondegeneracy result in weighted Sobolev spaces.

1. Introduction

Let us consider weak $C^1(\overline{\Omega})$ solutions of the problem

$$\begin{cases} -\Delta_m(u) &= f(u) \quad \text{in } \Omega \\ \quad u &> 0 \quad \text{in } \Omega \\ \quad u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded, smooth domain in $\mathbb{R}^N$, $N \geqslant 2$,

$$\Delta_m(u) = \operatorname{div}\left(|Du|^{m-2}Du\right)$$

is the m-Laplace operator, $1 < m < \infty$, and f is a locally Lipschitz-continuous function. It is well known that, since the m-Laplace operator is singular or degenerate elliptic (respectively if $1 < m < 2$ or $m > 2$) in the critical set

$$Z \equiv \{x \in \Omega : D(u)(x) = 0\}, \tag{1.2}$$

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solutions of (1.1) belong generally to the class $C^{1,\tau}$ with $\tau < 1$ (see [10, 18]), and solve (1.1) only in the weak sense.

In the recent paper [8] the authors prove regularity properties of positive solutions u of (1.1) when f is a positive, locally Lipschitz-continuous function. More precisely, they get summability properties of $\frac{1}{|Du|}$ and Sobolev- and Poincaré-type inequalities in weighted Sobolev spaces with weight $\rho = |Du|^{m-2}$. These results are then exploited to prove monotonicity and symmetry results that in Section 2 we recall briefly.

The same estimates are used in [9] to prove a Harnack-type inequality for solutions v of the linearized equation at a fixed solution u of (1.1), as well as a Harnack-type comparison inequality for two solutions of (1.1).

In particular in [9], using this Harnack-type inequality together with symmetry and monotonicity arguments, the following result is obtained:

Theorem 1.1. *Let Ω be a bounded, smooth domain in $\mathbb{R}^N$, convex and symmetric with respect to N orthogonal directions e_i, $i = 1, \ldots, N$, $N \geqslant 2$.*

Assume $\frac{2N+2}{N+2} < m < 2$ or $m > 2$, and let $u \in C^1(\overline{\Omega})$ be a weak solution of (1.1) *with f positive and locally Lipschitz continuous. Then*

$$Z \equiv \{x \in \Omega : D(u)(x) = 0\} = \{0\}. \tag{1.3}$$

Consequently $u \in C^2(\overline{\Omega} \setminus \{0\})$.

Let us recall that the linearized operator at a fixed solution u of (1.1), $L_u(v, \varphi)$, is well defined, for every v and φ belonging to the weighted Sobolev space $H^{1,2}_\rho(\Omega)$ (see Section 2 for details) with weight $\rho \equiv |Du|^{m-2}$, by

$$L_u(v, \varphi) \equiv$$
$$\int_\Omega [|Du|^{m-2}(Dv, D\varphi) + (m-2)|Du|^{m-4}(Du, Dv)(Du, D\varphi) - f'(u)v\varphi]dx.$$

Moreover, $v \in H^{1,2}_\rho(\Omega)$ is a weak solution of the linearized equation if

$$L_u(v, \varphi) = 0 \tag{1.4}$$

for any $\varphi \in H^{1,2}_{0,\rho}(\Omega)$.

More generally, $v \in H^{1,2}_\rho(\Omega)$ is a weak supersolution (subsolution) of (1.4) if $L_u(v, \varphi) \geqslant 0\,(\leqslant 0)$ for any nonnegative $\varphi \in H^{1,2}_{0,\rho}(\Omega)$.

In the present paper we use the Poincarè-type inequality proved in [8], to get a weak maximum principle in small domains for the linearized operator L_u (see Proposition 3.2). Since L_u is naturally defined in the space $H^{1,2}_\rho(\Omega)$, we prove the weak maximum principle under minimal assumption on the solution v. In particular we assume $v \in H^{1,2}_\rho(\Omega)$ if $m > 2$ and $v \in H^{1,2}_\rho(\Omega) \cap C^0(\overline{\Omega})$ if $\frac{2N+2}{N+2} < m < 2$ (the condition $m > \frac{2N+2}{N+2}$ guarantees that the weight ρ is integrable so that the weighted Sobolev space is well defined). By the weak maximum principle in small domains, using u_{x_i} as comparison function, we get the following:

Theorem 1.2 (Weak maximum principle)**.** *Let $u \in C^1(\overline{\Omega})$ be a weak solution of (1.1), where Ω is a bounded, smooth domain of $\mathbb{R}^N$ which is convex in the e_i direction and symmetric with respect to $T_o^{e_i} = \{x \in \mathbb{R} : x \cdot e_i = 0\}$ for N orthogonal directions $e_1, \dots, e_N$. Suppose that f is a locally Lipschitz-continuous function such that $f(s) > 0$ for $s > 0$.*

Let $\Omega_i^- = \{x \in \Omega : x \cdot e_i < 0\}$ and suppose that $v \in H_\rho^{1,2}(\Omega_i^-)$ if $m > 2$ or $v \in H_\rho^{1,2}(\Omega_i^-) \cap C^0(\overline{\Omega_i^-})$ if $\frac{2N+2}{N+2} < m < 2$. Then, if

$$L_u(v, \varphi) = 0 \tag{1.5}$$

for any $\varphi \in H_{0,\rho}^{1,2}(\Omega_i^-)$, we get that v is continuous in $\Omega_i^- \setminus \{0\}$ and, if $v \geqslant 0$ on $\partial\Omega_i^- \setminus \{0\}$ pointwise, it follows $v \geqslant 0$ in Ω_i^-.

Analogously, if $\Omega' \subset \Omega_i^-$ and $v \geqslant 0$ on $\partial\Omega' \setminus \{0\}$ we get $v \geqslant 0$ in Ω'.

As an application we consider in $\mathbb{R}^2$ the case of $f(s) = s^q$ with $q > \max\{1, (m-1)\}$ (the case $q \leqslant m-1$ has been well studied) and we prove the following nondegeneracy result:

Theorem 1.3. *Let $u \in C^1(\overline{\Omega})$ be a weak solution of (1.1), where Ω is a bounded, smooth domain in $\mathbb{R}^2$ which is convex in the e_i direction and symmetric with respect to $T_o^{e_i} = \{x \in \mathbb{R} : x \cdot e_i = 0\}$ for N orthogonal directions $e_1, \dots, e_N$. Suppose that $f(s) = s^q$ with $q > \max\{1, (m-1)\}$ and that $v \in H_{0,\rho}^{1,2}(\Omega)$ if $m > 2$ or $v \in H_{0,\rho}^{1,2}(\Omega) \cap C^0(\overline{\Omega})$ if $\frac{3}{2} < m < 2$. Then, if*

$$L_u(v, \varphi) = 0 \tag{1.6}$$

for any $\varphi \in H_{0,\rho}^{1,2}(\Omega)$, it follows that $v \equiv 0$ in Ω.

Remark 1.1. The same proof as for Theorem 1.3 would apply in the case when Ω is a ball in $\mathbb{R}^N$, $\frac{N+2}{N+1} < m < \infty$, and v belongs to the space of radial functions of $H_{0,\rho}^{1,2}(\Omega)$. Anyway, this would only be a particular case of a more general result proved in [1].

In the proof of Theorem 1.3 we exploit Theorem 1.2 and follow the techniques developed in [6] where the case of Laplace equations is considered. Since u is generally only of class $C^{1,\tau}$ with $\tau < 1$ we encounter a further difficulty since we can not apply the divergence theorem as done in [6] and (1.1) holds in the weak sense. Indeed, also some versions of the divergence theorem for nonsmooth vector fields known in the literature (see for example [4] and the references therein) do not work in our case. We overcome this difficulty exploiting some properties of the critical set Z obtained in [8, 9] and proving some a priori estimates for the derivatives of the solutions of (1.4) (see Lemma (4.2)).

If Ω is a ball in $\mathbb{R}^N$ a more general result in weighted Sobolev spaces of radial functions is proved in the pioneer work of A. Aftalion and F. Pacella [1]. In that paper in particular the idea of using $|Du|^{m-2}$ as a weight function is introduced and then used, together with radial-symmetry arguments, to study the Morse index and the uniqueness of solutions of (1.1).

Anyway in the general case, we have weaker information on the regularity of the solutions of (1.1) and of (1.4), and the approach of [1] fails.

The lack of regularity of the solutions of (1.1) and of (1.4) is the greatest difficulty we encounter in these problems. To our knowledge, there are not other nondegeneracy results for degenerate elliptic operators when Ω is not a ball.

The paper is organized as follows: In Section 2 we recall some preliminary results about the regularity, monotonicity, and symmetry properties of the solutions of (1.1) proved in [8, 9]. In Section 3 we prove a weak maximum principle in small domains for the linearized operator and we exploit it to prove Theorem 1.2. In Section 4 we prove Theorem 1.3 and some related results.

2. Preliminaries

In what follows, as in [15, 19], if $\rho \in L^1(\Omega)$, the space $H^{1,p}_\rho(\Omega)$ is defined as the completion of $C^1(\Omega)$ (or $C^\infty(\Omega)$) under the norm

$$\|v\|_{H^{1,p}_\rho} = \|v\|_{L^p(\Omega)} + \|Dv\|_{L^p(\Omega,\rho)} \tag{2.1}$$

and

$$\|Dv\|^p_{L^p(\Omega,\rho)} = \int_\Omega |Dv|^p \rho \, dx.$$

In this way $H^{1,p}_\rho(\Omega)$ is a Banach space and $H^{1,2}_\rho(\Omega)$ is a Hilbert space.

In [8] the authors prove that if u is a weak solution of (1.1), then $u_{x_i} \in H^{1,2}_\rho(\Omega)$. This result is then used to study the linearized operator L_u associated to problem (1.1), proving in particular that

$$\begin{aligned} L_u(u_{x_i}, \varphi) \equiv \int_\Omega [&|Du|^{m-2}(Du_{x_i}, D\varphi) \\ &+ (m-2)|Du|^{m-4}(Du, Du_{x_i})(Du, D\varphi) - f'(u)u_{x_i}\varphi]dx \end{aligned}$$

is well defined for every $\varphi \in H^{1,2}_{0,\rho}(\Omega)$ and the following equation holds:

$$L_u(u_{x_i}, \varphi) = 0 \qquad \forall \varphi \in H^{1,2}_{0,\rho}(\Omega), \quad i = 1, \ldots, N. \tag{2.2}$$

In other words each derivative u_{x_i} is a weak solution of the linearized equation.

Remark 2.1. In general by [10, 18] a solution u of (1.1) belongs only to the class $C^{1,\tau}(\Omega)$ (see also [14] for the estimates on the boundary). By standard elliptic estimates we also know that $u \in C^2(\Omega \setminus Z)$ (Z as in (1.2)), since in $\Omega \setminus Z$ the m-Laplace operator is uniformly elliptic. Since for f positive Z has zero measure, we can compute all the derivatives in the classical sense almost everywhere (see [8] for details).

In [8] some regularity results on the second derivatives of any solution u are obtained as well as some summability properties of $\frac{1}{|Du|}$.

These results have been exploited in [8] to prove weighted Sobolev- and Poincarè-type inequalities. In particular we have the following:

Theorem 2.1. *Let $u \in C^1(\overline{\Omega})$ be a weak solution of (1.1) with $f(s) > 0$ if $s > 0$, $m > 2$. Then, if we consider $\rho = |Du|^{m-2}$ we get, for every $p \geqslant 2$,*

$$\|v\|_{L^p(\Omega)} \leqslant C(|\Omega|)\|Dv\|_{L^p(\Omega,\rho)} \qquad \text{for every } v \in H^{1,p}_{0,\rho}(\Omega), \tag{2.3}$$

where $C(|\Omega|) \to 0$ if $|\Omega| \to 0$.

In particular (2.3) holds for every $v \in H^{1,2}_{0,\rho}(\Omega)$.

Our techniques will be based also on monotonicity and symmetry properties of the solutions obtained in [8]. For the reader's convenience we recall these results. To this aim, let us first recall some notation.

Let ν be a direction in $\mathbb{R}^n$. For a real number λ we define

$$T^\nu_\lambda = \{x \in \mathbb{R} : x \cdot \nu = \lambda\} \tag{2.4}$$

$$\Omega^\nu_\lambda = \{x \in \Omega : x \cdot \nu < \lambda\} \tag{2.5}$$

$$x^\nu_\lambda = R^\nu_\lambda(x) = x + 2(\lambda - x \cdot \nu)\nu, \qquad x \in \mathbb{R}^N \tag{2.6}$$

and

$$a(\nu) = \inf_{x \in \Omega} x \cdot \nu. \tag{2.7}$$

If $\lambda > a(\nu)$ then Ω^ν_λ is nonempty; thus, we set

$$(\Omega^\nu_\lambda)' = R^\nu_\lambda(\Omega^\nu_\lambda). \tag{2.8}$$

Following [11] we observe that for $\lambda - a(\nu)$ small, $(\Omega^\nu_\lambda)'$ is contained in Ω and will remain in it, at least until one of the following occurs:

(i) $(\Omega^\nu_\lambda)'$ becomes internally tangent to $\partial\Omega$.
(ii) T^ν_λ is orthogonal to $\partial\Omega$.

Let $\Lambda_1(\nu)$ be the set of those $\lambda > a(\nu)$ such that for each $\mu < \lambda$ none of the conditions (i) and (ii) holds, and define

$$\lambda_1 = \sup \Lambda_1(\nu). \tag{2.9}$$

Moreover, let

$$\Lambda_2(\nu) = \{\lambda > a(\nu) : (\Omega_\mu^\nu)' \subseteq \Omega \quad \forall \mu \in (a(\nu), \lambda]\} \tag{2.10}$$

and

$$\lambda_2(\nu) = \sup \Lambda_2(\nu). \tag{2.11}$$

Finally, define

$$\Lambda_0(\nu) = \{\lambda > a(\nu) : u \leqslant u_\lambda^\nu \quad \forall \mu \in (a(\nu), \lambda]\} \tag{2.12}$$

and

$$\lambda_0(\nu) = \sup \Lambda_0(\nu). \tag{2.13}$$

In [8], using the Alexandrov-Serrin moving-plane method [16] (see also [2, 11]), the problem of monotonicity (and symmetry) of any fixed solution of (1.1) in convex (and symmetric) domains when the nonlinearity f is positive is considered. In particular the following result is proved there:

Theorem 2.2. *Let Ω be a bounded, smooth domain in $\mathbb{R}^N$, $N \geqslant 2$, $1 < m < \infty$, $f : [0, \infty) \to \mathbb{R}$ a locally Lipschitz-continuous function such that $f(s) > 0$ for $s > 0$, and $u \in C^1(\overline{\Omega})$ a weak solution of* (1.1).

For any direction ν and for λ in the interval $(a(\nu), \lambda_2(\nu)]$ we have

$$u(x) \leqslant u(x_\lambda^\nu) \quad \forall x \in \Omega_\lambda^\nu. \tag{2.14}$$

Moreover, for any λ with $a(\nu) < \lambda < \lambda_2(\nu)$ we have

$$u(x) < u(x_\lambda^\nu) \quad \forall x \in \Omega_\lambda^\nu \setminus Z_\lambda^\nu, \tag{2.15}$$

where $Z_\lambda^\nu \equiv \{x \in \Omega_\lambda^\nu : Du(x) = Du_\lambda^\nu(x) = 0\}$. Finally,

$$\frac{\partial u}{\partial \nu}(x) > 0 \quad \forall x \in \Omega_{\lambda_2(\nu)}^\nu \setminus Z, \tag{2.16}$$

where $Z = \{x \in \Omega : Du(x) = 0\}$.

Corollary 2.1. *If the domain Ω is convex with respect to a direction ν and symmetric with respect to the hyperplane $T_0^\nu = \{x \in \mathbb{R}^N : x \cdot \nu = 0\}$, then u is symmetric, i.e., $u(x) = u(x_0^\nu)$, and nondecreasing in the ν direction in Ω_0^ν with $\frac{\partial u}{\partial \nu}(x) > 0$ in $\Omega_0^\nu \setminus Z$.*

In particular if Ω is a ball then u is radially symmetric and $\frac{\partial u}{\partial r} < 0$ in $\Omega \setminus \{0\}$, where $\frac{\partial u}{\partial r}$ is the derivative in the radial direction.

Remark 2.2. In the case of Lipschitz-continuous nonlinearities (not necessarily positive) and $1 < m < 2$, Theorem 2.2 had been previously proved in [7] for the case of a strictly convex domain ($\lambda_2(\nu)$ replaced by $\lambda_1(\nu)$). The proof given in [8] extends the result to the case $m > 2$ and, at the same time, allows for $1 < m < +\infty$ to consider a larger class of domains (e.g. the smoothed rectangle). Therefore, since we consider the case of positive nonlinearities, we will refer to Theorem 2.2. A different approach is used in [13], where the case of f continuous and positive is considered when Ω is a ball and $m = N$. In [3], with the aid of the so-called "continuous Steiner symmetrization," the author proved that solutions of (1.1), in the ball, are radially symmetric under a fairly weak assumption on the nonlinearity.

One of the main tools used in [8] is the fact, proved there, that if f is positive then $\Omega \setminus Z$ is connected. Since this will also be crucial in our setting, let us recall the precise statement:

Theorem 2.3. *Let $u \in C^1(\overline{\Omega})$ be a weak solution of (1.1), where Ω is a general bounded domain, and suppose that $f(s) > 0$ if $s > 0$. Then $\Omega \setminus Z$ does not contain any connected component C which is compactly contained in Ω. Moreover, if we assume that Ω is a smooth, bounded domain with connected boundary, it follows that $\Omega \setminus Z$ is connected.*

3. Weak maximum principle

In this section we prove a weak maximum principle in small domains for the linearized operator L_u and then we exploit it to get a weak maximum principle in Ω_i^- (or more generally in regions contained in Ω_i^-).

The proof of the weak maximum principle in small domains is based on a weighted Poincarè inequality proved in [8], and there used to prove a weak comparison principle in small domains for C^1 solutions of (1.1).

In our case we encounter a further difficulty since we consider solutions of (1.4) which are not smooth. More precisely, since the linearized operator L_u is naturally defined in $H^{1,2}_\rho(\Omega)$, we assume only $v \in H^{1,2}_\rho(\Omega)$ if $m > 2$. If instead $\frac{2N+2}{N+2} < m < 2$, we will need to assume that v is continuous. Note that by the results of [8] the condition $m > \frac{2N+2}{N+2}$ guarantees that the weight $\rho \equiv |Du|^{m-2}$ belongs to $L^1(\Omega)$ so that $H^{1,2}_\rho(\Omega)$ is well defined.

Remark 3.1. In what follows we use the fact that v is regular in $\Omega \setminus Z$ as follows by the regularity of the coefficients of L_u and by standard elliptic estimates (see e.g. [12], Theorem 8.22 and Theorem 8.10). The regularity of v up to the boundary follows by Theorem 8.13 of [12] once we note that there exists a region near $\partial\Omega$ where by Hopf's lemma [20] $Du(x) \neq 0$ (see also [14] for the estimates on the boundary).

A crucial point in this work consists in the fact that to prove nondegeneracy results for the linearized operator in its natural space of definition we have to work with functions belonging to weighted Sobolev spaces. Therefore we have to take care about what it means that a function is positive, or negative on the boundary of subdomains of Ω. We start here following [15] and [19] and giving an abstract definition which leads to some abstract maximum principle results. Later we will show how to use these results in our context.

Definition 3.1. Let $v \in H^{1,2}_{\rho}(\Omega')$. Then we say that $v \geqslant 0$ on $\partial\Omega'$ if $v^- \in H^{1,2}_{0,\rho}(\Omega')$. In the same way, $v \leqslant 0$ on $\partial\Omega'$ if $v^+ \in H^{1,2}_{0,\rho}(\Omega')$.

Proposition 3.1. *Let $u \in C^1(\overline{\Omega})$ be a weak solution of (1.1), where Ω is a bounded, smooth domain of $\mathbb{R}^N$, and suppose that f is a locally Lipschitz-continuous function such that $f(s) > 0$ for $s > 0$, $\frac{2N+2}{N+2} < m < \infty$. Let $\rho \equiv |Du|^{m-2}$ and suppose $v \in H^{1,2}_{\rho}(\Omega)$ and*

$$L_u(v, \varphi) \geqslant 0 \tag{3.1}$$

for any nonnegative function $\varphi \in H^{1,2}_{0,\rho}(\Omega)$.

Then, there exists $\delta > 0$ such that, if $v \geqslant 0$ on $\partial\Omega'$ in the sense of Definition 3.1 and $\Omega' \subseteq \Omega$ is such that $|\Omega'| < \delta$, it follows that $v \geqslant 0$ in Ω'.

Proof. Let us first suppose $m > 2$. By the hypothesis we have $v^- \in H^{1,2}_{0,\rho}(\Omega')$. We can therefore use it as a test function in (3.1) and get

$$\begin{aligned}\int_{\Omega'} [|Du|^{m-2}(Dv, D(v)^-) + (m-2)|Du|^{m-4}(Du, Dv)(Du, D(v)^-)]\, dx \\ - \int_{\Omega'} f'(u)v(v)^-\, dx \geqslant 0;\end{aligned} \tag{3.2}$$

i.e.,

$$\int_{\Omega'} |Du|^{m-2}|Dv^-|^2\, dx \leqslant \int_{\Omega'} f'(u)|v^-|^2\, dx, \tag{3.3}$$

where we have used the fact that $|Du|^{m-4}(Du, Dv^-)^2 \geqslant 0$ in Ω'.

By the hypothesis on f we obtain

$$\int_{\Omega'} |Du|^{m-2}|D(v)^-|^2\, dx \leqslant c_0 \int_{\Omega'} |(v)^-|^2\, dx. \tag{3.4}$$

By Theorem 2.1 we have that

$$\int_{\Omega'} |(v)^-|^2\,dx \leqslant C(|\Omega'|) \int_{\Omega'} |Du|^{m-2}|D(v)^-|^2\,dx, \tag{3.5}$$

where $C(|\Omega'|)$ tends to zero if $|\Omega'|$ tends to zero. Therefore by (3.4) and (3.5) we get

$$\int_{\Omega'} |Du|^{m-2}|D(v)^-|^2\,dx \leqslant C_0(|\Omega'|) \int_{\Omega'} |Du|^{m-2}|D(v^-)|^2\,dx, \tag{3.6}$$

where $C_0(|\Omega'|)$ tends to zero if $|\Omega'|$ tends to zero. So there exists $\delta > 0$ such that if $|\Omega'| < \delta$ then $C_0(|\Omega'|) < 1$ and a contradiction occurs. Thus the theorem is proved for the case $m > 2$.

Let us now consider the case $\frac{2N+2}{N+2} < m < 2$. Using v^- as test function as above, we get

$$(m-1)\int_{\Omega'} |Du|^{m-2}|Dv^-|^2\,dx \leqslant \int_{\Omega'} f'(u)(v^-)^2\,dx. \tag{3.7}$$

In this case, since $u \in C^1(\overline{\Omega})$, we get

$$\int_{\Omega'} |Dv^-|^2\,dx \leqslant c\int_{\Omega'} |Du|^{m-2}|Dv^-|^2\,dx.$$

Therefore, by a classic Poincarè's inequality, we get

$$\int_{\Omega'} |Dv^-|^2\,dx \leqslant C_1\int_{\Omega'} (v^-)^2\,dx \leqslant C(|\Omega'|)\int_{\Omega'} |D(v^-)|^2\,dx, \tag{3.8}$$

where $C(|\Omega'|)$ tends to zero if $|\Omega'|$ tends to zero. If $C(|\Omega'|) < 1$ we get the thesis as in the case $m > 2$. $\square$

In the sequel we will need to use Proposition 3.1 for functions v which are continuous, except possibly for isolated points, and satisfy pointwise inequalities on the boundary.

In general we recall that by standard arguments, we have the following:

Remark 3.2. If v is a continuous function and $v \geqslant 0$ ($v \leqslant 0$) on $\partial\Omega'$ pointwise, then $v \geqslant 0$ ($v \leqslant 0$) on $\partial\Omega'$ in the sense of Definition 3.1.

In the case $m > 2$, in our setting, we have better results. In particular we prove the following:

Lemma 3.1. *Let $u \in C^1(\overline{\Omega})$ be a weak solution of (1.1), where Ω is a bounded, smooth domain of $\mathbb{R}^N$, and suppose that f is a locally Lipschitz-continuous function such that $f(s) > 0$ for $s > 0$, $m > 2$. Assume that*

$v \in H^{1,2}_\rho(\Omega)$ with $\rho \equiv |Du|^{m-2}$, and suppose that

$$Z \equiv \{x \in \Omega : D(u)(x) = 0\} = \{z_0\}.$$

Let $\Omega' \subseteq \Omega$ and suppose v is continuous in $\overline{\Omega'} \setminus \{z_0\}$ and $v \geqslant 0$ ($v \leqslant 0$) pointwise on $\partial\Omega' \setminus \{z_0\}$. Then $v \geqslant 0$ ($v \leqslant 0$) on $\partial\Omega'$ in the sense of Definition 3.1.

Proof. Let $T_k(s)$ be defined by

$$\begin{cases} T_k(s) = s & \text{if } |s| \leqslant k \\ T_k(s) = k & \text{if } |s| \geqslant k \end{cases}$$

so that T_k is a Lipschitz-continuous function. We claim that $T_k(v)^- \in H^{1,2}_{0,\rho}(\Omega')$.

To prove this let us consider $\varphi_\epsilon \in C^\infty_c(\Omega)$ such that $\varphi_\epsilon \equiv 0$ in $B(z_0, \epsilon)$ and $\varphi_\epsilon \equiv 1$ outside $\Omega \setminus B(z_0, 2\epsilon)$. Moreover assume $|D\varphi_\epsilon| \leqslant \frac{2}{\epsilon}$. With these definitions, we have that $\varphi_\epsilon T_k(v)^- \in H^{1,2}_{0,\rho}(\Omega')$ since it is continuous on $\partial\Omega'$ by the assumptions and identically zero there. Moreover,

$$\begin{aligned} &\|D(\varphi_\epsilon T_k(v)^-) - D(T_k(v)^-)\|_{H^{1,2}_\rho(\Omega')} \\ &\leqslant |\varphi_\epsilon - 1| \|D(T_k(v)^-)\|_{H^{1,2}_\rho(\Omega')} + k\|D(\varphi_\epsilon)\|_{H^{1,2}_\rho(\Omega')}; \end{aligned} \tag{3.9}$$

therefore, since for $\epsilon \to 0$ we have that

$$\sup_{x \in \Omega'} (\varphi_\epsilon(x) - 1) \to 0 \quad \text{and} \quad \|D(\varphi_\epsilon)\|_{H^{1,2}_\rho(\Omega')} \to 0$$

(this is not true if $m < 2$), we get that $\varphi_\epsilon T_k(v)^-$ approximates $T_k(v)^-$ in $H^{1,2}_\rho(\Omega')$ so that

$$T_k(v)^- \in H^{1,2}_{0,\rho}(\Omega'). \tag{3.10}$$

We will now show that $T_k(v)^-$ approximates v^- in $H^{1,2}_\rho(\Omega')$. To prove this let us note that *supp*$(T_k(v)^-) \equiv$ *supp*(v^-) and $(T_k(v)^-) \equiv v^-$ if $|v| \leqslant k$. Therefore, $\|(T_k(v)^-) - v^-\|_{L^2(\Omega')} \leqslant c\|v\|_{L^2(\{|v| \geqslant k\})}$, and then

$$\|(T_k(v)^-) - v^-\|_{L^2(\Omega')} \to 0 \quad \text{if} \quad k \to \infty.$$

In the same way we prove that

$$\|D(T_k(v)^-) - D(v^-)\|_{L^2_\rho(\Omega')} \to 0 \quad \text{if} \quad k \to \infty,$$

showing that $v^- \in H^{1,2}_{0,\rho}(\Omega')$. □

Proposition 3.2 (Weak maximum principle in small domains). *Let $u \in C^1(\overline{\Omega})$ be a weak solution of* (1.1), *where Ω is a bounded, smooth domain of $\mathbb{R}^N$, and suppose that f is a locally Lipschitz-continuous function such that $f(s) > 0$ for $s > 0$, $m > 2$. Assume that*

$$Z \equiv \{x \in \Omega : D(u)(x) = 0\} = \{z_0\}.$$

Suppose $v \in H^{1,2}_\rho(\Omega)$ and

$$L_u(v, \varphi) = 0 \tag{3.11}$$

for any $\varphi \in H^{1,2}_{0,\rho}(\Omega)$ (consequently $v \in C^0(\overline{\Omega} \setminus \{z_0\})$).

Then, for any $\Omega' \subseteq \Omega$, there exists $\delta > 0$ such that, if $v \geqslant 0$ on $\partial\Omega' \setminus \{z_0\}$ pointwise and Ω' is such that $|\Omega'| < \delta$, it follows $v \geqslant 0$ in Ω'.

If $\frac{2N+2}{N+2} < m < 2$, the same result follows if we assume that $v \in H^{1,2}_\rho(\Omega) \cap C^0(\overline{\Omega'})$ (no assumptions on the critical set Z are needed in this case).

Proof. If $m > 2$ and $v \in H^{1,2}_\rho(\Omega)$, since L_u is strictly elliptic in $\Omega \setminus Z$, by standard elliptic regularity (see [12]), $v \in C^0(\overline{\Omega} \setminus \{z_0\})$. Therefore by Lemma 3.1 we get that $v \geqslant 0$ on $\partial\Omega'$ in the sense of Definition 3.1.

If on the other hand $\frac{2N+2}{N+2} < m < 2$ and $v \in H^{1,2}_\rho(\Omega) \cap C^0(\overline{\Omega'})$ (see Remark 3.2), it follows immediately $v \geqslant 0$ on $\partial\Omega'$ in the sense of Definition 3.1.

The thesis follows now by Proposition 3.1. □

Remark 3.3. Note that in our applications we will consider a bounded, smooth domain Ω in $\mathbb{R}^N$ which is convex in the e_i direction and symmetric with respect to $T^{e_i}_o$ for N orthogonal directions $e_1, \dots, e_N$. In this case, by Theorem 1.1, we have

$$Z \equiv \{x \in \Omega : D(u)(x) = 0\} = \{0\},$$

assuming that 0 is the center of symmetry, so that the assertion of Proposition 3.2 holds in this case.

We now exploit Proposition 3.2 to get a weak maximum principle for the linearized operator in regions where the solution u is monotone.

Proof of Theorem 1.2. Suppose first $\frac{2N+2}{N+2} < m < 2$. The same proof will work for the case $m > 2$ with simple changes explained below.

Consider an open set $A \subset \Omega_i^-$ such that $Z \subset A$. Let us define $A_\xi \equiv \{x \in \Omega_i^- : \xi < x_i < 0\}$. Since $|Z| = 0$ we can take ξ sufficiently small such that we can apply Proposition 3.2 in $A_1 \equiv A \cup A_\xi$. Moreover, since by Hopf's lemma (see [20]) $Z \cap \partial\Omega = \emptyset$, we can suppose that there are not points on ∂A_1 where the gradient of u vanishes. Let K be a compact set contained in Ω_i^- such that $\Omega_i^- \setminus K$ has small measure. By Corollary 2.1 of [8] we have $u_{x_i} > 0$ on $\overline{K \setminus A_1}$, and, since L_u is not degenerate in in $\Omega \setminus Z$, we have that v is regular in $\overline{K \setminus A_1}$. Therefore there exists $t > 0$ such that $v + tu_{x_i} > 0$ in $\overline{K \setminus A_1}$. Moreover by Proposition 3.2, since $v + tu_{x_i} \geqslant 0$ on $\partial(\Omega_i^- \setminus (K \cup A_1) \cup A_1)$ and still satisfies the linearized equation, we get $v + tu_{x_i} \geqslant 0$ in $\Omega_i^- \setminus (K \cup A_1) \cup A_1$ and hence, $v + tu_{x_i} \geqslant 0$ in Ω_i^-. Let us now put

$$t_o \equiv \inf_t \{t \in \mathbb{R} : v + tu_{x_i} \geqslant 0 \quad \text{in} \quad \Omega_i^-\}. \tag{3.12}$$

We will prove our result by showing that $t_0 = 0$.

Suppose on the contrary $t_0 > 0$. By continuity we have $v + t_o u_{x_i} \geqslant 0$ in Ω_i^-. Now, since $\Omega \setminus Z$ is connected, by symmetry also $\Omega_i^- \setminus Z$ is connected. Then, by the strong maximum principle for uniformly elliptic operators in $\Omega_i^- \setminus Z$, since $v + t_o u_{x_i} > 0$ on $\partial\Omega_i^- \setminus T_0^{e_i}$, we get $v + t_o u_{x_i} > 0$ in $\Omega_i^- \setminus Z$. Therefore $v + t_o u_{x_i} > \gamma > 0$ in $K \setminus A_1$. By continuity we find $\epsilon > 0$ such that $v + (t_o - \epsilon)u_{x_i} > 0$ in $\overline{K \setminus A_1}$. Arguing as above we get $v + (t_o - \epsilon)u_{x_i} \geqslant 0$ in Ω_i^-, which contradicts the definition of t_0. Therefore $t_0 = 0$ and consequently $v \geqslant 0$ in Ω_i^-.

Let us consider now the case $m > 2$. In this case, by [9] (see Theorem 1.1), we know that $Z \equiv \{0\}$; therefore, the hypotheses of Proposition 3.2 are fulfilled and we can exploit it as above. Moreover in this case we can consider $B(0, \epsilon)$ instead of A_1, and the thesis follows more easily using standard elliptic estimates to conclude that v is continuous in $\Omega \setminus \{0\}$.

Finally, if we consider $\Omega' \subset \Omega$, then the thesis follows exactly as above if $m > 2$. If on the other hand $\frac{2N+2}{N+2} < m < 2$, then we only have to note that, even if $\Omega' \setminus Z$ is not connected, we know that any connected component C_0 of $\Omega' \setminus Z$ intersects $\partial\Omega'$ at least in one point where $u_{x_i} > 0$ (if not we would have $\partial C_0 \subset Z$). Therefore we can exclude regions where $v + t_o u_{x_i} = 0$, arguing as above. □

Corollary 3.1. *Let $u \in C^1(\overline{\Omega})$ be a weak solution of (1.1), where Ω is a bounded, smooth domain of $\mathbb{R}^N$ which is convex in the e_i direction and symmetric with respect to $T_o^{e_i}$ for N orthogonal directions $e_1, \ldots, e_N$. Suppose that f is a locally Lipschitz-continuous function such that $f(s) > 0$ for $s > 0$. Suppose that $v \in H^{1,2}_{0,\rho}(\Omega)$ if $m > 2$ or $v \in H^{1,2}_{0,\rho}(\Omega) \cap C^0(\overline{\Omega})$ if $\frac{2N+2}{N+2} < m < 2$.*

Then, if

$$L_u(v, \varphi) = 0 \tag{3.13}$$

for any nonnegative function $\varphi \in H^{1,2}_{0,\rho}(\Omega)$, v is symmetric with respect to $T_o^{e_i}$ for any direction e_i with $i = 1, \ldots, N$.

Proof. It is sufficient to apply Theorem 1.2 to $w(x) \equiv (v(x) - v(x_o^{e_i}))$. □

4. Nondegeneracy

In a recent paper, [1], by means of techniques based on radial symmetry, the Morse index of radial solutions is studied. In the same paper some regularity results on the solutions of (1.4) are obtained (see also [17]) and then used to prove that the solutions are nondegenerate in the space of radial functions of $H^{1,2}_{0,\rho}$. In this section we exploit the results obtained in Section 3 to prove that if $f(s) = s^q$ with $q > \max\{1, (m-1)\}$, and Ω is a bounded, smooth domain of $\mathbb{R}^2$ which is convex in the e_i direction and symmetric with respect to $T_o^{e_i}$ for N orthogonal directions $e_1, \dots, e_N$, then any solution of (1.1) is nondegenerate.

Since in our setting we have very weak regularity information we can not follow the approach of [1]. We will therefore extend here to the case of degenerate operators some of the proofs in [6], where semilinear elliptic equations involving the regular Laplace operator are considered.

Proposition 4.1. *Let $u \in C^1(\overline{\Omega})$ be a weak solution of (1.1), where Ω is a bounded, smooth domain in $\mathbb{R}^2$ which is convex in the e_i direction and symmetric with respect to $T_o^{e_i}$ for N orthogonal directions $e_1, \dots, e_N$. Suppose that f is a locally Lipschitz-continuous function such that $f(s) > 0$ for $s > 0$. Suppose $v \in H^{1,2}_{0,\rho}(\Omega)$ if $m > 2$ and $v \in H^{1,2}_{0,\rho}(\Omega) \cap C^0(\overline{\Omega})$ if $\frac{3}{2} < m < 2$. Assume that*

$$L_u(v, \varphi) = 0 \tag{4.1}$$

for any $\varphi \in H^{1,2}_{0,\rho}(\Omega)$. Then, either there exists a neighborhood of $\partial\Omega$ where $v > 0$, or there exists a neighborhood of $\partial\Omega$ where $v < 0$.

Proof. We consider the cases $m > 2$ and $\frac{3}{2} < m < 2$ simultaneously, and we suppose that Ω is a bounded, smooth domain in $\mathbb{R}^2$.

Assume for the sake of contradiction that the assertion does not hold. Let C_0 be a connected component of $U_0^+ \equiv \{x \in \Omega : v(x) > 0\}$. Since $v = 0$ on $\partial\Omega$, then $v = 0$ on $\partial C_0 \setminus \{0\}$. Here it is crucial that, by the geometric assumption on the domain and by Theorem 1.1, $Z = \{0\}$ and $v \in C^0(\overline{\Omega} \setminus \{0\})$.

By Theorem 1.2 C_0 cannot be contained in Ω_i^-, and by symmetry (Corollary 3.1) we can construct a closed, simple curve Γ_0 which is symmetric with respect to the axis where $v > 0$. Let U_1 be the component of $\Omega \setminus \Gamma_0$ which does not contains the origin. If U_1 does not contains points where $v < 0$, then there exists a neighborhood of $\partial\Omega$ where $v \geqslant 0$, and therefore, since L_u is not degenerate near $\partial\Omega$ (by Hopf's lemma [20] $Z \cap \partial\Omega = \emptyset$), we have $v > 0$ or $v = 0$ there. By the construction of Γ_0 and by the strong maximum principle for strictly elliptic operators in $\Omega \setminus Z$, taking into account Theorem 2.3, we can easily prove that the case $v = 0$ is impossible. Therefore this would prove the thesis.

Otherwise let C_1 be a connected component of

$$U_1^- \equiv \{x \in U_1 : v(x) < 0\}.$$

As above we have $v = 0$ on ∂C_1, and we can construct in C_1 a closed, simple arch Γ_1 which is symmetric. Arguing in this way we get infinitely many components $C_n \subset \Omega$ with the property that $v = 0$ on ∂C_n and v does not change sign in C_n (with v not identically zero in C_n).

Of course for any $\delta > 0$ there exists n_δ such that, for any $n \geqslant n_\delta$, $|C_n| < \delta$. Therefore, taking δ small, by Proposition 3.2 v would be identically zero in the corresponding components C_n for $n \geqslant n_\delta$. This contradiction proves the thesis. $\square$

Lemma 4.1. *Let $u \in C^1(\overline{\Omega})$ be a weak solution of (1.1), where Ω is a general, bounded, smooth domain in $\mathbb{R}^N$, and assume that $f(s) = s^q$ with $q > \max\{1, (m-1)\}$, $\frac{2N+2}{N+2} < m < \infty$. If $v \in H^{1,2}_{0,\rho}(\Omega)$ is such that*

$$L_u(v, \varphi) = 0 \tag{4.2}$$

for any $\varphi \in H^{1,2}_{0,\rho}(\Omega)$, then

$$\int_\Omega u^q v \, dx = 0. \tag{4.3}$$

Proof. By density arguments, we can use v as test function in (1.1) and get

$$\int_\Omega |Du|^{m-2}(Du, Dv)\, dx = \int_\Omega u^q v \, dx.$$

Moreover, using u as test function in (1.4) we get

$$(m-1)\int_\Omega |Du|^{m-2}(Du, Dv)\, dx = q\int_\Omega u^q v \, dx.$$

Therefore,

$$(1 - \frac{q}{m-1})\int_\Omega u^q v \, dx = 0.$$

By the assumption on q, we get $(1 - \frac{q}{m-1}) \neq 0$, and the thesis follows. $\square$

To prove our main result we will need to exploit the divergence theorem. Anyway, since we deal with nonsmooth vector fields, this is possible only away from the critical set Z, where all the functions considered are smooth. This causes the appearing of extra boundary terms. We will overcome this difficulty proving some a priori estimates:

Lemma 4.2. *Let $u \in C^1(\overline{\Omega})$ be a weak solution of (1.1), where Ω is a bounded, smooth domain in $\mathbb{R}^N$, and suppose that f is locally Lipschitz continuous, $\frac{2N+2}{N+2} < m < \infty$. Suppose $v \in H^{1,2}_{0,\rho}(\Omega)$ weakly solves*

$$L_u(v, \varphi) = 0 \tag{4.4}$$

for any $\varphi \in H^{1,2}_{0,\rho}(\Omega)$. Then for any open, smooth set $A \subset\subset \Omega$ such that $Z \subset A$ we have

$$\int_{\partial A} (|Du|^{m-2}\frac{\partial v}{\partial \eta} + (m-2)|Du|^{m-4}(Du, Dv)\frac{\partial u}{\partial \eta})\, d\sigma \leqslant K, \tag{4.5}$$

where K does not depend on A.

Proof. By the assumption on A, we have that v is regular in $\Omega \setminus A$ (see Remark 3.1). Therefore we can apply the divergence theorem in $\Omega \setminus A$ to the vector field $W \equiv |Du|^{m-2}Dv + (m-2)|Du|^{m-4}(Du, Dv)Du$, obtaining

$$\begin{aligned} &\int_{\partial A} (|Du|^{m-2}\frac{\partial v}{\partial \eta} + (m-2)|Du|^{m-4}(Du, Dv)\frac{\partial u}{\partial \eta})\, d\sigma \\ &= -\int_{\partial \Omega} (|Du|^{m-2}\frac{\partial v}{\partial \eta} + (m-2)|Du|^{m-4}(Du, Dv)\frac{\partial u}{\partial \eta})\, d\sigma \\ &+ \int_{\Omega \setminus A} div(|Du|^{m-2}Dv + (m-2)|Du|^{m-4}(Du, Dv)Du)\, dx. \end{aligned} \tag{4.6}$$

Now, let us note that, by Hopf's lemma (see [20]), there are not points of the critical set Z on $\partial\Omega$, so that W is regular up to $\partial\Omega$ and

$$\Big| \int_{\partial \Omega} (|Du|^{m-2}\frac{\partial v}{\partial \eta} + (m-2)|Du|^{m-4}(Du, Dv)\frac{\partial u}{\partial \eta})\, d\sigma \Big| \leqslant K_0,$$

where K_0 does not depend on A . Moreover, by (1.4), we have

$$div(|Du|^{m-2}Dv + (m-2)|Du|^{m-4}(Du, Dv)Du) \equiv f'(u)v$$

almost everywhere in $\Omega \setminus A$. By the assumptions on f and on v this implies

$$\begin{aligned} &\Big| \int_{\Omega \setminus A} div(|Du|^{m-2}Dv + (m-2)|Du|^{m-4}(Du, Dv)Du)\, dx \Big| \\ &\leqslant \Big| \int_{\Omega \setminus A} f'(u)v\, dx \Big| \leqslant \int_{\Omega} |f'(u)v|\, dx \leqslant K_1, \end{aligned}$$

where K_1 does not depend on A. Taking $K = K_o + K_1$, we prove the result. $\square$

Let us now prove the nondegeneracy result:

Proof of Theorem 1.3. Let us consider the auxiliary function

$$\xi(x) \equiv x_1 u_{x_1} + x_2 u_{x_2}.$$

Since u_{x_1} and u_{x_2} weakly solve (1.4), then easy calculations show that $\xi(x)$ weakly solves (1.4) with a different zero-order term. More precisely we have

$$\begin{aligned}\int_\Omega [|Du|^{m-2}(D\xi, D\varphi) + (m-2)|Du|^{m-4}(Du, D\xi)(Du, D\varphi)]\, dx \\ = \int_\Omega (mu^q + qu^{q-1}\xi)\varphi\, dx\end{aligned} \tag{4.7}$$

for any $\varphi \in H^{1,2}_{0,\rho}(\Omega)$. Let A_ϵ be such that $Z \subset A_\epsilon \subset\subset \Omega$, satisfying $dist(x, Z) < \epsilon$ for every $x \in A_\epsilon$. By the regularity of v in $\Omega \setminus A_\epsilon$ (see Remark 3.1) we can apply the divergence theorem and get

$$\begin{aligned}&\int_{\Omega\setminus A_\epsilon} [|Du|^{m-2}(D\xi, Dv) + (m-2)|Du|^{m-4}(Du, D\xi)(Du, Dv)]\, dx \\ &+ \int_{\Omega\setminus A_\epsilon} [div(|Du|^{m-2}Dv + (m-2)|Du|^{m-4}(Du, Dv)Du]\xi\, dx \\ =&\int_{\partial\Omega} \xi(|Du|^{m-2}\frac{\partial v}{\partial \eta} + (m-2)|Du|^{m-4}(Du, Dv)\frac{\partial u}{\partial \eta})\, d\sigma \\ &+ \int_{\partial A_\epsilon} \xi(|Du|^{m-2}\frac{\partial v}{\partial \eta} + (m-2)|Du|^{m-4}(Du, Dv)\frac{\partial u}{\partial \eta})\, d\sigma.\end{aligned} \tag{4.8}$$

By density arguments, we can use v as test function in (4.7) and prove that

$$\int_\Omega [|Du|^{m-2}(D\xi, Dv) + (m-2)|Du|^{m-4}(Du, D\xi)(Du, Dv)]\, dx$$

is bounded. Therefore, we get

$$\begin{aligned}&\int_{\Omega\setminus A_\epsilon} [|Du|^{m-2}(D\xi, Dv) + (m-2)|Du|^{m-4}(Du, D\xi)(Du, Dv)]\, dx \\ &\underset{\epsilon\to 0}{\longrightarrow} \int_\Omega [|Du|^{m-2}(D\xi, Dv) + (m-2)|Du|^{m-4}(Du, D\xi)(Du, Dv)]\, dx \\ &\equiv \int_\Omega (mu^q + qu^{q-1}\xi)v\, dx.\end{aligned} \tag{4.9}$$

Moreover, since ξ tends to zero uniformly in A_ϵ, by Lemma 4.2 we get

$$\int_{\partial A_\epsilon} \xi[|Du|^{m-2}\frac{\partial v}{\partial \eta} + (m-2)|Du|^{m-4}(Du, Dv)\frac{\partial u}{\partial \eta}]\, d\sigma \underset{\epsilon\to 0}{\longrightarrow} 0. \tag{4.10}$$

Let us also note that, since v is a strong solution of L_u in $\Omega \setminus A_\epsilon$, we have

$$\begin{aligned} &\int_{\Omega\setminus A_\epsilon} [div(|Du|^{m-2}Dv + (m-2)|Du|^{m-4}(Du,Dv)Du]\xi\, dx \\ &= -\int_{\Omega\setminus A_\epsilon}[qu^{q-1}v\xi]\,dx \underset{\epsilon\to 0}{\longrightarrow} -\int_\Omega [qu^{q-1}v\xi]\,dx. \end{aligned} \tag{4.11}$$

Therefore, by (4.8), (4.9), (4.10), and (4.11), we get

$$\begin{aligned} &\int_{\partial\Omega} \xi(|Du|^{m-2}\frac{\partial v}{\partial \eta} + (m-2)|Du|^{m-4}(Du,Dv)\frac{\partial u}{\partial \eta})\,d\sigma \\ &= m\int_\Omega (u^q v)\,dx = 0. \end{aligned} \tag{4.12}$$

Let us now use this preliminary result to prove the theorem. Since u is constant on $\partial\Omega$, $Du \equiv \frac{\partial u}{\partial \eta}\eta$ (where η is the outer normal). Therefore we have

$$\int_{\partial\Omega}(x,Du)(|\frac{\partial u}{\partial \eta}|^{m-2}\frac{\partial v}{\partial \eta}) + (m-2)|\frac{\partial u}{\partial \eta}|^{m-4}(Du,Dv)\frac{\partial u}{\partial \eta}\,d\sigma = 0 \tag{4.13}$$

and

$$\begin{aligned} &\int_{\partial\Omega}(x,\eta)\left[|\frac{\partial u}{\partial \eta}|^{m-2}\frac{\partial v}{\partial \eta}\frac{\partial u}{\partial \eta} + (m-2)|\frac{\partial u}{\partial \eta}|^{m-4}(Du,Dv)(\frac{\partial u}{\partial \eta})^2\right]\,d\sigma \\ &= (m-1)\int_{\partial\Omega}(x,\eta)|\frac{\partial u}{\partial \eta}|^{m-2}\frac{\partial u}{\partial \eta}\frac{\partial v}{\partial \eta}\,d\sigma = 0. \end{aligned} \tag{4.14}$$

Let us note that by Hopf's lemma (see [20]) the linearized operator is regular near the boundary. Moreover, since v is continuous in a neighborhood of the boundary, having assumed $v \in H^{1,2}_{0,\rho}(\Omega)$, it follows that $v = 0$ on $\partial\Omega$, and we can apply Proposition 4.1 to show that v does not change sign in a neighborhood of $\partial\Omega$. Therefore, by Hopf's lemma we have $\frac{\partial v}{\partial \eta} < 0$ or $\frac{\partial v}{\partial \eta} > 0$ on the boundary. The same arguments show that $\frac{\partial u}{\partial \eta} < 0$ on the boundary. Moreover, by the geometric assumptions on Ω, we have $(x,\eta) > 0$. Therefore the last identity is possible only if $v \equiv 0$. $\square$

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References

[1] A. Aftalion and F. Pacella, *Morse index and uniqueness for positive solutions of radial p-Laplace equations*, Trans. Amer. Math. Soc., 356 (2004), 4225–4272.

[2] H. Berestycki and L. Nirenberg, *On the method of moving planes and the sliding methods*, Bol. Soc. Brasil. Mat., 22 (1991), 1–37.

[3] F. Brock, *Radial symmetry for nonnegative solutions of semilinear elliptic problems involving the p-Laplacian*, in "Calculus of Variations, Applications and Computations," proceedings of a conference in Pont-á-Mousson, 1997.

[4] M. Cuesta and P. Takàc, *A strong comparison principle for positive solutions of degenerate elliptic equations*, Differ. Integral Equ., 13 (2000), 721–746.

[5] L. Damascelli, *Comparison theorems for some quasilinear degenerate elliptic operators and applications to symmetry and monotonicity results*, Ann. Inst. H. Poincaré, Analyse non linéaire, 15 (1998), 493–516.

[6] L. Damascelli, M. Grossi, and F. Pacella, *Qualitative properties of positive solutions of semilinear elliptic equations in symmetric domains via the maximum principle*, Ann. Inst. H. Poincaré, Analyse non linéaire, 16 (1999), 631–652.

[7] L. Damascelli and F. Pacella, *Monotonicity and symmetry of solutions of p-Laplace equations,* $1 < p < 2$*, via the moving plane method*, Ann. Scuola Norm. Sup. Pisa Cl. Sci, 26 (1998), 689–707.

[8] L. Damascelli and B. Sciunzi, *Regularity, monotonicity and symmetry of positive solutions of* m*-Laplace equations*, J. Diff. Equations, to appear.

[9] L. Damascelli and B. Sciunzi, *Harnack inequalities, maximum and comparison principles, and regularity of positive solutions of* m*-Laplace equations*, to appear

[10] E. DiBenedetto, $C^{1+\alpha}$ *local regularity of weak solutions of degenerate elliptic equations*, Nonlinear Anal., 7 (1983), 827–850.

[11] B. Gidas, W. M. Ni, and L. Niremberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys., 68 (1979), 209–243.

[12] D. Gilbarg and N. S. Trudinger, "Elliptic Partial Differential Equations of Second Order," Springer, second edition, 1983.

[13] S. Kesavan and F. Pacella, *Symmetry of positive solutions of a quasilinear elliptic equation via isoperimetric inequality*, Appl. Anal., 54 (1994), 27–37.

[14] G. M. Lieberman, *Boundary regularity for solutions of degenerate elliptic equations*, Nonlinear Anal., 12 (1988), 1203–1219.

[15] M. K. V. Murthy and G. Stampacchia, *Boundary value problems for some degenerate-elliptic operators*, Ann. Mat. Pura Appl., 80 (1968), 1–122.

[16] J. Serrin, *A symmetry problem in potential theory*, Arch. Rational Mech. Anal., 43 (1971), 304–318.

[17] J. Serrin and M. Tang *Uniqueness of ground states for quasilinear elliptic equations*, Indiana Univ. Math. J., 49 (2000), 897– 923.

[18] P. Tolksdorf, *Regularity for a more general class of quasilinear elliptic equations*, J. Diff. Equations, 51 (1984), 126–150.

[19] N. S. Trudinger, *Linear elliptic operators with measurable coefficients*, Ann. Scuola Norm. Sup. Pisa, 27 (1973), 265–308.

[20] J. L. Vazquez, *A strong maximum principle for some quasilinear elliptic equations*, Appl. Math. Optim. (1984), 191–202.