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**Calculus of Variations**

# Monotonicity and one-dimensional symmetry for solutions of $-\Delta_p u = f(u)$ in half-spaces

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**Abstract** We prove a weak comparison principle in narrow domains for sub-super solutions to $-\Delta_p u = f(u)$ in the case $1 < p \le 2$ and $f$ locally Lipschitz continuous. We exploit it to get the monotonicity of positive solutions to $-\Delta_p u = f(u)$ in half spaces, in the case $\frac{2N+2}{N+2} < p \le 2$ and $f$ positive. Also we use the monotonicity result to deduce some Liouville-type theorems. We then consider a class of sign-changing nonlinearities and prove a monotonicity and a one-dimensional symmetry result, via the same techniques and some general a-priori estimates.

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## 1 Introduction and statement of the main results

In this paper we consider the problem

$$\begin{cases} -\Delta_p u = -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = f(u), & \text{in } \mathbb{R}^N_+ \\ u(x', y) > 0, & \text{in } \mathbb{R}^N_+ \\ u(x', 0) = 0, & \text{on } \partial\mathbb{R}^N_+ \end{cases} \tag{1.1}$$

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where we denote a generic point belonging to $\mathbb{R}^N_+$ by $(x', y)$ with $x' = (x_1, x_2, \dots, x_{N-1})$ and $y = x_N$. It is well known that solutions of $p$-Laplace equations are generally of class $C^{1,\alpha}$ (see [14,23,29]), and the equation has to be understood in the weak sense.

We first study the monotonicity of the solutions. This is an important task that naturally occurs in many applications: blow-up analysis, a-priori estimates and also in the proofs of Liouville type theorems.

The study of the monotonicity of the solutions was started in the semilinear nondegenerate case in a series of papers. We refer the readers to [2–4] and to [11,12] and [16]. Also we point out some interesting results recently obtained in [6] and in [18]. We also refer the readers to [26] and to the references therein, for the case of fully nonlinear operators.

The technique which is mostly used in this topic is the well known moving plane method which goes back to the seminal works of Alexandrov [1] and Serrin [27]. See also the celebrated papers [5,21].

The moving plane technique was adapted to the case of the $p$-laplacian operator in *bounded* domains firstly in [7] for the case $1 < p < 2$ and later in [8] for the case of positive nonlinearities and $p > 2$. Actually the technique used in [7,8] follows more closely the arguments in [5], where the application of the moving plane technique is carried out using only the weak comparison principle *in small domains* and the strong comparison principle. We recall that one of the difficulties encountered working with nonlinear operators is due to the fact that comparison principles are not equivalent to maximum principles, as for the semilinear case.

When considering the case of the half-space, the application of the moving plane technique is much more delicate since weak comparison principles in small domains have to be substituted by weak comparison principles in narrow unbounded domains.

Also the strong comparison principle does not applies simply as in the case when bounded domains are considered. In the semilinear case $p = 2$ many arguments exploited in the literature are very much related to the linear and nondegenerate nature of the operator, so that it is not possible to extend these arguments to the case of equations involving nonlinear degenerate operators. These are the main reasons for which there are no general results in the literature when dealing with the case of the $p$-laplacian.

In [10] it is considered the *two dimensional* case for positive solutions of $-\Delta_p u = f(u)$ with a positive nonlinearity $f$. It is there used a geometric technique which goes back to [3]. It seems not possible to adapt this technique to the higher dimensional case for geometric reasons.

In this paper we use a new approach based on a new weak comparison principle in narrow domains stated in the following:

**Theorem 1.1** *We suppose $N \geq 2$, $1 < p \leq 2$, $\lambda > 0$ and assume that $f$ is locally Lipschitz continuous. Set*

$$\Sigma_{(\lambda, y_0)} := \left\{ \mathbb{R}^{N-1} \times [y_0 - \frac{\lambda}{2}, y_0 + \frac{\lambda}{2}] \right\}, \quad y_0 \geq \frac{\lambda}{2}.$$

*Consider $u, v \in C^{1,\alpha}_{loc}(\Sigma_{(\lambda, y_0)})$ and $u, \nabla u, v, \nabla v \in L^\infty(\Sigma_{(\lambda, y_0)})$ such that*

$$\begin{cases} -\Delta_p u \leq f(u), & in\ \Sigma_{(\lambda, y_0)}, \\ -\Delta_p v \geq f(v), & in\ \Sigma_{(\lambda, y_0)}, \\ u \leq v, & on\ \partial\Sigma_{(\lambda, y_0)}. \end{cases} \tag{1.2}$$

*Then there exists* $\lambda_0 = \lambda_0(N, p, ||\nabla u||_\infty, ||\nabla v||_\infty, ||u||_\infty, ||v||_\infty, f) > 0$[1] *such that if,* $0 < \lambda < \lambda_0$*, it follows that*

$$u \leq v \qquad in\ \Sigma_{(\lambda, y_0)}.$$

*If u and v are not assumed to be bounded, the same conclusion holds, if we assume that the nonlinearity f is globally Lipschitz continuous.*[2]

*Remark 1.2* (i) Theorem 1.1 is proved assuming only $1 < p \leq 2$, and $f$ locally Lipschitz continuous. The proof, based on an iterative argument, is also new in the semilinear case $p = 2$. To the best of our knowledge, this is the first general weak comparison principle in narrow domains for the $p$-laplacian.

(ii) A more general class of equations can be considered exploiting exactly the same technique. More precisely, our proofs works if we replace the $p$-Laplace operator with a general operator in divergence form $\mathrm{div}(A(x\,,\,\nabla u))$, assuming that $A \in C^0(\mathbb{R}^N, \mathbb{R}^N; \mathbb{R}^N) \cap C^1(\mathbb{R}^N, \mathbb{R}^N \setminus \{0\}; \mathbb{R}^N)$ with $A(x, 0) = 0$ and

$$\sum_{i,j}^{N} |\frac{\partial A_j}{\partial \eta_i}(x, \eta)| \quad \leq \quad \Gamma|\eta|^{p-2} \qquad \forall \eta \in \mathbb{R}^N \backslash \{0\}$$

$$\sum_{i,j}^{N} \frac{\partial A_j}{\partial \eta_i}(x, \eta)\xi_i\xi_j \geq \gamma|\eta|^{p-2}|\xi|^2 \qquad \forall \eta \in \mathbb{R}^N \backslash \{0\}$$

(iii) More general domains can be considered. For instance, domains like $\Omega = \mathbb{R}^{N-K} \times \omega$ with $\omega \subset \mathbb{R}^K$ of small measure, are admissible.

We exploit Theorem 1.1 together with a translation argument which goes back to [3] that allows to recover compactness in the application of the strong comparison principle. The application of this procedure in our context is complicated by the fact that comparison principle for the limiting equations are not known in full generality, see Sect. 4. We overcome this problem in Proposition 4.1, by studying the limiting problem in the half-space and exploiting also the properties of $p$-harmonic functions. This allows us to get the following:

**Theorem 1.3** *Let* $u \in C^{1,\alpha}_{loc}(\overline{\mathbb{R}^N_+})$ *be a positive solution of* (1.1) *with* $|\nabla u| \in L^\infty(\mathbb{R}^N_+)$ *and* $\frac{2N+2}{N+2} < p \leq 2$*. Assume that the nonlinearity f is locally Lipschitz continuous and positive, that is* $f(s) > 0$ *for* $s > 0$ *with* $f(0) \geq 0$*.*

*Then u is monotone increasing w.r.t. the* $x_N$*-direction, and moreover*

$$\frac{\partial u}{\partial x_N} = \frac{\partial u}{\partial y} > 0 \quad in \quad \mathbb{R}^N_+.$$

*Remark 1.4* An important consequence of Theorem 1.3, is that actually

$$u \in C^{2,\alpha}_{loc}(\overline{\mathbb{R}^N_+}).$$

Indeed, the property $\frac{\partial u}{\partial x_N} > 0$ implies that the set of critical points $\{\nabla u = 0\}$ is empty, and consequently the equation is nondegenerate everywhere. The regularity follows therefore by standard regularity results.

[1] $\lambda_0$ will actually depend on the Lipschitz constant $L_f$ of $f$ in the interval $[-\max\{\|u\|_\infty, \|v\|_\infty\}, \max\{\|u\|_\infty, \|v\|_\infty\}]$.

[2] In this case $\lambda_0$ will depend on the Lipschitz constant of $f$ in $\mathbb{R}$.

Exploiting Theorem 1.1 and arguing as in the proof of Theorem 1.3 we also deduce the following:

**Theorem 1.5** *Let $u \in C^{1,\alpha}_{loc}(\mathbb{R}^{N-1} \times [0,h])$ be a positive solution to*

$$\begin{cases} -\Delta_p u = f(u), & in \quad \mathbb{R}^{N-1} \times (0,h) \\ u(x',0) = u(x',h) = 0, & for\ any \quad x' \in \mathbb{R}^{N-1}. \end{cases} \tag{1.3}$$

*Assume that $|\nabla u|$ is bounded and that $\frac{2N+2}{N+2} < p \le 2$. Assume also that the nonlinearity $f$ is locally Lipschitz continuous and positive, that is $f(s) > 0$ for $s > 0$ with $f(0) \ge 0$.*

*Then $u$ is symmetric with respect to the hyperplane $\{y = h/2\}$ with*

$$\frac{\partial u}{\partial x_N} = \frac{\partial u}{\partial y} > 0 \quad \forall x' \in \mathbb{R}^{N-1} \qquad \forall y \in [0, h/2)$$

*Furthermore, for every $\theta \in (0, h/2)$, we have*

$$u(x',y) < u_\theta(x',y) = u(x', 2\theta - y) \quad \forall x' \in \mathbb{R}^{N-1} \qquad \forall y \in [0,\theta)$$

In Theorem 7.1, we will discuss some consequences of Theorem 1.3 for the case $N = 3$, the case $N = 2$ being already considered in [10]. In particular, we assume $\frac{8}{5} < p \le 2$, that is the condition $\frac{2N+2}{N+2} < p \le 2$ for $N = 3$. Assuming also that the nonlinearity $f$ is positive, that is $f(s) > 0$ for $s > 0$, and that $f(0) = 0$, we show that the equation $-\Delta_p u = f(u)$, under zero Dirichlet boundary condition, has no bounded non-negative non-trivial solutions in the half space.

In particular in the proof of Theorem 7.1 we will use some recent results in [19] (see also [20]) to show that the solution $u$ must have one-dimensional symmetry, that is, $u$ must depend only on the variable $x_N$. A consequently ODE analysis allows to get the desired Liouville type theorem and shows that actually the only nonnegative solution is the trivial one.
We consider the higher dimensional case in Theorem 8.1, where we show that there are no bounded non-negative non-trivial solutions to $-\Delta_p u = f(u)$ in the half-space, if $N \ge 3$, $\frac{2N+2}{N+2} < p < 2$, and the nonlinearity $f$ is positive and subcritical, w.r.t. the Sobolev critical exponent in $\mathbb{R}^{N-1}$ (see Remark 8.2). The proof of Theorem 8.1 is a consequence of Theorem 1.3, which is exploited following some ideas from [16].

The monotonicity of the solution $u$ allows to define a limiting profile, which is a non-negative bounded solution to the equation $-\Delta_p u = f(u)$ in $\mathbb{R}^{N-1}$. Theorem 8.1 then follows using the Liouville-type results in [28].

In Theorem 8.3, by making use of a similar strategy, we prove that there are no bounded non-negative non-trivial solutions to $-\Delta_p u = f(u)$ in the half-space, if $N \ge 3$, $\frac{2N+2}{N+2} < p < 2$, and the nonlinearity $f$ is positive with $f(s) \ge \lambda s^{\frac{(N-1)(p-1)}{N-1-p}}$ in $[0,\delta]$, for some $\lambda, \delta > 0$.

The results in Theorem 7.1, Theorem 8.1 and Theorem 8.3 are summarized in the following:

**Theorem 1.6** *Let $u \in C^1(\overline{\mathbb{R}^N_+}) \cap W^{1,\infty}(\mathbb{R}^N_+)$ be a non-negative weak solution of (1.1) in $\mathbb{R}^N_+$, with $\frac{2N+2}{N+2} < p < 2$. Assume that one of the following holds* [3]:

(a) *$N = 3$ and $f(s) > 0$ for $s > 0$, with $f(0) = 0$,*
(b) *$N \ge 3$, $f(s) > 0$ for $s > 0$, $f(0) = 0$ and $f$ is subcritical w.r.t. the Sobolev critical exponent in $\mathbb{R}^{N-1}$,*

[3] The case $N = 2$ was already considered in [10].

(c) (c) $N \geq 3 f(s) > 0$ *for* $s > 0$, $f(0) = 0$ *and* $f(s) \geq \lambda s^{\frac{(N-1)(p-1)}{N-1-p}}$ *in* $[0, \delta]$, *for some* $\lambda, \delta > 0$,

*Then* $u = 0$.

*On the other hand, if* $N \geq 3$ *and* $f(0) > 0$, *then there are no non-negative solutions of* (1.1).

We refer the reader to [10,13,31] for others Liouville-type theorems in half-spaces and in the nonlinear degenerate setting.

Generally, supercritical problems do have non-trivial solutions, which therefore turns out to be monotone by our Theorem 1.3. Nontrivial solutions also exist for changing-sign nonlinearities $f$. A particular and interesting class of such nonlinear functions arises in phase transitions models, see Remark 1.9 below.

To deal with this class of problems, we prove the following:

**Theorem 1.7** *Assume* $N \geq 1$, $p > 1$. *Suppose that* $f : \mathbb{R} \to \mathbb{R}$ *be a continuous function satisfying*:

$$\exists z > 0 \quad : \qquad u > z \Rightarrow f(u) < 0$$

*and let* $u \in C^1(\overline{\mathbb{R}^N_+})$ *be a solution of*

$$\begin{cases} -\Delta_p u \leq f(u) & \text{in } \mathbb{R}^N_+ \\ u \leq 0 & \text{on } \partial\mathbb{R}^N_+ \\ u \leq C & \text{in } \mathbb{R}^N_+ \end{cases} \tag{1.4}$$

*Then,* $u \leq z$.

*Furthermore, if* $f$ *is locally Lipschitz and* $1 < p \leq 2$, *then* $u < z$.

The combination of Theorem 1.7 and the techniques developed in this work, leads to the following

**Theorem 1.8** *Let* $u \in C^{1,\alpha}_{loc}(\overline{\mathbb{R}^N_+}) \cap W^{1,\infty}(\mathbb{R}^N_+)$ *be a positive solution of* (1.1), *with* $\frac{2N+2}{N+2} < p \leq 2$. *Assume that*

$$\exists z > 0 \quad : \quad 0 < u < z \Rightarrow f(u) > 0 \quad \text{and} \quad u > z \Rightarrow f(u) < 0.$$

*Then* $u$ *is monotone increasing w.r.t. the* $x_N$*-direction, and moreover*

$$\frac{\partial u}{\partial x_N} > 0 \quad \text{in} \quad \mathbb{R}^N_+.$$

*Furthermore, if* $N = 2$ *or* $N = 3$, *it follows that* $u$ *has one-dimensional symmetry. More precisely*

$$u(x', y) = \bar{u}(y).$$

Some results related to Theorem 1.8, and obtained under stronger assumptions on the nonlinearity $f$, can be found in [15].

*Remark 1.9* Theorem 1.8 applies for example to the problem

$$\begin{cases} -\Delta_p u = u(1-u^2)|1-u^2|^q, & \text{in } \mathbb{R}^N_+ \\ u(x', y) > 0, & \text{in } \mathbb{R}^N_+ \\ u(x', 0) = 0, & \text{on } \partial\mathbb{R}^N_+. \end{cases} \tag{1.5}$$

where $q \geq 0$. The above equation reduces to the equation

$$-\Delta u = u(1 - u^2)$$

when $p = 2$ and $q = 0$, that is the equation arising in a famous conjecture of De Giorgi. We refer the readers to [17] for a survey on this topic.

The paper is organized as follows: in Sect. 2 we prove Lemma 2.1, that will be crucial in the proof of Theorem 1.1. In Sect. 3 we prove Theorem 1.1. In Sect. 4 we get an auxiliary proposition which allows us to exploit the moving plane technique. In Sect. 5 we prove Theorem 1.3. In Sect. 6 we demonstrate Theorem 1.5. In Sect. 7 we prove a Liouville-type theorem in low dimension while Liouville-type theorems in higher dimensions are considered in Sect. 8. In Sect. 9 we obtain some a priori estimates which are used in Sect. 10 to prove Theorem 1.7. In Sect. 11 we give the proof of Theorem 1.8.

## 2 Preliminary results

We start proving a lemma that will be useful in the proof of Theorem 1.1:

**Lemma 2.1** *Let $\theta > 0$ and $\gamma > 0$ such that $\theta < 2^{-\gamma}$. Moreover let $R_0 > 0, c > 0$ and*

$$\mathcal{L} : (R_0, +\infty) \to \mathbb{R}$$

*a non-negative and non-decreasing function such that*

$$\begin{cases} \mathcal{L}(R) \leq \theta\mathcal{L}(2R) + g(R) & \forall R > R_0, \\ \mathcal{L}(R) \leq CR^\gamma & \forall R > R_0, \end{cases} \tag{2.1}$$

*where $g : (R_0, +\infty) \to \mathbb{R}^+$ is such that*

$$\lim_{R \to +\infty} g(R) = 0.$$

*Then*

$$\mathcal{L}(R) = 0.$$

*Proof* It is sufficient to prove that

$$l := \lim_{R \to +\infty} \mathcal{L}(R) = 0.$$

By contradiction suppose that $l \neq 0$ and choose $\theta_1$ such that $\theta < \theta_1 < 2^{-\gamma}$. This implies the exixtence of $R_1 = R_1(\theta_1) \geq R_0$ such that

$$(\theta - \theta_1)\mathcal{L}(2R) + g(R) < 0 \quad \forall R \geq R_1,$$

and then

$$\mathcal{L}(R) \leq \theta_1\mathcal{L}(2R) \quad \forall R \geq R_1. \tag{2.2}$$

By (2.2) we have: $\forall \bar{l} \in \mathbb{N}^*, \ \forall R \geq R_1$

$$\begin{aligned} \mathcal{L}(R) &\leq \theta_1^{\bar{l}}\mathcal{L}(2^{\bar{l}}R) \\ &\leq C\theta_1^{\bar{l}}(2^{\bar{l}}R)^\gamma \\ &= C(2^\gamma\theta_1)^{\bar{l}}R^\gamma, \end{aligned} \tag{2.3}$$

where we have used that $\mathcal{L}(R) \leq CR^{\gamma}$ for $R > R_0$, by (2.1).

Since $0 < \theta_1 < 2^{-\gamma}$, by (2.3) we obtain

$$\mathcal{L}(R) \leq \lim_{\bar{l} \to +\infty} C(2^{\gamma}\theta_1)^{\bar{l}} R^{\gamma} = 0 \quad \forall R \geq R_1,$$

getting the contradiction. □

Below we recall some known results regarding $p$-Laplace equations. Referring to [30] for the case of the $p$-Laplace operator, and to [25] for the case of a broad class of quasilinear elliptic operators, we recall the following:

**Theorem 2.2** (Strong Maximum Principle and Hopf's Lemma). *Let $\Omega$ be a domain in $\mathbb{R}^N$ and suppose that $u \in C^1(\Omega)$, $u \geq 0$ in $\Omega$, weakly solves*

$$-\Delta_p u + cu^q = g \geq 0 \quad in \quad \Omega$$

*with $1 < p < \infty, q \geq p - 1, c \geq 0$ and $g \in L^{\infty}_{loc}(\Omega)$. If $u \neq 0$ then $u > 0$ in $\Omega$. Moreover for any point $x_0 \in \partial\Omega$ where the interior sphere condition is satisfied, and such that $u \in C^1(\Omega \cup \{x_0\})$ and $u(x_0) = 0$ we have that $\frac{\partial u}{\partial s} > 0$ for any inward directional derivative (this means that if $y$ approaches $x_0$ in a ball $B \subseteq \Omega$ that has $x_0$ on its boundary, then $\lim_{y \to x_0} \frac{u(y)-u(x_0)}{|y-x_0|} > 0$).*

Also we will make repeated use of the following strong comparison principle (see [9]):

**Theorem 2.3** (Strong Comparison Principle) *Let $u, v \in C^1(\overline{\Omega})$ where $\Omega$ is a bounded smooth domain of $\mathbb{R}^N$ with $\frac{2N+2}{N+2} < p < \infty$. Suppose that either $u$ or $v$ is a weak solution of $-\Delta_p(w) = f(w)$ with $f$ positive and locally Lipschitz continuous. Assume*

$$-\Delta_p(u) - f(u) \leq -\Delta_p(v) - f(v) \qquad u \leq v \quad in \quad \Omega \tag{2.4}$$

*Then $u \equiv v$ in $\Omega$ or $u < v$ in $\Omega$.*

Let us recall that the linearized operator $L_u(v, \varphi)$ at a fixed solution $u$ of $-\Delta_p(u) = f(u)$ is well defined, for every $v, \varphi \in H^{1,2}_{\rho}(\Omega)$ with $\rho \equiv |\nabla u|^{p-2}$(see [8] for details), by

$$L_u(v, \varphi) \equiv \int_{\Omega} [|\nabla u|^{p-2}(\nabla v, \nabla\varphi) + (p-2)|\nabla u|^{p-4}(\nabla u, \nabla v)(\nabla u, \nabla\varphi) - f'(u)v\varphi]dx$$

Moreover, $v \in H^{1,2}_{\rho}(\Omega)$ is a weak solution of the linearized equation if

$$L_u(v, \varphi) = 0 \tag{2.5}$$

for any $\varphi \in H^{1,2}_{0,\rho}(\Omega)$.

By [8] we have $u_{x_i} \in H^{1,2}_{\rho}(\Omega)$ for $i = 1, \dots, N$, and $L_u(u_{x_i}, \varphi)$ is well defined for every $\varphi \in H^{1,2}_{0,\rho}(\Omega)$, with

$$L_u(u_{x_i}, \varphi) = 0 \qquad \forall \varphi \in H^{1,2}_{0,\rho}(\Omega). \tag{2.6}$$

In other words, the derivatives of $u$ are weak solutions of the linearized equation. Consequently by a strong maximum principle for the linearized operator (see [9]) we have the following:

**Theorem 2.4** *Let $u \in C^1(\overline{\Omega})$ be a weak solution of $-\Delta_p(u) = f(u)$ in a bounded smooth domain $\Omega$ of $\mathbb{R}^N$ with $\frac{2N+2}{N+2} < p < \infty$, and $f$ positive and locally Lipschitz continuous. Then, for any $i \in \{1, \dots, N\}$ and any domain $\Omega' \subset \Omega$ with $u_{x_i} \geq 0$ in $\Omega'$, we have either $u_{x_i} \equiv 0$ in $\Omega'$ or $u_{x_i} > 0$ in $\Omega'$.*

## 3 A weak comparison principle in narrow domains

Proof of Theorem 1.1

We prove here Theorem 1.1. We therefore assume that $N \geq 2$, $1 < p \leq 2$, $\lambda > 0$ and that $f$ is locally Lipschitz continuous. We set

$$\Sigma_{(\lambda,y_0)} := \left\{ \mathbb{R}^{N-1} \times [y_0 - \frac{\lambda}{2}, y_0 + \frac{\lambda}{2}] \right\}, \quad y_0 \geq \frac{\lambda}{2},$$

and we consider $u, v \in C^{1,\alpha}_{loc}$ with $u, \nabla u, v, \nabla v \in L^\infty(\Sigma_{(\lambda,y_0)})$ such that $u, v$ weakly solve (1.2).

We want to show that there exists $\lambda_0 > 0$ such that if $0 < \lambda < \lambda_0$, then

$$u \leq v \qquad \text{in } \Sigma_{(\lambda,y_0)}.$$

We carry out the proof in the case $u, v \in L^\infty(\Sigma_{(\lambda,y_0)})$.The same proof works when $u$ and $v$ may be not bounded, but $f$ is *globally* Lipschitz continuous.

In the sequel we further use the following inequalities:
$\forall \eta, \eta' \in \mathbb{R}^N$ with $|\eta| + |\eta'| > 0$ there exists positive constants $C_1, C_2$ depending on $p$ such that

$$\begin{aligned} [|\eta|^{p-2}\eta - |\eta'|^{p-2}\eta'][\eta - \eta'] &\geq C_1(|\eta| + |\eta'|)^{p-2}|\eta - \eta'|^2, \\ ||\eta|^{p-2}\eta - |\eta'|^{p-2}\eta'| &\leq C_2|\eta - \eta'|^{p-1}, \quad \text{if } 1 < p \leq 2. \end{aligned} \tag{3.1}$$

First of all we remark that $(u - v)^+ \in L^\infty(\Sigma_{(\lambda,y_0)})$ since we assumed $u, v$ to be bounded in $\Sigma_{(\lambda,y_0)}$.

Let us now define

$$\Psi = [(u - v)^+]^\alpha \varphi^2, \tag{3.2}$$

where $\alpha > 1$, and $\varphi(x', y) = \varphi(x') \in C^\infty_c(\mathbb{R}^{N-1})$, $\varphi \geq 0$ such that

$$\begin{cases} \varphi \equiv 1, & \text{in } B'(0, R) \subset \mathbb{R}^{N-1}, \\ \varphi \equiv 0, & \text{in } \mathbb{R}^{N-1} \setminus B'(0, 2R), \\ |\nabla\varphi| \leq \frac{C}{R}, & \text{in } B'(0, 2R) \setminus B'(0, R) \subset \mathbb{R}^{N-1}. \end{cases} \tag{3.3}$$

We note that $\Psi \in W^{1,p}_0(\Sigma_{(\lambda,y_0)})$ by (3.3) and since $u \leq v$ on $\partial\Sigma_{(\lambda,y_0)}$.

Let us define the cylinder

$$\mathcal{C}(R) := \left\{ \Sigma_{(\lambda,y_0)} \cap \overline{\{B'(0, R) \times \mathbb{R}\}} \right\}.$$

Then using $\Psi$ as test function in both equations of problem (1.2) and substracting we get

$$\begin{aligned} &\alpha \int_{\mathcal{C}(2R)} \left\langle \left( |\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v \right), \nabla(u - v)^+ \right\rangle [(u - v)^+]^{\alpha-1}\varphi^2 \\ &\quad + \int_{\mathcal{C}(2R)} \left\langle \left( |\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v \right), \nabla\varphi^2 \right\rangle [(u - v)^+]^\alpha \\ &= \int_{\mathcal{C}(2R)} (f(u) - f(v))[(u - v)^+]^\alpha \varphi^2. \end{aligned}$$

Taking into account (3.1) and the fact that $p \leq 2$, we have

$$
\begin{aligned}
&\alpha C_1 \int\limits_{\mathcal{C}(2R)} (|\nabla u| + |\nabla v|)^{p-2} |\nabla (u-v)^+|^2 [(u-v)^+]^{\alpha-1} \varphi^2 \\
&\leq \alpha \int\limits_{\mathcal{C}(2R)} \left\langle \left( |\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right), \nabla (u-v)^+ \right\rangle [(u-v)^+]^{\alpha-1} \varphi^2 \\
&= - \int\limits_{\mathcal{C}(2R)} \left\langle \left( |\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right), \nabla \varphi^2 \right\rangle [(u-v)^+]^{\alpha} \\
&\quad + \int\limits_{\mathcal{C}(2R)} (f(u) - f(v)) [(u-v)^+]^{\alpha} \varphi^2 \\
&\leq \int\limits_{\mathcal{C}(2R)} \left| \left\langle \left( |\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right), \nabla \varphi^2 \right\rangle \right| [(u-v)^+]^{\alpha} \\
&\quad + \int\limits_{\mathcal{C}(2R)} |(f(u) - f(v))| \, [(u-v)^+]^{\alpha} \varphi^2 \\
&\leq C_2 \int\limits_{\mathcal{C}(2R)} |\nabla (u-v)|^{p-1} |\nabla \varphi^2| [(u-v)^+]^{\alpha} \\
&\quad + \int\limits_{\mathcal{C}(2R)} \left| \frac{(f(u) - f(v))}{(u-v)} \right| [(u-v)^+]^{\alpha+1} \varphi^2. 
\end{aligned} \tag{3.4}
$$

Then, since $u, v \in C^{1,\alpha}_{loc}$ have bounded gradient by assumption, one has

$$
\begin{aligned}
&\alpha c_1 \int\limits_{\mathcal{C}(2R)} |\nabla (u-v)^+|^2 [(u-v)^+]^{\alpha-1} \varphi^2 \\
&\leq c_2 \int\limits_{\mathcal{C}(2R)} [(u-v)^+]^{\alpha} |\nabla \varphi^2| + \int\limits_{\mathcal{C}(2R)} L_f [(u-v)^+]^{\alpha+1} \varphi^2 \\
&:= c_2 I_1 + L_f I_2,
\end{aligned} \tag{3.5}
$$

where

$$
c_1 = (||\nabla u||_\infty + ||\nabla v||_\infty)^{p-2} C_1,
$$

$$
c_2 = (||\nabla u||_\infty + ||\nabla v||_\infty)^{p-1} C_2.
$$

$L_f$ is the Lipschitz constant of $f$ in the interval $[-\max\{\|u\|_\infty, \|v\|_\infty\}, \max\{\|u\|_\infty, \|v\|_\infty\}]$. We now evaluate the term

$$
I_1 = \int\limits_{\mathcal{C}(2R)} [(u-v)^+]^{\alpha} |\nabla \varphi^2|.
$$

$$
I_1 \leq 2 \int\limits_{\mathcal{C}(2R)} [(u-v)^+]^{\alpha} \varphi |\nabla \varphi| = 2 \int\limits_{\mathcal{C}(2R)} [(u-v)^+]^{\alpha} \varphi |\nabla \varphi|^{\frac{1}{2}} |\nabla \varphi|^{\frac{1}{2}}
$$

$$
\begin{aligned}
&\le 2 \int_{\mathcal{C}(2R)} \frac{[(u-v)^+]^{\alpha+1} \varphi^{\frac{\alpha+1}{\alpha}} |\nabla \varphi|^{\frac{\alpha+1}{2\alpha}}}{\frac{\alpha+1}{\alpha}} + 2 \int_{\mathcal{C}(2R)} \frac{|\nabla \varphi|^{\frac{\alpha+1}{2}}}{\alpha+1} \\
&\le 2 \int_{\mathbb{R}^{N-1}} \left( \int_{y_0-\frac{\lambda}{2}}^{y_0+\frac{\lambda}{2}} \left( [(u-v)^+]^{\frac{\alpha+1}{2}} \right)^2 dy \right) \varphi^{\frac{\alpha+1}{\alpha}} |\nabla \varphi|^{\frac{\alpha+1}{2\alpha}} dx' + 2 \int_{\mathcal{C}(2R)} |\nabla \varphi|^{\frac{\alpha+1}{2}} \\
&\le C_p^2(\lambda) \frac{(\alpha+1)^2}{2} \int_{\mathcal{C}(2R)} [(u-v)^+]^{\alpha-1} |\partial_y (u-v)^+|^2 \varphi^{\frac{\alpha+1}{\alpha}} |\nabla \varphi|^{\frac{\alpha+1}{2\alpha}} + 2 \int_{\mathcal{C}(2R)} |\nabla \varphi|^{\frac{\alpha+1}{2}} \\
&\le C_p^2(\lambda) \frac{(\alpha+1)^2}{2} \int_{\mathcal{C}(2R)} [(u-v)^+]^{\alpha-1} |\nabla (u-v)^+|^2 \varphi^{\frac{\alpha+1}{\alpha}} |\nabla \varphi|^{\frac{\alpha+1}{2\alpha}} + 2 \int_{\mathcal{C}(2R)} |\nabla \varphi|^{\frac{\alpha+1}{2}}.
\end{aligned} \tag{3.6}
$$

In (3.6) we used Young inequality with conjugate exponents $\left(\frac{\alpha+1}{\alpha}, \alpha+1\right)$, a Poincaré inequality in the set $[y_0 - \frac{\lambda}{2}, y_0 + \frac{\lambda}{2}]$, denoting with $C_p$ the associated constant and the fact that $\varphi = \varphi(x')$.

We now evaluate the term

$$
\begin{aligned}
I_2 &= \int_{\mathcal{C}(2R)} [(u-v)^+]^{\alpha+1} \varphi^2 \\
I_2 &= \int_{\mathcal{C}(2R)} \left( [(u-v)^+]^{\frac{\alpha+1}{2}} \right)^2 \varphi^2 \\
&= \int_{\mathbb{R}^{N-1}} \left( \int_{y_0-\frac{\lambda}{2}}^{y_0+\frac{\lambda}{2}} \left( [(u-v)^+]^{\frac{\alpha+1}{2}} \right)^2 dy \right) (\varphi(x'))^2 dx' \\
&\le C_p^2(\lambda) \int_{\mathbb{R}^{N-1}} \left( \int_{y_0-\frac{\lambda}{2}}^{y_0+\frac{\lambda}{2}} \left( \frac{\alpha+1}{2} \right)^2 [(u-v)^+]^{\alpha-1} |\partial_y (u-v)^+|^2 dy \right) (\varphi(x'))^2 dx' \\
&= C_p^2(\lambda) \left( \frac{\alpha+1}{2} \right)^2 \int_{\mathcal{C}(2R)} [(u-v)^+]^{\alpha-1} |\nabla (u-v)^+|^2 \varphi^2
\end{aligned} \tag{3.7}
$$

Now we are going to choose the constants $\alpha > 1$ and $\lambda > 0$ in such a way

$$
L_f C_p^2(\lambda) \left(\frac{\alpha+1}{2}\right)^2 < \frac{\alpha c_1}{2} \tag{3.8}
$$

so that from (3.5) we have

$$
\begin{aligned}
&\alpha \frac{c_1}{2} \int_{\mathcal{C}(2R)} |\nabla (u-v)^+|^2 [(u-v)^+]^{\alpha-1} \varphi^2 \\
&\quad \le c_2 \int_{\mathcal{C}(2R)} [(u-v)^+]^{\alpha} |\nabla \varphi^2| = c_2 I_1.
\end{aligned} \tag{3.9}
$$

From (3.3) one has that

$$\alpha\frac{c_1}{2}\int\limits_{\mathcal{C}(R)}|\nabla(u-v)^+|^2(u-v)^{\alpha-1}$$
$$\leq \alpha\frac{c_1}{2}\int\limits_{\mathcal{C}(2R)}|\nabla(u-v)^+|^2(u-v)^{\alpha-1}\varphi^2\leq c_2 I_1. \tag{3.10}$$

Consequently we obtain

$$\int\limits_{\mathcal{C}(R)}|\nabla(u-v)^+|^2[(u-v)^+]^{\alpha-1}$$
$$\leq \frac{c_2}{\alpha c_1}C_p^2(\lambda)(\alpha+1)^2\int\limits_{\mathcal{C}(2R)}[(u-v)^+]^{\alpha-1}|\nabla(u-v)^+|^2\varphi^{\frac{\alpha+1}{\alpha}}|\nabla\varphi|^{\frac{\alpha+1}{2\alpha}}$$
$$+4\frac{c_2}{\alpha c_1}\int\limits_{\mathcal{C}(2R)}|\nabla\varphi|^{\frac{\alpha+1}{2}}. \tag{3.11}$$

From (3.11), setting $\alpha=2N+1$, one has

$$\int\limits_{\mathcal{C}(R)}|\nabla(u-v)^+|^2(u-v)^{\alpha-1}$$
$$\leq\theta\int\limits_{\mathcal{C}(2R)}|\nabla(u-v)^+|^2[(u-v)^+]^{\alpha-1}+4\frac{c_2}{\alpha c_1}C\lambda R^{N-1}R^{-(N+1)}$$
$$=\theta\int\limits_{\mathcal{C}(2R)}|\nabla(u-v)^+|^2[(u-v)^+]^{\alpha-1}+c_3R^{-2}, \tag{3.12}$$

where

$$c_3=4\frac{c_2}{\alpha c_1}C\lambda\in\mathbb{R}^+,$$
$$\frac{c_2}{\alpha c_1}C_p^2(\lambda)(2N+2)^2=\theta<2^{-N}.$$

In particular to do this, recalling that $C_p^2(\lambda)\simeq\lambda^2$, $\lambda>0$ will be taken such that

$$\frac{c_2}{\alpha c_1}C_p^2(\lambda)(\alpha+1)^2<2^{-N}. \tag{3.13}$$

Let us set

$$\mathcal{L}(R)=\int\limits_{\mathcal{C}(R)}|\nabla(u-v)^+|^2[(u-v)^+]^{\alpha-1},$$

and

$$g(R)=c_3R^{-2}.$$

Then one has

$$\begin{cases}\mathcal{L}(R)\leq\theta\mathcal{L}(2R)+g(R) & \forall R>0,\\ \mathcal{L}(R)\leq CR^N & \forall R>0,\end{cases}$$

and from Lemma 2.1 with $\gamma = N$, since we assumed $\theta < 2^{-N}$, we get $\mathcal{L}(R) = 0$ and consequently the thesis.

## 4 Recovering compacteness

Let us consider $u$ to be a positive solution of (1.1), that is

$$\begin{cases} -\Delta_p u = f(u), & \text{in } \mathbb{R}^N_+ \\ u(x', y) > 0, & \text{in } \mathbb{R}^N_+ \\ u(x', 0) = 0, & \text{on } \partial\mathbb{R}^N_+ \end{cases}$$

As in Theorem 1.3 let $u \in C^{1,\alpha}_{loc}$, $\nabla u \in L^\infty(\mathbb{R}^N_+)$ and assume $\frac{2N+2}{N+2} < p \leq 2$. Assume that the nonlinearity $f$ is locally Lipschitz continuous and positive, that is $f(s) > 0$ for $s > 0$, with $f(0) \geq 0$. Let us set $\bar\lambda = \sup \Lambda$ where

$$\Lambda \equiv \{t > 0 \mid u \leq u_\theta \quad \text{in} \quad \Sigma_\theta \quad \forall \theta \leq t\}$$

with $\Sigma_\theta = \{0 < y < \theta\}$, and as usual

$$u_\theta(x', y) = u(x', 2\theta - y).$$

By Theorem 1.1 we know that $\Lambda$ is not empty, since $t \in \Lambda$ for sufficiently small $t$. Our aim is to show that actually $\bar\lambda = \sup \Lambda = \infty$ which actually implies monotonicity. We assume therefore from now on by contradiction that $\bar\lambda < \infty$. It follows therefore that $u$ is bounded in $\Sigma_{\bar\lambda}$ by the Dirichlet condition on the boundary and the fact that $|\nabla u|$ is bounded by assumption. In the same way it follows that $u_\theta$ is bounded in $\Sigma_{\bar\lambda}$ too.

By continuity $u \leq u_{\bar\lambda}$, and consequently

$$u < u_{\bar\lambda}$$

by the Strong Comparison Principle (see Theorem 2.3). Note that the Strong Comparison Principle actually says that $u < u_{\bar\lambda}$ unless $u = u_{\bar\lambda}$, but the latter case is not possible by the Dirichlet assumption $u(x', 0) = 0 < u_{\bar\lambda}(x', 0)$.

Let us now define

$$W_\varepsilon = (u - u_{\bar\lambda+\varepsilon}) \cdot \chi_{\{y \leq \bar\lambda+\varepsilon\}},$$

with $\chi(\cdot)$ denoting the characteristic function of a set. We have the following

**Proposition 4.1** *Given* $0 < \delta < \frac{\bar\lambda}{2}$, *we find* $\varepsilon_0$ *such that, for any* $\varepsilon \leq \varepsilon_0$, *it follows*

$$Supp\ W_\varepsilon^+ \subset \{0 \leq y \leq \delta\} \cup \{\bar\lambda - \delta \leq y \leq \bar\lambda + \varepsilon\}.$$

*That is,* $W_\varepsilon^+$ *vanishes outside the set* $\{0 \leq y \leq \delta\} \cup \{\bar\lambda - \delta \leq y \leq \bar\lambda + \varepsilon\}$.

*Proof* Assume by contradiction that the thesis is false, so that there exists $\delta > 0$, with $0 < \delta < \frac{\bar\lambda}{2}$, in such a way that, given any $\varepsilon_0 > 0$, we find $\varepsilon \leq \varepsilon_0$ so that there exists a corresponding $x_\varepsilon = (x'_\varepsilon, y_\varepsilon)$ such that $u(x'_\varepsilon, y_\varepsilon) \geq u_{\bar\lambda+\varepsilon}(x'_\varepsilon, y_\varepsilon)$ and $\delta \leq y_\varepsilon \leq \bar\lambda - \delta$.

Take now $\varepsilon_0 = \frac{1}{n}$, then there exists $\varepsilon_n \leq \varepsilon_0$ going to zero, and a corresponding $x_n = (x'_n, y_n) = (x'_{\varepsilon_n}, y_{\varepsilon_n})$ with

$$u(x'_n, y_n) \geq u_{\bar\lambda+\varepsilon_n}(x'_n, y_n)$$

with $\delta \leq y_n \leq \bar\lambda - \delta$. Up to subsequences, let us assume that

$$y_n \to y_0 \text{ with } \delta \leq y_0 \leq \bar{\lambda} - \delta.$$

Let us now define

$$\tilde{u}_n(x', y) = u(x' + x'_n, y)$$

so that $\|\tilde{u}_n\|_\infty = \|u\|_\infty \leq C$.
Let us first consider the case $f(0) = 0$. In this situation we can consider $u$, (and consequently $\tilde{u}_n$) defined on the entire space $\mathbb{R}^N$ by odd reflection. That is

$$u(x', y) = -u(x', -y) \quad \text{in} \quad \{y < 0\}$$

with consequently $f(t) = -f(-t) \quad \text{if} \quad \{t < 0\}$.

By standard regularity theory, see [14,29], since $\|\tilde{u}_n\|_\infty = \|u\|_\infty \leq C$, we have that

$$\|\tilde{u}_n\|_{C^{1,\alpha}_{loc}(\mathbb{R}^N)} \leq C$$

for some $0 < \alpha < 1$. By Ascoli's Theorem we have

$$\tilde{u}_n \overset{C^{1,\alpha'}_{loc}(\mathbb{R}^N)}{\longrightarrow} \tilde{u} \tag{4.1}$$

up to subsequences, for $\alpha' < \alpha$. We consider $\tilde{u}$ in the entire space $\mathbb{R}^N$ constructed by a standard diagonal process.

We claim now that

- $\tilde{u} \geq 0$ in $\mathbb{R}^N_+$, with $\tilde{u}(x, 0) = 0$
- $\tilde{u} \leq \tilde{u}_{\bar{\lambda}}$ in $\Sigma_{\bar{\lambda}}$
- $\tilde{u}(0, y_0) \geq \tilde{u}_{\bar{\lambda}}(0, y_0)$ (with actually $\tilde{u}(0, y_0) = \tilde{u}_{\bar{\lambda}}(0, y_0)$).

The fact that $\tilde{u} \geq 0$ in $\mathbb{R}^N_+$ follows immediately by the uniform convergence on compact sets and the fact that $\tilde{u}_n(x', y)$ are positive by construction. Also we have that $\tilde{u} \leq \tilde{u}_{\bar{\lambda}}$ in $\Sigma_{\bar{\lambda}}$ by the definition of $\bar{\lambda}$ and taking again into account the uniform convergence. Also, passing to the limit, it is easy to see that

$$\tilde{u}(0, y_0) \geq \tilde{u}_{\bar{\lambda}}(0, y_0).$$

Since $\tilde{u} \leq \tilde{u}_{\bar{\lambda}}$ as shown above, actually we have $\tilde{u}(0, y_0) = \tilde{u}_{\bar{\lambda}}(0, y_0)$.

It is now standard to see that $-\Delta_p \tilde{u} = f(\tilde{u})$ in $\mathbb{R}^N_+$, that is

$$\int_{\mathbb{R}^N_+} |\nabla \tilde{u}|^{p-2}(\nabla \tilde{u}, \nabla \varphi) = \int_{\mathbb{R}^N_+} f(\tilde{u})\varphi \quad \forall \varphi \in C^\infty_c(\mathbb{R}^N_+).$$

Since $\tilde{u} \geq 0$ in $\mathbb{R}^N_+$, by the Strong Maximum Principle (see [25,30]), it follows now that $\tilde{u} > 0$ or $\tilde{u} = 0$. We reduce therefore to consider the case $\tilde{u} > 0$ and the case $\tilde{u} = 0$. We will prove the thesis showing that in both cases we have a contradiction.

*Case-1* ($\tilde{u} > 0$ ). In this case, if $f(0) = 0$, we have

$$-\Delta_p \tilde{u} = f(\tilde{u}) \qquad \text{in} \qquad \mathbb{R}^N_+$$

and we recall that we are assuming

- $f(s) > 0$ for $s > 0$
- $\frac{2N+2}{N+2} < p \leq 2$

so that by the Strong Comparison Principle (see Theorem 2.3), we get that $\tilde{u} \leq \tilde{u}_{\bar{\lambda}}$ implies $\tilde{u} < \tilde{u}_{\bar{\lambda}}$, since the case $\tilde{u} = \tilde{u}_{\bar{\lambda}}$ is clearly impossible being $\tilde{u}(x, 0) = 0$. This is a contradiction since $\tilde{u}(0, y_0) = \tilde{u}_{\bar{\lambda}}(0, y_0)$.

If else $f(0) > 0$, again by standard interior estimates, we have, proceeding as before,

$$\tilde{u}_n \longrightarrow \tilde{u} \tag{4.2}$$

in $C^{1,\alpha'}_{loc}(\mathbb{R}^N_+) \cap C^0(\overline{\mathbb{R}^N_+})$ up to subsequences for $\alpha' < \alpha$.

Since $(0, y_0)$ and it's reflection $(0, 2\bar{\lambda} - y_0)$ belong to $\mathbb{R}^N_+$, we still have a contradiction by the Strong Comparison Principle, since we should have $\tilde{u} < \tilde{u}_{\bar{\lambda}}$, by the fact that $-\Delta_p \tilde{u} = f(\tilde{u})$ in $\mathbb{R}^N_+$, $\tilde{u}(x, 0) = 0$ and we recall that we are assuming $f(s) > 0$ for $s \geq 0$ with $\frac{2N+2}{N+2} < p \leq 2$. But this is a contradiction since again by construction $\tilde{u}(0, y_0) = \tilde{u}_{\bar{\lambda}}(0, y_0)$.

*Case-2* ($\tilde{u} = 0$ ). This case is possible only if $f(0) = 0$, since on the contrary $\tilde{u}$ could not be a solution of $-\Delta_p \tilde{u} = f(\tilde{u})$. We set

$$\bar{u}_n \equiv \frac{\tilde{u}_n(x', y)}{\tilde{u}_n(0, y_n)} = \frac{u(x' + x'_n, y)}{u(x'_n, y_n)}$$

so that

$$\bar{u}(0, y_n) = 1,$$

and $\tilde{u}_n$ uniformly converges to 0 on compact sets by construction. It is easily seen that

$$-\Delta_p \bar{u}_n = \frac{f(\tilde{u}_n)}{\tilde{u}_n^{p-1}} \cdot \bar{u}_n^{p-1} = c_n(x) \cdot \bar{u}_n^{p-1}, \tag{4.3}$$

with $c_n(x)$ uniformly bounded. In fact, $\lim\limits_{t \to 0^+} \frac{f(t)}{t^{p-1}} = 0$ since $f(0) = 0$, $f$ is locally Lipschitz continuous, $p - 1 \leq 1$, and $\tilde{u}_n$ uniformly converges to 0.

We can therefore exploit Harnack inequality, see Theorem 7.2.2 in [25], and get, for any compact set $\mathcal{K}$,

$$\sup_{\mathcal{K} \cap \{y \geq \delta\}} \bar{u}_n \leq C_H \inf_{\mathcal{K} \cap \{y \geq \delta\}} \bar{u}_n \leq C_H.$$

Also, by the monotonicity of $u$ in $\Sigma_{\bar{\lambda}}$ we have

$$\sup_{\mathcal{K} \cap \{y \geq 0\}} \bar{u}_n \leq \sup_{\mathcal{K} \cap \{y \geq \delta\}} \bar{u}_n \leq C_H.$$

We can therefore use $C^{1,\alpha}$ estimates, Ascoli's Theorem and a standard diagonal process, as above, to show that

$$\bar{u}_n \overset{C^{1,\alpha'}_{loc}(\overline{\mathbb{R}^N_+})}{\longrightarrow} \bar{u}$$

up to subsequences, for $\alpha' < \alpha$. Arguing exactly as above, we see that $\bar{u} \geq 0$ in $\overline{\mathbb{R}^N_+}$, $\bar{u} \leq \bar{u}_{\bar{\lambda}}$ in $\Sigma_{\bar{\lambda}}$ and $\bar{u}(0, y_0) = \bar{u}_{\bar{\lambda}}(0, y_0)$.

Since $\lim_{t\to 0^+} \frac{f(t)}{t^{p-1}} = 0$ and $\tilde{u}_n$ uniformly converges to 0, passing to the limit in (4.3), we obtain

$$\Delta_p \bar{u} = 0 \quad \text{in} \quad \mathbb{R}^N_+.$$

By the Strong Maximum Principle [30], we therefore get that $\bar{u} > 0$ since the case $\bar{u} = 0$ is not possible in view of the fact that $\bar{u}(0, y_0) = 1$. Actually, by construction, we have

$$\begin{cases} -\Delta_p \bar{u} = 0, & \text{in } \mathbb{R}^N_+ \\ \bar{u} > 0, & \text{in } \mathbb{R}^N_+ \\ \bar{u}(x', 0) = 0, & \text{on } \partial\mathbb{R}^N_+. \end{cases} \tag{4.4}$$

By [22], it follows now that $\bar{u}$ is affine linear, which implies

$$\bar{u}(x', y) = ky$$

for some $k > 0$, by the Dirichlet assumption. This is a contradiction since by construction we would also get $\bar{u}(0, y_0) = \bar{u}_{\bar{\lambda}}(0, y_0)$. □

## 5 Proof of Theorem 1.3

We prove here Theorem 1.3. To this end, let us note that by Proposition 4.1 we have that, given $0 < \delta < \frac{\bar{\lambda}}{2}$, we find $\varepsilon_0$ such that, for any $\varepsilon \leq \varepsilon_0$, it follows

$$Supp\, W^+_\varepsilon \subset \{0 \leq y \leq \delta\} \cup \{\bar{\lambda} - \delta \leq y \leq \bar{\lambda} + \varepsilon\},$$

where $W^+_\varepsilon = (u - u_{\bar{\lambda}+\varepsilon})^+ \cdot \chi_{\{y \leq \bar{\lambda}+\varepsilon\}}$. In particular we have that $u - u_{\bar{\lambda}+\varepsilon} \leq 0$ in $\{\delta \leq y \leq \bar{\lambda} - \delta\}$. Now, we choose $\delta$ sufficiently small such that Theorem 1.1 applies in $\{0 \leq y \leq \delta\}$ and in $\{\bar{\lambda} - \delta \leq y \leq \bar{\lambda} + \varepsilon\}$, getting that actually $W^+_\varepsilon = 0$ for any $\varepsilon$ sufficiently small. This is a contradiction, in view of the definition of $\bar{\lambda}$, and consequently we deduce that $\bar{\lambda} = \infty$. This implies the monotonicity of $u$, that is $\frac{\partial u}{\partial x_N}(x', y) \geq 0$ in $\mathbb{R}^N_+$. By Theorem 2.4, it follows

$$\frac{\partial u}{\partial x_N}(x', y) > 0 \quad \text{in} \quad \mathbb{R}^N_+.$$

## 6 Proof of Theorem 1.5

Under the assumptions of Theorem 1.5, let $u \in C^{1,\alpha}_{loc}$ be a positive solution to

$$\begin{cases} -\Delta_p u = f(u), & \text{in} \quad \mathbb{R}^{N-1} \times (0, h) \\ u(x', 0) = 0 = u(x', h), & \text{for any} \quad x' \in \mathbb{R}^{N-1}. \end{cases} \tag{6.1}$$

Arguing as in the proof of Theorem 1.3, let us set $\bar{\lambda} = \sup \bar{\Lambda}$ where

$$\bar{\Lambda} \equiv \{0 < t < h/2 \mid u \leq u_\theta \quad \text{in} \quad \Sigma_\theta \quad \forall \theta \leq t\}$$

By Theorem 1.1, we know that $\bar{\Lambda}$ is not empty, since it follows that $t \in \bar{\Lambda}$ for sufficiently small $t$. This also implies that $u$ is monotone increasing in the $y$-direction in $\Sigma_\theta$ for sufficiently small $\theta < h/2$.

Analogously, we set $\tilde{\lambda} = \inf \tilde{\Lambda}$ where

$$\tilde{\Lambda} \equiv \{h/2 < t < h \mid u \leq u_\theta \quad \text{in} \quad \Sigma_h \setminus \Sigma_\theta \quad \forall\ t < \theta < h\}.$$

Again, by Theorem 1.1 we obtain that $\tilde{\Lambda}$ is not empty and that $u$ is monotone decreasing in the $y$-direction in $\Sigma_h \setminus \Sigma_\theta$ for $\theta$ sufficiently close to $h$.

The result will be proved once we show that actually

$$\bar{\lambda} = \sup \bar{\Lambda} = h/2 \qquad \text{and} \qquad \tilde{\lambda} = \inf \tilde{\Lambda} = h/2$$

We will only prove that $\bar{\lambda} = h/2$, the proof of $\tilde{\lambda} = h/2$ being analogous. We assume therefore by contradiction that $\bar{\lambda} < h/2$ and note that, by continuity, $u \leq u_{\bar{\lambda}}$, and consequently $u < u_{\bar{\lambda}}$ by the Strong Comparison Principle and the Dirichlet assumption.

**Proposition 6.1** *Assume $\bar{\lambda} < h/2$. Given $0 < \delta < \frac{\bar{\lambda}}{2}$, we find $\varepsilon_0 \in (0,\ h/2 - \bar{\lambda})$ such that, for any $\varepsilon \leq \varepsilon_0$, it follows*

$$Supp\ W_\varepsilon^+ \subset \{0 \leq y \leq \delta\} \cup \{\bar{\lambda} - \delta \leq y \leq \bar{\lambda} + \varepsilon\}$$

*where $W_\varepsilon = (u - u_{\bar{\lambda}+\varepsilon}) \cdot \chi_{\{y \leq \bar{\lambda}+\varepsilon\}}$.*

*Proof* The case $f(0) > 0$ can be carried out exactly in the same way as in Proposition 4.1.

We therefore assume $f(0) = 0$. In this case we consider $u$ defined in $\mathbb{R}^N$ as follows

$$u(x', y) = \begin{cases} u(x', y - 2kh) & \text{if} \quad y \in [2kh\,,\ (2k+1)h] \quad k \in \mathcal{Z} \\ -\,u(x', (2k+2)h - y) & \text{if} \quad y \in [(2k+1)h\,,\ (2k+2)h] \quad k \in \mathcal{Z} \end{cases} \tag{6.2}$$

so that, setting $f(t) = -f(-t)$ for $t < 0$, we have that $-\Delta_p u = f(u)$ in the entire space. Assume now by contradiction that the thesis is false, and note that consequently there exist points $x_n = (x_n', y_n)$ with $u(x_n', y_n) \geq u_{\bar{\lambda}+\varepsilon_n}(x_n', y_n)$ with $\varepsilon_n$ converging to zero, $y_n \to y_0$ and $\delta \leq y_0 \leq \bar{\lambda} - \delta$.

Recalling now the definition $\tilde{u}_n(x', y) = u(x' + x_n', y)$ and noticing that, $\|\tilde{u}_n\|_\infty = \|u\|_\infty \leq C$, we can argue as in Proposition 4.1. Exploiting standard regularity theory, Ascoli's Theorem and a standard diagonal process we can assume that

$$\tilde{u}_n \overset{C^{1,\alpha'}_{loc}(\mathbb{R}^N)}{\longrightarrow} \tilde{u}, \tag{6.3}$$

up to subsequences, for $\alpha' < \alpha$. Then $\tilde{u}$ is a solution to $-\Delta_p \tilde{u} = f(\tilde{u})$ in the entire space. It follows now that $\tilde{u} \geq 0$ in $\Sigma_h$, with $\tilde{u}(x, 0) = 0$ and $\tilde{u} \leq \tilde{u}_{\bar{\lambda}}$ in $\Sigma_{\bar{\lambda}}$. Also $\tilde{u}(0, y_0) = \tilde{u}_{\bar{\lambda}}(0, y_0)$. Now since $\tilde{u}$ is non-negative, we get by the strong maximum principle that either $\tilde{u} = 0$ or $\tilde{u} > 0$ in $\Sigma_h$.

If $\tilde{u} > 0$ in $\Sigma_h$ we get a contradiction by the fact that $\tilde{u}(0, y_0) = \tilde{u}_{\bar{\lambda}}(0, y_0)$ since $\tilde{u} < \tilde{u}_{\bar{\lambda}}$ in $\Sigma_{\bar{\lambda}}$ as we can prove via the strong comparison principle, exploited exactly as in the proof of Case 1 in Proposition 4.1.

We are therefore reduced to consider the case $\tilde{u} = 0$. We set in this case

$$\bar{u}_n \equiv \frac{\tilde{u}_n(x', y)}{\tilde{u}_n(0, y_n)} = \frac{u(x' + x_n', y)}{u(x_n', y_n)}$$

so that $\bar{u}(0, y_n) = 1$, and $\tilde{u}_n$ uniformly converges to 0 on compact sets by construction. It is easily seen that $-\Delta_p \bar{u}_n = c_n(x) \cdot \bar{u}_n^{p-1}$ With $c_n(x)$ uniformly bounded. We can therefore exploit Harnack inequality in any compact set $\mathcal{K}$, see Theorem 7.2.2 in [25], and get

$$\sup_{\mathcal{K}\cap\{0\leq y\leq h\}} \bar{u}_n \leq \sup_{\mathcal{K}\cap\{\delta\leq y\leq h-\delta\}} \bar{u}_n \leq C_H \inf_{\mathcal{K}\cap\{\delta\leq y\leq h-\delta\}} \bar{u}_n \leq C_H$$

where, to deduce that $\sup_{\mathcal{K}\cap\{0\leq y\leq h\}} \bar{u}_n \leq \sup_{\mathcal{K}\cap\{\delta\leq y\leq h-\delta\}} \bar{u}_n$ we used that $u$ is monotone increasing in the $y$-direction in $\Sigma_\theta$ for sufficiently small $\theta < h/2$ and that $u$ is monotone decreasing in the $y$-direction in $\Sigma_h \setminus \Sigma_\theta$ for $\theta$ sufficiently close to $h$.

We can therefore use $C^{1,\alpha}$ estimates, Ascoli's Theorem and a standard diagonal process as in the proof of Case 2 of Proposition 4.1, to show that

$$\bar{u}_n \overset{C^{1,\alpha'}_{loc}(\mathbb{R}^N_+)}{\longrightarrow} \bar{u}$$

with $\Delta_p \bar{u} = 0$ in $\mathbb{R}^N$ (and in particular in $\mathbb{R}^N_+$). By [22] we know that $\bar{u}$ is affine linear, which implies

$$\bar{u} = 0$$

taking into account the fact that $\bar{u}(x', 0) = \bar{u}(x', h) = 0$.

This is not possible since $\bar{u}(0, y_0) = 1$. The contradiction concludes the proof. □

Using Proposition 6.1, the proof of Theorem 1.5 may be now concluded via the moving hyperplane technique exploited exactly as in the proof of Theorem 1.3 (see Sect. 5). Note that Theorem 1.1 applies exactly in the same way, and that by the moving hyperplane technique follows that $\frac{\partial u}{\partial x_N}(x', y) \geq 0$ in $\mathbb{R}^{N-1} \times (0, \frac{h}{2})$. By Theorem 2.4, it follows consequently

$$\frac{\partial u}{\partial x_N}(x', y) > 0 \quad \text{in} \quad \mathbb{R}^{N-1} \times (0, h/2).$$

## 7 A Liouville type theorem in low dimension

The result that follows is a Liouville type theorem in $\mathbb{R}^3$. The analogous result in $\mathbb{R}^2$ was already proved in [10].

**Theorem 7.1** *Let $u \in C^1(\overline{\mathbb{R}^3_+}) \cap W^{1,\infty}(\mathbb{R}^3_+)$ be a non-negative solution of*

$$\begin{cases} -\Delta_p u = f(u), & \text{in } \mathbb{R}^3_+ \\ u(x', 0) = 0, & \text{on } \partial\mathbb{R}^3_+ \end{cases} \tag{7.1}$$

*where $\mathbb{R}^3_+$ is the half-space in $\mathbb{R}^3$. Assume that $\frac{8}{5} < p \leq 2$. Assume also that the nonlinearity $f$ is positive, that is $f(s) > 0$ for $s > 0$, and $f(0) = 0$.*

*Then*

$$u = 0.$$

*Proof* Let us start assuming that there exists a non-trivial non-negative solution $u$. In this case, by the strong maximum principle [30], it follows that $u > 0$. In this case, by Theorem 1.3, we have that $u$ is monotone. We now follows some arguments used in [19], providing some details for the readers convenience. For any $(x_1, x_2, y) \in \mathbb{R}^3$ and $t \in \mathbb{R}$, we define

$$u^\star(x_1, x_2, y) := \begin{cases} u(x_1, x_2, y) & \text{if } y \geq 0, \\ -u(x_1, x_2, -y) & \text{if } y \leq 0, \end{cases}$$

and

$$f^\star(t) := \begin{cases} f(t) & \text{if } t \geq 0, \\ -f(-t) & \text{if } t \leq 0. \end{cases}$$

It follows, taking into account that $f(0) = 0$, that

$$-\Delta_p u^\star = f^\star(u^\star). \tag{7.2}$$

Moreover $u^\star$ is monotone with $u^\star_y > 0$ by construction. Also since the gradient of $u$ is bounded, the gradient of $u^\star$ is bounded too. We can therefore exploit Theorem 1.2 in [19] to get that $u^\star$ has one-dimensional symmetry in the sense that there exists $\bar{u} : \mathbb{R} \to \mathbb{R}$ and $\omega \in S^1$ in such a way that $u^\star(z) = \bar{u}(\omega \cdot z)$, for any $z \in \mathbb{R}^3$. Since in this case the level sets of our solution are parallel hyperplanes, and since the zero level set $\{u = 0\}$ is $\{y = 0\}$, it follows that necessarily $u^\star = \bar{u}(y)$.
Therefore in $\mathbb{R}^+$, $\bar{u}$ is a solution of the one dimensional problem

$$\begin{cases} -\left(\bar{u}'^{(p-1)}\right)' = f(\bar{u}), & \text{in } \mathbb{R}^+ \\ \bar{u} > 0, & \text{in } \mathbb{R}^+ \\ \bar{u}(0) = 0, & \\ \bar{u}' > 0, & \text{in } \mathbb{R}^+ \cup \{0\} \end{cases} \tag{7.3}$$

A simple ODE analysis of the problem shows that actually $\bar{u} = 0$ and the thesis follows. □

## 8 Liouville type theorems in higher dimensions

In this section we prove a Liouville type theorem for solutions of (1.1) in $\mathbb{R}^N_+$, with $N \geq 3$. The idea is that, the monotonicity of the solutions allows us to define and to study the limiting profile of the solutions in $\mathbb{R}^{N-1}$ following some ideas from [16].

**Theorem 8.1** *Let $u \in C^1(\overline{\mathbb{R}^N_+}) \cap W^{1,\infty}(\mathbb{R}^N_+)$ be a non-negative weak solution of* (1.1) *in $\mathbb{R}^N_+$, with $\frac{2N+2}{N+2} < p < 2$. Assume that $N \geq 3$ and $f(s) > 0$ for $s > 0$, and that $f$ is subcritical w.r.t. the Sobolev critical exponent in $\mathbb{R}^{N-1}$* (*see Remark* 8.2)*, then $u = 0$.*

*Proof* Assume by contradiction that $u$ is not identically zero. Therefore, $u > 0$ in $\mathbb{R}^N_+$ by the strong maximum principle [30]. By Theorem 1.3, we consequently get that

$$\frac{\partial u}{\partial x_N} > 0 \quad \text{in} \quad \mathbb{R}^N_+.$$

Since $u$ is bounded, we can now define

$$w(x') := \lim_{t \to \infty} u(x', y + t). \tag{8.1}$$

Such limit exists and is finite for any $x' \in \mathbb{R}^{N-1}$. Also, the limit in (8.1) holds in $C^1_{\text{loc}}(\mathbb{R}^N)$ and $w$ is a weak solution of

$$-\Delta_p w = f(w) \quad \text{in} \quad \mathbb{R}^{N-1}, \tag{8.2}$$

see for example [19].

It turns out that $w$ is a bounded positive solution to $-\Delta_p w = f(w)$ in $\mathbb{R}^{N-1}$, with $f$ positive and subcritical w.r.t the Sobolev critical exponent in $\mathbb{R}^{N-1}$. By [28, Theorem III pag. 84], it follows $w = 0$. This contradicts the construction of $w$ and shows that actually $u = 0$. □

*Remark 8.2* Theorem 8.1 holds true (accordingly to [28]) if $f$ is subcritical w.r.t. the Sobolev critical exponent in $\mathbb{R}^{N-1}$, that is

$$(\alpha - 1) f(s) - s f'(s) \geq 0 \quad \text{for} \quad s > 0$$

for some $1 < \alpha < p^*(N-1) := \frac{(N-1)p}{(N-1)-p}$. Note that, $p < 2 \leq N-1$ and that the nonlinearity $f(s) = s^q$ is subcritical, w.r.t. the Sobolev critical exponent in $\mathbb{R}^{N-1}$, for $q < p^*(N-1) - 1$.

**Theorem 8.3** *Let $u \in C^1(\overline{\mathbb{R}^N_+}) \cap W^{1,\infty}(\mathbb{R}^N_+)$ be a nonnegative weak solution of (1.1) in $\mathbb{R}^N_+$, with $\frac{2N+2}{N+2} < p < 2$. Assume that $N \geq 3$ and $f(s) > 0$ for $s > 0$, $f(0) = 0$ and that*

$$f(s) \geq \lambda s^{\frac{(N-1)(p-1)}{N-1-p}} \qquad in \quad [0, \delta]$$

*for some $\lambda, \delta > 0$.*
*Then $u = 0$.*

*If else we assume that $f(0) > 0$, then we conclude without any further assumptions that there are no non-negative solutions.*

*Proof* Assume again by contradiction that $u$ is not identically zero, and conclude that $u > 0$ in $\mathbb{R}^N_+$ with $\frac{\partial u}{\partial x_N} > 0$ in $\mathbb{R}^N_+$. Define as above

$$w(x') := \lim_{t \to \infty} u(x', y + t) \tag{8.3}$$

We see that $w$ is a bounded positive solution to $-\Delta_p w = f(w)$ in $\mathbb{R}^{N-1}$, with $f$ positive such that $f(s) \geq \lambda s^{\frac{(N-1)(p-1)}{N-1-p}}$ in $[0, \delta]$ for some $\lambda, \delta > 0$. Since $w$ is bounded by construction, and $f(s) > 0$ for $s > 0$, it follows that actually $f(s) \geq \tilde{\lambda} s^{\frac{(N-1)(p-1)}{N-1-p}}$ for any $s \in [0, \|w\|_\infty]$ for some $\tilde{\lambda} > 0$. Consequently $w$ is a solution of the inequality

$$-\Delta_p w \geq \tilde{\lambda} w^{\frac{(N-1)(p-1)}{N-1-p}}$$

By [24] it follows $w = 0$ and the thesis.

If else we assume that $f(0) > 0$, then it follows that the condition $f(s) \geq \tilde{\lambda} s^{\frac{(N-1)(p-1)}{N-1-p}}$ for any $s \in [0, \|w\|_\infty]$ for some $\tilde{\lambda} > 0$ is automatically fulfilled, and we can argue as above, concluding in this case that there are no non-negative solutions, since $u = 0$ is not a solution in this case. □

## 9 Some a priori estimates

**Lemma 9.1** *Assume $N \geq 1$, $p > 1$, $\gamma \geq 0$. Let $u \in C^1(\overline{\mathbb{R}^N_+})$ be a solution of*

$$\begin{cases} \Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u) \geq g(u) & \text{in} \quad \mathbb{R}^N_+ \\ u(x) \leq 0 & \text{on} \quad \partial \mathbb{R}^N_+ \end{cases} \tag{9.1}$$

*where $g : \mathbb{R} \to \mathbb{R}$, $g$ continuous.*

*Then, for every $\alpha \geq 1$, for every $\beta \geq p$ and for every $\varphi \in C^{0,1}_c(\mathbb{R}^N)$, $\varphi \geq 0$ we have:*

$$\int g(u)[(u-\gamma)^+]^\alpha \varphi^\beta \leq C(\alpha, p, \beta) \int [(u-\gamma)^+]^{p+\alpha-1} |\nabla \varphi|^p \varphi^{\beta - p}, \tag{9.2}$$

*where $C(\alpha, p, \beta) = \frac{1}{p}\left(\frac{p-1}{p\alpha}\right)^{p-1} \beta^p$ and $v^+$ denotes the positive part of the function $v$.*

*Proof* By a standard approximation argument one can use test functions in (9.1) of the form $\psi = [(u-\gamma)^+]^\alpha \varphi^\beta$, where $\varphi \in C_c^{0,1}(\mathbb{R}^N)$, $\varphi \geq 0$ we have:

$$\begin{aligned}
&\int g(u)[(u-\gamma)^+]^\alpha \varphi^\beta \leq -\int |\nabla u|^{p-2}\nabla u \nabla \psi \\
&= -\int \alpha |\nabla u|^p [(u-\gamma)^+]^{\alpha-1}\text{sign}^+(u-\gamma)\varphi^\beta - \int |\nabla u|^{p-2}\nabla u \nabla \varphi \beta \varphi^{\beta-1}[(u-\gamma)^+]^\alpha \\
&\leq -\int \alpha |\nabla u|^p [(u-\gamma)^+]^{\alpha-1}\text{sign}^+(u-\gamma)\varphi^\beta + \int \beta |\nabla u|^{p-1}|\nabla \varphi|\varphi^{\beta-1}[(u-\gamma)^+]^\alpha \\
&= -\int \alpha |\nabla u|^p [(u-\gamma)^+]^{\alpha-1}\text{sign}^+(u-\gamma)\varphi^\beta \\
&\quad + \int \beta |\nabla u|^{p-1}|\nabla \varphi|\varphi^{\beta-1}[(u-\gamma)^+]^\alpha \text{sign}^+(u-\gamma) \\
&= -\int \alpha |\nabla u|^p [(u-\gamma)^+]^{\alpha-1}\text{sign}^+(u-\gamma)\varphi^\beta \\
&\quad + \int [\beta[(u-\gamma)^+]^{1+\frac{\alpha-1}{p}}|\nabla \varphi|\varphi^{\frac{\beta}{p}-1}][|\nabla u|^{p-1}\varphi^{\frac{\beta}{p'}}[(u-\gamma)^+]^{\frac{\alpha-1}{p'}}\text{sign}^+(u-\gamma)] \\
&\leq C(\alpha, p, \beta)\int [(u-\gamma)^+]^{p+\alpha-1}|\nabla \varphi|^p \varphi^{\beta-p},
\end{aligned}$$

where we have used the weighted Young's inequality $xy \leq \frac{\varepsilon^p}{p}x^p + \frac{1}{p'\varepsilon^{p'}}y^{p'}$ with $\varepsilon^p = \left(\frac{p-1}{p\alpha}\right)^{p-1} > 0$. □

**Theorem 9.2** *Assume $N \geq 1$, $p > 1$. Suppose that $g : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying:*

$$\exists u_0 \geq 0 \quad : \quad u \geq u_0 \Rightarrow g(u) \geq \epsilon_0 > 0 \tag{9.3}$$

*and let $u \in C^1(\overline{\mathbb{R}^N_+})$ be a solution of (1) satisfying*

$$u(x) = o\left(|x|^{\frac{p}{p-1}}\right) \quad as \quad |x| \to +\infty.$$

*Then, $u \leq u_0$.*

*Proof* *Apply inequality (9.2) with $\varphi = \varphi_R$ ($\varphi_R$ standard cut-off), $\alpha \geq 1$, $\beta = p$, $\gamma = u_0$ to obtain, for every $R > 0$,*

$$\begin{aligned}
\varepsilon_0 \int_{B_R} [(u-\gamma)^+]^\alpha &\leq \int_{B_{2R}} [(u-\gamma)^+]^\alpha \varphi^p \leq C(\alpha, p, p)\int_{B_{2R}} [(u-\gamma)^+]^{p+\alpha-1}|\nabla \varphi|^p \\
&\leq C(\alpha, p, p)\int_{B_{2R}} [(u-\gamma)^+]^\alpha [(u-\gamma)^+]^{p-1}|\nabla \varphi|^p \\
&\leq C(\alpha, p, p)R^{-p}\int_{B_{2R}\setminus B_R} [(u-\gamma)^+]^\alpha [(u-\gamma)^+]^{p-1}.
\end{aligned}$$

*Now we use the growth assumption on $u$ to find that, for every $R > R_\gamma > 0$,*

$$\int_{B_R} [(u-\gamma)^+]^\alpha \leq \varepsilon_0^{-1} C(\alpha, p, p)\eta_\gamma(2R)\int_{B_{2R}\setminus B_R} [(u-\gamma)^+]^\alpha, \tag{9.4}$$

*where* $\eta_\gamma = \eta_\gamma(t)$ *is a function satisfying* $\lim_{t\to+\infty}\eta_\gamma(t) = 0$.

*On the other hand, an application of inequality* (9.2)*, with* $\varphi = \varphi_R, \alpha = 1, \beta = p, \gamma = u_0$*, gives for every* $R > R_\gamma$,

$$\int_{B_R}[(u-\gamma)^+] \le \int_{B_{2R}}[(u-\gamma)^+]\varphi^p \le \varepsilon_0^{-1}C(1,p,p)\int_{B_{2R}}[(u-\gamma)^+]^p|\nabla\varphi|^p$$
$$\le \varepsilon_0^{-1}C(1,p,p)|B_1|(2^N-1)R^{-p+N+p\frac{p}{p-1}} = C(p,C_A,N,C_{\mathcal{A}})R^\delta. \tag{9.5}$$

*For* $R > 0$*, set* $h(R) = \int_{B_R}[(u-\gamma)^+] = \int_{B_R}[(u-u_0)^+]$ *and observe that* (9.4)*, with* $\alpha = 1$*, implies the existence of* $R_1 > R_\gamma$ *such that:*

$$\forall\, R > R_1 \qquad h(R) \le 2^{-\delta}h(2R),$$

*and therefore Lemma* 2.1 *yields* $h \equiv 0$*, which implies that* $u \le u_0$. □

## 10 Proof of Theorem 1.7

Assume $N \ge 1$, $p > 1$. Suppose that $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying:

$$\exists z > 0 \quad : \qquad u > z \Rightarrow f(u) < 0$$

and let $u \in C^1(\overline{\mathbb{R}^N_+})$ be a solution of

$$\begin{cases} -\Delta_p u = -\mathrm{div}(|\nabla u|^{p-2}\nabla u) \le f(u) & \text{in } \mathbb{R}^N_+ \\ u(x) \le 0 & \text{on } \partial\mathbb{R}^N_+ \\ u(x) \le C & \text{in } \mathbb{R}^N_+ \end{cases} \tag{10.1}$$

We want to show that $u \le z$. To do this assume by contradiction that $\sup u > z$ and set $M := \sup u + 1$. Hence, $M \in (z+1, +\infty)$ since $u$ is bounded from above. Moreover $f(M) < 0$ by the assumptions on $f$. Now we consider the new nonlinear function $\tilde{f}$ defined as follows: $\tilde{f} = f$ for $u \le M$ and $\tilde{f} = f(M)$ for $u \ge M$, and therefore, $u$ is a solution of the same problem, but with the new nonlinear function $\tilde{f}$. Now, observe that for every $u_0 > z$ there is $\epsilon_0 > 0$ such that $u \ge u_0 \Rightarrow -\tilde{f}(u) \ge \epsilon_0$ and thus, an application of Theorem 9.2 to the function $u$ with $g = -\tilde{f}$ gives $u \le u_0$. The desired conclusion then follows by taking $u_0 \to z$.

Furthermore, if $f$ is locally Lipschitz and $1 < p \le 2$, then $u < z$ as easily follows from the strong maximum principle.

## 11 Proof of Theorem 1.8

Let $u \in C^{1,\alpha}_{loc}(\overline{\mathbb{R}^N_+}) \cap W^{1,\infty}(\mathbb{R}^N_+)$ be a positive solution of (1.1), with $\frac{2N+2}{N+2} < p \le 2$. Assume that

$$\exists z > 0 \quad : \qquad u > z \Rightarrow f(u) < 0 \qquad \text{and} \qquad u < z \Rightarrow f(u) > 0$$

We want to show that $u$ is monotone increasing w.r.t. the $x_N$-direction, with $\frac{\partial u}{\partial x_N} > 0$ in $\mathbb{R}^N_+$.

To do this note that by Theorem 1.7 actually

$$0 < u < z$$

in $\mathbb{R}^N_+$. We can therefore argue as in the case of a positive nonlinearity. We only point out that in the construction of the limiting solutions $\tilde{u}$ and $\bar{u}$ as in Proposition 4.1, we only get by construction $\tilde{u} \le z$ and $\bar{u} \le z$. We then get $\tilde{u} < z$ and $\bar{u} < z$, by the strong maximum principle as remarked here above.The rest of the proof is exactly the same as in Proposition 4.1 and Theorem 1.3.

If $N = 2$ or $N = 3$, it follows now that $u$ has one-dimensional symmetry by exploiting the results in [19] exactly as in Theorem 7.1, and the rest of the theorem is proved.

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