SIGN-CHANGING STATIONARY SOLUTIONS AND BLOWUP FOR A NONLINEAR HEAT EQUATION IN DIMENSION TWO

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Abstract. Consider the nonlinear heat equation

$$v_t - \Delta v = |v|^{p-1} v \tag{NLH}$$

in the unit ball of $\mathbb{R}^2$, with Dirichlet boundary condition. Let $u_{p,\mathcal{K}}$ be a radially symmetric, sign-changing stationary solution having a fixed number $\mathcal{K}$ of nodal regions. We prove that the solution of (NLH) with initial value $\lambda u_{p,\mathcal{K}}$ blows up in finite time if $|\lambda - 1| > 0$ is sufficiently small and if p is sufficiently large. The proof is based on the analysis of the asymptotic behavior of $u_{p,\mathcal{K}}$ and of the linearized operator $L = -\Delta - p|u_{p,\mathcal{K}}|^{p-1}$.

1. Introduction

Let us consider the nonlinear heat equation

$$\begin{cases} v_t - \Delta v = |v|^{p-1}v, & \text{in } \ \Omega \times (0,T) \\ v = 0, & \text{on } \partial\Omega \times (0,T) \\ v(0) = v_0, & \text{in } \ \Omega\,, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \in \mathbb{N}$, is a bounded domain, $p > 1$, $T \in (0, +\infty]$ and

$$v_0 \in C_0(\Omega) = \{w \in C(\overline{\Omega}), w = 0 \text{ on } \partial\Omega\}.$$

It is well known that there exists a unique classical solution of (1.1) which is defined over a maximal time interval $[0, T_{v_0})$. It is also well known that (1.1) admits both nontrivial global solutions and blowup solutions for any $p > 1$. In fact, given $\varphi \in C_0(\Omega)$ and $\lambda \in \mathbb{R}$, let us consider $v_\lambda(\varphi)$ the solution of (1.1) corresponding to $v_0 = \lambda\varphi$. For $|\lambda|$ small, using that the first eigenvalue of the Laplace-Dirichlet operator is positive, it is easy to construct global sub and supersolutions of (1.1), ensuring that $v_\lambda(\varphi)$ is globally defined. On the other hand, $v_\lambda(\varphi)$ has negative energy for large $|\lambda|$ and, as a consequence, it blows up,

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see [3] or [16]. An interesting question is to understand what happens for intermediate values of λ. The case of positive functions $\Psi \geq 0$, $\Psi \not\equiv 0$, is better understood. It follows immediately from the maximum principle for the heat equation that there exists $\lambda^* > 0$ such that $v_\lambda(\Psi)$ is global if $0 < \lambda < \lambda^*$ and $v_\lambda(\Psi)$ blows up if $\lambda > \lambda^*$. (The borderline case $\lambda = \lambda^*$ may correspond to either globality [9], [10], [21] or to blowup [20].)

In other words, defining

$$\mathcal{G} = \{v_0 \in C_0(\Omega), T_{v_0} = \infty\},$$

it holds that $\mathcal{G}^+ = \{v_0 \in \mathcal{G}, v_0 \geq 0\}$ is star-shaped with respect to 0. (In fact, $\mathcal{G}^+$ is convex.) In general, however, $\mathcal{G}$ is not star-shaped. In fact, consider the stationary problem

$$\begin{cases} -\Delta\, u\, = |u|^{p-1}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

where $p > 1$ and Ω is the unit ball in $\mathbb{R}^N$, $N > 2$. In [5] the authors showed that there exists $p^* < p_S := (N+2)/(N-2)$ with the following property. If u is a radial sign-changing solution of the Lane Emden problem (1.2) (for subcritical p there are countable many), there exists $\varepsilon > 0$ such that if $p^* < p < p_S$ and if $0 < |1-\lambda| < \varepsilon$ then $\lambda u \notin \mathcal{G}$, i.e., $v_\lambda(u)$ blows up in finite time for λ slightly greater or slightly smaller then 1. Note that $u \in \mathcal{G}$, so that $\mathcal{G}$ is not star-shaped. Let us point out that an analogous result was proven for $N = 3$ and p close to 1, see [8]. The results in [5] have been extended to case of general non symmetric domains in [19]. Further analysis of the structure of the set $\mathcal{G}$ and of its complementary set

$$\mathcal{B} = \{v_0 \in C_0(\Omega), T_{v_0} < \infty\} \tag{1.3}$$

can be found in [6] and [7].

The results of [5] and [8] do not apply in the case $N = 1$. In fact, for $N = 1$ and $p > 1$ $v_\lambda(u)$ is global and converges uniformly to zero if $|\lambda| < 1$, while $v_\lambda(u)$ blows up if $|\lambda| > 1$. This is due to the anti-periodic structure of the one-dimensional problem, which implies that $v_\lambda(u)$ does not change sign between two consecutive nodes of u. In this way, in the one-dimensional case there is no essential difference in considering u with or without a definite sign.

In this paper we treat the case $N = 2$, which was left open in [5]. We recall that for any $p > 1$ and $\mathcal{K} \in \mathbb{N}$ there exists a unique (up to a sign) radial solution $u_{p,\mathcal{K}} \in C^2(\overline{\Omega})$ of (1.2) with $\mathcal{K}$ nodal regions. The main goal of this work is to establish the following result.

Theorem 1.1. *Let $u_{p,\mathcal{K}}$ be a sign-changing radial stationary solution of (1.1) (see (1.2)) with $\mathcal{K}$ nodal regions. Then there exists $p^* = p^*(\mathcal{K}) > 1$ and $\varepsilon = \varepsilon(p, \mathcal{K}) > 0$ such that if $p > p^*$ and $0 < |1-\lambda| < \varepsilon$, then*

$$\lambda u_{p,\mathcal{K}} \in \mathcal{B}.$$

Our result is analogous in spirit to the one in [5] cited above. In fact, the proofs are based on similar strategies. They are both consequences of the following proposition, which is a particular case of Theorem 2.3 of [7].

Proposition 1.2. *Let u be a sign changing solution of* (1.2) *and let φ_1 be a positive eigenvector of the self-adjoint operator L given by $L\varphi = -\Delta\varphi - p|u|^{p-1}\varphi$, for $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$. Assume that*

$$\int_\Omega u\,\varphi_1 \neq 0. \tag{1.4}$$

Then there exists $\varepsilon > 0$ such that if $0 < |1-\lambda| < \varepsilon$, then the solution $v_\lambda(u)$ of (1.1) *with the initial value λu blows up in finite time.*

Proposition 1.2 says that the linear instability of the stationary solution expressed by (1.4) yields not only nonlinear instability, but also blowup. A similar result for positive solutions of the nonlinear heat equation and of the nonlinear wave equation may be found in [15]. In view of Proposition 1.2, Theorem 1.1 holds if we prove the following:

Theorem 1.3. *Given $\mathcal{K} \geq 2$, let u be a radial solution to* (1.2) *having $\mathcal{K}$ nodal regions. Then there exists $p^* = p^*(\mathcal{K})$ such that for $p > p^*$*

$$\int_\Omega u\,\varphi_1 > 0,$$

where φ_1 is the first positive eigenfunction of the linearized operator L at u.

The proof of Theorem 1.1 relies on the fact that, in an appropriate sense, the limit problem of the Lane Emden problem (1.2) is the Liouville problem

$$\begin{cases} -\Delta u = e^u, & \text{in } \mathbb{R}^2 \\ e^u \in L^1(\mathbb{R}^2), \end{cases} \tag{1.5}$$

see [1], [13], [14]. To be more precise, we consider a suitable scaling $\tilde{u}$ of u, which is defined on a ball $\tilde{\Omega}$ of radius $r(p)$ such that $r(p) \to \infty$ as $p \to \infty$. We define as well a rescaling $\tilde{L}$ of the linear operator L, possessing a first eigenvector $\tilde{\varphi}_1$ associated to a first eigenvalue $\tilde{\lambda}_1$. Extending $\tilde{u}$ and $\tilde{\varphi}_1$ identically equal to zero outside $\tilde{\Omega}$, it turns out that

$$|\tilde{u}|^{p-1}\tilde{u} \underset{p\to\infty}{\longrightarrow} e^{z^*}, \tag{1.6}$$

uniformly over the compact sets of $\mathbb{R}^2$, where z^* is the unique radial solution of (1.5) such that $z^*(0) = 0$ and $\nabla z^*(0) = 0$. Moreover, the linearized limit operator $L^* = -\Delta - e^{z^*}$ has a negative first eigenvalue λ_1^* and a positive corresponding eigenfunction φ_1^* and

$$\tilde{\lambda}_1 \underset{p\to\infty}{\longrightarrow} \lambda_1^*, \tag{1.7}$$

$$\tilde{\varphi}_1 \underset{p\to\infty}{\longrightarrow} \varphi_1^* \qquad \text{in } L^2(\mathbb{R}^2)\,. \tag{1.8}$$

Using (1.6) and (1.8) we show that

$$\int_{\tilde{\Omega}} |\tilde{u}|^{p-1}\tilde{u}\,\varphi_1^* \underset{p\to\infty}{\longrightarrow} \int_{\mathbb{R}^2} e^{z^*}\varphi_1^*\,. \tag{1.9}$$

Since both e^{z^*} and φ_1^* are positive, the integral at the left hand side of (1.9) is positive for large p. By a simple computation, this allows to conclude that (1.4) holds. Then Theorem 1.1 follows from Proposition 1.2 and Theorem 1.3.

To obtain (1.6)-(1.9) we exploit the analysis of [14] concerning the case of two nodal regions. For $\mathcal{K} = 2$, the limit problem associated to u^+, the positive part of u, is a regular Liouville problem in the whole space $\mathbb{R}^2$ (while the negative part u^- is associated to a singular Liouville problem). Using the results of [14], we have been able to prove that (1.6) holds for solutions having any fixed number $\mathcal{K}$ of nodal regions. There are two crucial steps in the proofs of (1.7)-(1.9) for general $\mathcal{K}$, the variational characterization (2.10) of u, which is a consequence of the results of [4], and the energy estimate (2.1).

For $N \geq 3$ and subcritical $p < p_S$, it was shown in [8] that $\lambda u \in \mathcal{B}$ if $|1-\lambda|$ and $p_S - p$ are small enough ($\lambda \neq 1$), independently of the number $\mathcal{K}$ of oscillations of the stationary solution u. We were not able to obtain here an analogous result, since p and λ depend on $\mathcal{K}$ in Theorem 1.1. There is a distinguished difference between the two cases. In the case $N \geq 3$, the limit problem of (1.2) for $p \to p_S$ is still the same problem (1.2) for $p = p_S$, which has a (unique, up to dilations and translations) positive regular solution. However, in the present case $N = 2$, there is qualitative, other than quantitative, transformation when passing to the limit $p \to \infty$. This explains why the analysis here is more involved.

The rest of the paper is organized as follows. In Section 2 we obtain some preliminary results that will be useful in the sequel. In particular, we obtain the energy estimate in Proposition 2.1 and the variational characterization in Proposition 2.4. In Section 3, we carry out an asymptotic spectral analysis, proving (1.7) and (1.8). Finally, in Section 4 we show (1.9), which yields Theorem 1.3 and Theorem 1.1.

2. Preliminary results

It is well known that, for $p > 1$ and $\mathcal{K} \geq 1$ (1.2) admits a unique radially symmetric solution $u_{p,\mathcal{K}} \in C^2(\overline{\Omega})$ having $\mathcal{K}$ nodal regions and such that $u_{p,\mathcal{K}}(0) > 0$, see e.g. [23]. In this section we establish bounds on the energy of $u_{p,\mathcal{K}}$ and on its C_0 norm which will be crucial for the proof of our main result. These estimates extend those in [18] for the case $\mathcal{K} = 2$.

Proposition 2.1. *There exist* $p^* = p^*(\mathcal{K}) \in \mathbb{R}$ *and* $\mathcal{E} = \mathcal{E}(\mathcal{K}) > 0$ *such that*

$$p \int_\Omega |u_{p,\mathcal{K}}|^{p+1}\, dx = p \int_\Omega |\nabla u_{p,\mathcal{K}}|^2\, dx \le \mathcal{E} \tag{2.1}$$

for $p > p^*$.

Proof. Consider the energy functional

$$E_p(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1}$$

for $u \in H^1_{0,r}(\Omega)$, the space of radial functions of $H^1_0(\Omega)$. Note that, if u is a solution of (1.2), then

$$E_p(u) = \frac{p-1}{2(p+1)} \int_\Omega |\nabla u|^2.$$

In this way, Proposition 2.1 will be proven once we bound $pE_p(u_{p,\mathcal{K}})$ uniformly in p. To do so, we first remark that the proofs of Theorem 1.2 and of Theorem 1.4 of [4] still hold when applied to the space $H^1_{0,r}(\Omega)$. As a consequence, we obtain a sequence of distinct solutions of (1.2) $\pm v_{p,j}$, $j \in \mathbb{N}$, such that

a) $\|v_{p,j}\|_{H^1_{0,r}(\Omega)} \to \infty$ as $j \to \infty$.
b) $v_{p,1}$ is positive and $v_{p,j}$ changes sign for $j \ge 2$. Moreover $v_{p,j}$ has at most j nodal regions.
c) $E_p(v_{p,j}) \le \beta_j$, where

$$\beta_j = \inf_{\substack{V \subset H^1_{0,r}(\Omega) \\ dim(V) \ge j}} \sup_{v \in V} E_p(v), \tag{2.2}$$

We next observe that, by the uniqueness (up to a sign) of the radial solution of (1.2) having j nodal regions, we may write that

$$v_{p,j} = u_{p,j} \tag{2.3}$$

for all j. We shall now use $c)$ here above to estimate $pE_p(u_{p,j})$ independently of p. Our arguments extend those employed in [18] for the case $j = 2$ of two nodal regions.

Given $\mathcal{K} \in \mathbb{N}$, fix $\alpha_1, \dots, \alpha_{\mathcal{K}-1}$ positive numbers satisfying $\alpha_j > \alpha_{j+1}$ for $j = 1, \dots, \mathcal{K}-1$ and set $\alpha_{\mathcal{K}} = 0$. Consider the $\mathcal{K}$-dimensional subspace $V^p_{\mathcal{K}}$ of $H^1_{0,r}(\Omega)$ spanned by the $\mathcal{K}$ linearly independent functions $g_{p,1}, \dots, g_{p,\mathcal{K}}$, defined in the following way.

1) $g_{p,1}$ is the unique positive radial solution to (1.2) in the ball
$$B_p = \{x \in \mathbb{R}^2 \ : \ |x| \le e^{-\alpha_1 p}\}.$$
2) For $2 \le j \le \mathcal{K}$, $g_{p,j}$ is the unique radial positive solution to (1.2) in the annulus
$$A_{p,j} = \{x \in \mathbb{R}^2 \ : \ e^{-\alpha_{j-1} p} \le |x| \le e^{-\alpha_j p}\}.$$

Let us assume for the moment that there exist $\overline{p} > 1$ and constants $c_1, \dots, c_{\mathcal{K}}$ such that

$$pE_p(g_{p,j}) \le c_j \qquad \forall p > \overline{p}, \qquad 1 \le j \le \mathcal{K}. \tag{2.4}$$

Since $g_{p,j}$ belongs to the Nehari manifold

$$\mathcal{N}_p = \{u \in H^1_{0,r}(\Omega) \setminus \{0\} \ : \ \|\nabla u\|_2^2 = \|u\|_{p+1}^{p+1}\}$$

it is easy to see that $E_p(tg_{p,j}) \le E_p(g_{p,j})$ for all $t \in \mathbb{R}$. By (2.4),

$$pE_p\Big(\sum_{j=1}^{\mathcal{K}} t_j g_{p,j}\Big) \le \sum_{j=1}^{\mathcal{K}} pE_p(g_{p,j}) \le \sum_{j=1}^{\mathcal{K}} c_j,$$

for all $t_j \in \mathbb{R}$ and $p > \overline{p}$. Hence, using (2.3) and $c)$ here above, we get

$$pE_p(u_{p,\mathcal{K}}) \le \sup_{v \in V^p_{\mathcal{K}}} pE_p(v) \le \sum_{j=1}^{\mathcal{K}} c_j,$$

showing (2.1) for any $p > \overline{p}$. To conclude the proof, it remains to show (2.4).

We start by estimating $pE_p(g_{p,1})$. Note that

$$g_{p,1}(|x|) = e^{\frac{2\alpha_1 p}{p-1}} w_p(e^{\alpha_1 p}|x|),$$

where w_p is the unique positive solution to (1.2) in the unit ball. Thus,

$$\int_{B_p} |\nabla g_{p,1}|^2 = e^{\frac{4\alpha_1 p}{p-1}} \int_{B_1} |\nabla w_p|^2. \tag{2.5}$$

Moreover, it follows from Lemma 2.1 of [1] that

$$p \int_{B_1} |\nabla w_p|^2 \underset{p \to \infty}{\longrightarrow} 8\pi e.$$

Therefore

$$pE_p(g_{p,1}) = \frac{2p(p+1)}{p-1} \int_{B_p} |\nabla g_{p,1}|^2 \underset{p \to \infty}{\longrightarrow} 16\pi e^{4\alpha_1 + 1},$$

and this gives (2.4) for $j = 1$.

We now estimate $pE_p(g_{p,j})$ for $j \ge 2$. Let $z_{p,j}$ be the positive (radial) solution of

$$\max_{H^1_{0,r}(A_{p,j})} \Big\{ \int_{A_{p,j}} |u|^{p+1}, \int_{A_{p,j}} |\nabla u|^2 = p^{-1} \Big\} =: I_{p,j}.$$

Then $z_{p,j}$ satisfies $-\Delta z_{p,j} = (pI_{p,j})^{-1}\, z^p_{p,j}$, so that $g_{p,j} = (pI_{p,j})^{-\frac{1}{p-1}} z_{p,j}$. Hence,

$$p \int_{A_{p,j}} |\nabla g_{p,j}|^2 = (pI_{p,j})^{-\frac{2}{p-1}}. \tag{2.6}$$

Next, inspired by the results in [12] on the asymptotic behavior of the radial positive solution in an annulus as $p \to \infty$, we set $\Delta_j = \alpha_{j-1} - \alpha_j$ and consider

$$w_{p,j}(x) = (2\pi\Delta_j)^{-\frac{1}{2}} p^{-1} \begin{cases} \alpha_{j-1}p + \log r, & e^{-\alpha_{j-1}p} \le r \le e^{-\frac{(\alpha_j+\alpha_{j-1})}{2}p}, \\ -\alpha_j\, p - \log r, & e^{-\frac{(\alpha_j+\alpha_{j-1})}{2}p} \le r \le e^{-\alpha_j p}, \end{cases}$$

where $r = |x|$. Since $w_{p,j} \in H^1_{0.r}(A_{p,j})$ and $\|\nabla w_{p,j}\|^2_{L^2(A_{p,j})} = p^{-1}$ we get

$$\int_{A_{p,j}} w_{p,j}^{p+1} \le I_{p,j}. \tag{2.7}$$

Then,

$$\int_{A_{p,j}} w_{p,j}^{p+1} \ge (2\pi)^{-\frac{p-1}{2}} \Delta_j^{-\frac{p+1}{2}} p^{-(p+1)} \int_{e^{-\alpha_{j-1}p}}^{e^{-\frac{(\alpha_j+\alpha_{j-1})}{2}p}} (\alpha_{j-1}p + \log r)^{p+1} r\, dr.$$

Through the change of variables $s = e^{\frac{\alpha_{j-1}+\alpha_j}{2}p}\, r$, we get

(2.8)

$$\int_{A_{p,j}} w_{p,j}^{p+1} \ge (2\pi)^{-\frac{p-1}{2}} \Delta_j^{-\frac{p+1}{2}} e^{(\alpha_{j-1}+\alpha_j)\,p} \int_{e^{-\frac{p\Delta_j}{2}}}^{1} \left(\frac{\Delta_j}{2} + p^{-1}\log s\right)^{p+1} s\, ds$$

$$= 2^{-\frac{3p+1}{2}} \pi^{-\frac{p-1}{2}} \Delta_j^{\frac{p+1}{2}} e^{(\alpha_{j-1}+\alpha_j)\,p} \int_{e^{-\frac{p\Delta_j}{2}}}^{1} \left(1 + \frac{2}{p\Delta_j}\log s\right)^{p+1} s\, ds.$$

Using the Dominated Convergence Theorem, we obtain

$$\int_{e^{-\frac{p\Delta_j}{2}}}^{1} \left(1 + \frac{2}{p\Delta_j}\log s\right)^{p+1} s\, ds \underset{p\to\infty}{\longrightarrow} \int_0^1 s^{2(\Delta_j)^{-1}+1}\, ds = \frac{\Delta_j}{2+2\Delta_j}. \tag{2.9}$$

It then follows from (2.6)-(2.9) that

$$pE_p(g_{p,j}) = \frac{2p(p+1)}{p-1} \int_{A_{p,j}} |\nabla g_{p,j}|^2 \le 5\pi(\Delta_j)^{-1} e^{-2(\alpha_{j-1}+\alpha_j)}$$

if p is large enough. This concludes the proof. □

Remark 2.2. *Note that* $\min_{j\le\mathcal{K}} \Delta_j \to 0$ *as* $\mathcal{K} \to \infty$. *Thus, the energy estimate* (2.1) *is not independent of* $\mathcal{K}$.

As a consequence of Proposition 2.1 and of Theorem 1.2 of [4] we can show a nice variational characterization of the radial solutions $u_{p,\mathcal{K}}$ of (1.2).

Proposition 2.3. *We have*

$$E_p(u_{p,\mathcal{K}}) = \inf_{\substack{V\subset H^1_{0,r}(\Omega)\\ dim(V)\geq \mathcal{K}}} \sup_{v\in V} E_p(v)\,. \tag{2.10}$$

Proof. Denoting by χ_1, χ_2, $\dots$, $\chi_{\mathcal{K}}$ the $\mathcal{K}$ the characteristic functions associated to the $\mathcal{K}$ disjoint nodal regions of $u_{p,\mathcal{K}}$, set $u^j_{p,\mathcal{K}} = u_{p,\mathcal{K}}\,\chi_j$ and define $V_{\mathcal{K}}$ as the subspace generated by $\{u^j_{p,\mathcal{K}}\}_{j\leq\mathcal{K}}$. Since $E(tu^j_{p,\mathcal{K}}) \leq E(u^j_{p,\mathcal{K}})$ for all $t\in\mathbb{R}$, we have that

$$E_p\big(\sum_{j=1}^{\mathcal{K}} t_j u^j_{p,\mathcal{K}}\big) \leq \sum_{j=1}^{\mathcal{K}} E_p(u_{p,\mathcal{K}}) = E_p(u_{p,\mathcal{K}}).$$

From (2.2) we get that $\beta_j \leq E_p(u_j)$. The reverse inequality was obtained in the proof of Proposition 2.1 and so (2.10) holds. $\square$

Since for general domains Ω there could be more solutions having the same number of nodal regions but different energy, as it is the case when Ω is a ball (see [2]), a characterization of type (2.10) does not hold for general stationary solutions in $H^1_0(\Omega)$.

Let now $\varepsilon_{p,\mathcal{K}}$ be such that

$$\varepsilon^{-2}_{p,\mathcal{K}} = p u_{p,\mathcal{K}}(0)^{p-1} \tag{2.11}$$

and set

$$0 < r_{p,\mathcal{K},1} < r_{p,\mathcal{K},2} < \cdots < r_{p,\mathcal{K},\mathcal{K}-1} < 1 \tag{2.12}$$

the nodal radii of $u_{p,\mathcal{K}}(|x|) = u_{p,\mathcal{K}}(r)$, $r=|x|$, in the ball.

Proposition 2.4. *We have the following.*

i) $\|u_{p,\mathcal{K}}\|_{L^\infty(\Omega)} = u_{p,\mathcal{K}}(0)$.
ii) *There exist* $\underline{c}>0$ *and* $C(\mathcal{K})>0$ *such that* $\underline{c}\leq u_{p,\mathcal{K}}(0)\leq C(\mathcal{K})$ *for all* $p>1$.
iii) $\frac{r_{p,\mathcal{K},1}}{\varepsilon_{p,\mathcal{K}}} \underset{p\to\infty}{\longrightarrow} \infty$.
iv) $\frac{\|u_{p,\mathcal{K}}\|_{L^\infty(\{|x|\geq r_{p,\mathcal{K},1}\})}}{u_{p,\mathcal{K}}(0)} \underset{p\to\infty}{\longrightarrow} \vartheta < \frac{1}{2}$.

Proof. Considering $u_{p,\mathcal{K}}$ as a function of $r=|x|$, it satisfies

$$u''_{p,\mathcal{K}} + \frac{N-1}{r}u'_{p,\mathcal{K}} + |u_{p,\mathcal{K}}|^{p-1}u_{p,\mathcal{K}} = 0.$$

Multiplying the equation by $u'_{p,\mathcal{K}}$, we get that $F'(r)\leq 0$, where

$$F(r) = \frac{1}{2}|u'_{p,\mathcal{K}}|^2 + \frac{1}{p+1}|u_{p,\mathcal{K}}|^{p+1}. \tag{2.13}$$

Thus F is nonincreasing. In particular, $F(0)\geq F(r)$ for all $r\geq 0$, which implies that $\|u_{p,\mathcal{K}}\|_{L^\infty(\Omega)} = u_{p,\mathcal{K}}(0)$. This also implies that the absolute values M_j, $j=1,2,\dots,\mathcal{K}$, of the local maxima of each nodal region of $u_{p,\mathcal{K}}$ decrease with j.

We next prove the lower bound in *ii*). Let us recall that this was shown to be true in Lemma 2.3 of [14] for the case $\mathcal{K} = 2$ of two nodal regions. This yields the result for general $\mathcal{K}$, since $u_{p,\mathcal{K}}(0) > u_{p,2}(0)$. Indeed, for $j < \mathcal{K}$

$$u_{p,j}(r) = r_{p,\mathcal{K},j}^{\frac{2}{p-1}} u_{p,\mathcal{K}}(r_{p,\mathcal{K},j} r). \tag{2.14}$$

Taking $j = 2$, we get $u_{p,\mathcal{K}}(0) = r_{p,\mathcal{K},2}^{-\frac{2}{p-1}} u_{p,2}(0) > u_{p,2}(0)$.

To obtain the upper bound, we see from (2.14) for $j = 1$ and from Proposition 2.1 that

$$\begin{aligned} p\int_0^1 u_{p,1}^{p+1}(r) r\, dr = p r_{p,\mathcal{K},1}^{\frac{2(p+1)}{p-1}} \int_0^1 u_{p,\mathcal{K}}^{p+1}(r_{p,\mathcal{K},1} r) r\, dr = \\ p r_{p,\mathcal{K},1}^{\frac{4}{p-1}} \int_0^{r_{p,\mathcal{K},1}} u_{p,\mathcal{K}}^{p+1}(s) s\, ds < p r_{p,\mathcal{K},1}^{\frac{4}{p-1}} \int_0^1 u_{p,\mathcal{K}}^{p+1}(s) s\, ds \le C r_{p,\mathcal{K},1}^{\frac{4}{p-1}}. \end{aligned} \tag{2.15}$$

for some $C = C(\mathcal{K})$. We next recall that

$$\lim_{p\to\infty} p\int_0^1 u_{p,1}^{p+1}(r) r\, dr = \frac{1}{2\pi} \lim_{p\to\infty} p\int_\Omega u_{p,1}^{p+1}\, dx = 4e, \tag{2.16}$$

see [1]. Using (2.15) and (2.16) we conclude that $r_{p,\mathcal{K},1}^{\frac{2}{p-1}}$ is uniformly bounded from below. Finally, we note from (2.14) that

$$r_{p,\mathcal{K},1}^{\frac{2}{p-1}} = \frac{u_{p,1}(0)}{u_{p,\mathcal{K}}(0)}. \tag{2.17}$$

Since $u_{p,1}(0) \to \sqrt{e}$, see [1], we conclude that $u_{p,\mathcal{K}}(0)$ is uniformly bounded from above. This completes the proof of ii).

To show *iii*), we use once again (2.14) to write that

$$r_{p,\mathcal{K},1} = r_{p,\mathcal{K},2} r_{p,2,1} \tag{2.18}$$

and that

$$u_{p,\mathcal{K}}^{\frac{p-1}{2}} = u_{p,2}^{\frac{p-1}{2}} r_{p,\mathcal{K},2}^{-1}. \tag{2.19}$$

From (2.18) and (2.19) we get

$$\frac{r_{p,\mathcal{K},1}}{\varepsilon_{p,\mathcal{K}}} = \sqrt{p}\, r_{p,\mathcal{K},1} u_{p,\mathcal{K}}^{\frac{p-1}{2}}(0) = \sqrt{p}\, r_{p,2,1} u_{p,2}^{\frac{p-1}{2}}(0) = \frac{r_{p,2,1}}{\varepsilon_{p,2}}. \tag{2.20}$$

Thus the result for general $\mathcal{K}$ follows from the one for $\mathcal{K} = 2$, which was proven in Proposition 2.7 of [14].

It remains to show *iv*). Since the absolute values of the local maxima of each nodal region of $u_{p,\mathcal{K}}$ decrease, it follows easily from (2.14) that the quotient in *iv*) does not depend on $\mathcal{K}$. For $\mathcal{K} = 2$, *iv*) was proven in Theorem 2 of [14]. This closes the proof. □

The next proposition gives a meaning to the statement that the Lane Emden problem has the Liouville problem as a limit.

Proposition 2.5. *Define the rescaled function*

$$z_{p,\mathcal{K}} = \frac{p}{u_{p,\mathcal{K}}(0)}(u_{p,\mathcal{K}}(\varepsilon_{p,\mathcal{K}}\, x) - u_{p,\mathcal{K}}(0)),$$

over the rescaled domain $\Omega_{\varepsilon_{p,\mathcal{K}}} = \varepsilon_{p,\mathcal{K}}^{-1}\Omega$ *and set* $z_{p,\mathcal{K}} = 0$ *outside* $\Omega_{\varepsilon_{p,\mathcal{K}}}$*. Then*

$$z_{p,\mathcal{K}} \underset{C^1_{loc}(\mathbb{R}^2)}{\longrightarrow} z^*\,, \tag{2.21}$$

where

$$z^* \;=\; \log\left((1+\frac{1}{8}|x|^2)^{-2}\right). \tag{2.22}$$

is the unique regular solution to the Liouville problem

$$\begin{cases} -\Delta\, z \;= e^z & \text{in } \mathbb{R}^2 \\ \int_{\mathbb{R}^2} e^z \; < +\infty, & z(0) = |\nabla z(0)| = 0. \end{cases} \tag{2.23}$$

Proof. The proof is similar to that of Theorem 2 in [13]. We outline the main steps for the reader's convenience. Using (2.11) it is easy to see that $z_{p,\mathcal{K}}$ solves

$$-\,\Delta z_{p,\mathcal{K}} \;=\; \left|1+\frac{z_{p,\mathcal{K}}}{p}\right|^{p-1}\left(1+\frac{z_{p,\mathcal{K}}}{p}\right) \qquad \text{in } \Omega_{\varepsilon_{p,\mathcal{K}}}\,,$$

with $|1+\frac{z_{p,\mathcal{K}}}{p}| \le 1$. By standard regularity theory it follows that $z_{p,\mathcal{K}}$ is uniformly bounded in $C^2_{loc}(\mathbb{R}^2)$ and hence (2.21) holds with z^* satisfying (2.23). Note that the uniform estimate of the energy obtained in Proposition 2.1 yields that $\int_{\mathbb{R}^2} e^z \; < +\infty$ (see the proof of Theorem 2 in [13] for details) and (2.22) follows by the classification of the solutions to (2.23). □

Remark 2.6. *Here is another argument for the proof of Proposition 2.5. It follows from* (2.14), (2.17) *and* (2.11) *that* $z_{p,\mathcal{K}} = z_{p,1}$ *in* $\Omega_{\varepsilon_{p,1}}$*. This yields* (2.21) *for general* $\mathcal{K}$*, since the case of positive solutions* $\mathcal{K} = 1$ *was shown to be true in* [1].

3. Asymptotic spectral analysis

As discussed in Section 2, an appropriate rescaling of $u_{p,\mathcal{K}}$ converges to the solution of the Liouville problem (2.23). In this section we consider the corresponding linearizations of the Lane Emden and of the Liouville problems and study their connections.

We first discuss the linearization of the limit problem. For $v \in H^2(\mathbb{R}^2)$ define

$$L^*(v) = -\Delta v - e^{z^*} v.$$

Consider the Rayleigh functional

$$\mathcal{R}(w) = \int_{\mathbb{R}^2} \left(|\nabla w|^2 - e^{z^*} w^2\right) dx$$

for $w \in H^1(\mathbb{R}^2)$ and define

$$\lambda_1^* = \inf_{\|w\|_{L^2(\mathbb{R}^2)}=1} \mathcal{R}(w)\,. \tag{3.1}$$

We remark that $\lambda_1^* > -\infty$, since e^{z^*} is bounded.

Proposition 3.1. *We have the following.*

i) $\lambda_1^* < 0$.

ii) *Every minimizing sequence of* (3.1) *has a subsequence which strongly converges in* $L^2(\mathbb{R}^2)$ *to a minimizer.*

iii) *There exists a unique positive minimizer* φ_1^* *to* (3.1) *which is radial and radially nonincreasing. Moreover,* λ_1^* *is an eigenvalue of* L *and* φ_1^* *is an eigenvector associated to* λ_1^*.

Proof. A direct computation gives that $e^{z^*} \in H^1(\mathbb{R}^2)$ and that

$$\mathcal{R}(e^{z^*}) = -\frac{1}{2}\int_{\mathbb{R}^2} e^{3z^*} = -\frac{4\pi}{5}\,,$$

so that λ_1^* is negative. This gives i).

To prove *ii*) let w_n be a minimizing sequence of (3.1). Clearly, w_n is bounded in $H^1(\mathbb{R}^2)$. Therefore, up to a subsequence, it converges weakly to some $w \in H^1(\mathbb{R}^2)$, and strongly in $L^2(\{|x| \le R\})$ for every $R > 0$. The weak lower semicontinuity of the norm gives

$$\int_{\mathbb{R}^2} |\nabla w|^2 \le \liminf_{n\to\infty} \int_{\mathbb{R}^2} |\nabla w_n|^2 \qquad \text{and} \qquad \|w\|_{L^2(\mathbb{R}^2)} \le 1\,.$$

Moreover, exploiting the decay properties of e^{z^*}, we get

$$\begin{aligned} \Big|\int_{\mathbb{R}^2} e^{z^*}(w_n^2 - w^2)\Big| &\le \int_{\mathbb{R}^2} e^{z^*}|w_n^2 - w^2| = \\ \int_{\{|x|\le R\}} e^{z^*}|w_n^2 - w^2| &+ \int_{\{|x\ge R|\}} e^{z^*}|w_n^2 - w^2| \\ \le C\,\|w_n - w\|_{L^2(|x|\le R)} &+ \frac{C}{R^4}\,, \end{aligned}$$

yielding

$$\int_{\mathbb{R}^2} e^{z^*} w_n^2 \to \int_{\mathbb{R}^2} e^{z^*} w^2\,.$$

Therefore $\mathcal{R}(w) \le \lambda_1^*$, so that $w \ne 0$. Letting

$$\hat{w} = \frac{w}{\|w\|_{L^2(\mathbb{R}^2)}}\,,$$

we have

$$\lambda_1^* \le \mathcal{R}(\hat{w}) = \frac{\mathcal{R}(w)}{\|w\|^2_{L^2(\mathbb{R}^2)}} \le \frac{\lambda_1^*}{\|w\|^2_{L^2(\mathbb{R}^2)}} \le \lambda_1^*.$$

Hence $\|w\|_{L^2(\mathbb{R}^2)} = 1$ and w is a minimizer. This also allows us to deduce that w_n converges to w in $L^2(\mathbb{R}^2)$ so that *ii*) holds.

The proof of *iii*) now uses standard arguments, including a rearrangement procedure (see [17]). □

We next consider the linearization of the Lane Emden problem. In the rest of this paper we fix $\mathcal{K} \geq 2$ and we denote for simplicity $u_{p,\mathcal{K}}$, $\varepsilon_{p,\mathcal{K}}$, etc. by u_p, ε_p, etc. Define for $v \in H^2(\Omega)$

$$L_p(v) = -\Delta v - p|u_p|^{p-1} v.$$

We denote by $\lambda_1(p)$ the first eigenvalue of L_p in Ω and by $\varphi_{1,p}$ the corresponding positive eigenfunction normalized such that $\varphi_{1,p} > 0$ and $\|\varphi_{1,p}\|_{L^2(\Omega)} = 1$. In particular, we have

$$-\Delta\varphi_{1,p} - p|u_p|^{p-1}\varphi_{1,p} \; = \; \lambda_1(p)\varphi_{1,p}. \tag{3.2}$$

Moreover, $\lambda_1(p) < 0$ for any $p > 1$, as it is easy to verify. Let us define $\tilde{\varphi}_{1,p}$ by

$$\tilde{\varphi}_{1,p} = \varepsilon_p \, \varphi_{1,p}(\varepsilon_p \, x) \qquad \text{in} \quad \Omega_{\varepsilon_p} \, ,$$

$\tilde{\varphi}_{1,p} = 0$ outside Ω_{ε_p}, ε_p being given by (2.11). Then $\tilde{\varphi}_{1,p}$ satisfies

$$-\Delta\tilde{\varphi}_{1,p} \; = \tilde{V}_p \, \tilde{\varphi}_{1,p} \; + \; \tilde{\lambda}_1(p)\tilde{\varphi}_{1,p} \qquad \text{in} \quad \Omega_{\varepsilon_p} \, ,$$

where

$$\tilde{V}_p(x) = \frac{|u_p(\varepsilon_p \, x)|^{p-1}}{u_p(0)^{p-1}} = \left| 1 + \frac{z_p}{p} \right|^{p-1} \tag{3.3}$$

and

$$\tilde{\lambda}_1(p) = \varepsilon_p^2 \lambda_1(p). \tag{3.4}$$

In other words, $\tilde{\varphi}_{1,p}$ is a first eigenfunction of the operator

$$\tilde{L}_p = -\Delta - \tilde{V}_p \, I \tag{3.5}$$

in $L^2(\Omega_{\varepsilon_p})$ with $D(\tilde{L}_p) = H^2(\Omega_{\varepsilon_p}) \cap H^1_0(\Omega_{\varepsilon_p})$, $\tilde{\lambda}_1(p)$ being the corresponding first eigenvalue.

Extending $\tilde{\varphi}_{1,p} \equiv 0$ outside Ω_{ε_p}, we have the following.

Lemma 3.2. *The set $\{\tilde{\varphi}_{1,p}, p > 1\}$ is bounded in $H^1_r(\mathbb{R}^2)$.*

Proof. We have that $\|\tilde{\varphi}_{1,p}\|_{L^2(\mathbb{R}^2)} = 1$. In addition, since $\tilde{\lambda}_1(p)$ is negative and $\|u_p\|_{L^\infty(\Omega)} = u_p(0)$,

$$\begin{aligned}
\int_{\mathbb{R}^2} |\nabla\tilde{\varphi}_{1,p}|^2 \; &= \varepsilon_p^4 \int_{\Omega_{\varepsilon_p}} |\nabla\varphi_{1,p}|^2(\varepsilon_p \, x) = \varepsilon_p^2 \int_\Omega |\nabla\varphi_{1,p}|^2 = \\
&= \varepsilon_p^2 \, p \int_\Omega |u_p|^{p-1}\varphi_{1,p}^2 + \varepsilon_p^2 \, \lambda_1(p) \int_\Omega \varphi_{1,p}^2 \\
&\leq \varepsilon_p^2 \, p \int_\Omega |u_p|^{p-1}\varphi_{1,p}^2 = \frac{1}{u_p(0)^{p-1}} \int_\Omega |u_p|^{p-1}\varphi_{1,p}^2 \leq 1 \, .
\end{aligned}$$

□

Remark 3.3. *Applying Strauss Lemma [22] for radial functions of $H^1_r(\mathbb{R}^2)$, we see from Lemma 3.2 that $\tilde{\varphi}_{1,p}(x) \to 0$ as $|x| \to \infty$ uniformly in p and $r = |x|$.*

We are now ready to discuss the convergence of the eigenvalues $\tilde{\lambda}_1(p)$.

Theorem 3.4. *We have*

$$\tilde{\lambda}_1(p) \underset{p\to+\infty}{\longrightarrow} \lambda_1^* . \tag{3.6}$$

Proof. We divide the proof in two steps.

Step 1 *: For $\epsilon > 0$ we have*

$$\lambda_1^* \leq \tilde{\lambda}_1(p) + \epsilon \quad \textit{for } p \textit{ sufficiently large.}$$

To prove this, we see that $\lambda_1^* \leq \mathcal{R}(\tilde{\varphi}_{1,p})$, since $\|\tilde{\varphi}_{1,p}\|_{L^2(\mathbb{R}^2)} = 1$. Thus,

(3.7)

$$\begin{aligned}
\lambda_1^* &\leq \int_{\mathbb{R}^2} |\nabla \tilde{\varphi}_{1,p}|^2 - e^{z^*} \tilde{\varphi}_{1,p}^2 = \int_{\Omega_{\varepsilon_p}} |\nabla \tilde{\varphi}_{1,p}|^2 - \tilde{V}_p \, \tilde{\varphi}_{1,p}^2 - \int_{\Omega_{\varepsilon_p}} \left(e^{z^*} - \tilde{V}_p\right) \tilde{\varphi}_{1,p}^2 \\
&= \tilde{\lambda}_1(p) - \int_{\Omega_{\varepsilon_p}} \left(e^{z^*} - \tilde{V}_p\right) \tilde{\varphi}_{1,p}^2 \\
&= \tilde{\lambda}_1(p) - \int_{|x|<R} \left(e^{z^*} - \tilde{V}_p\right) \tilde{\varphi}_{1,p}^2 - \int_{R<|x|<\varepsilon_p^{-1}} \left(e^{z^*} - \tilde{V}_p\right) \tilde{\varphi}_{1,p}^2
\end{aligned}$$

where $R > 0$. Using Hölder's inequality, (2.1), (2.11) and (2.22) we get

$$\begin{aligned}
&\int_{R<|x|<\varepsilon_p^{-1}} |e^{z^*} - \tilde{V}_p| \tilde{\varphi}_{1,p}^2 \leq \|e^{z^*}\|_{L^\infty(\{|x|\geq R\})} + \\
&C\|\tilde{\varphi}_{1,p}\|^2_{L^\infty(\{|x|\geq R\})} \, (u_p(0))^{-(p-1)} \left\{ \int_{R<|x|<\varepsilon_p^{-1}} u_p(\varepsilon_p x)^{p+1} \right\}^{\frac{p-1}{p+1}} \varepsilon_p^{-\frac{4}{p+1}} \\
&\leq 64R^{-4} + C\|\tilde{\varphi}_{1,p}\|^2_{L^\infty(\{|x|\geq R\})} \, (u_p(0))^{-(p-1)} \left(\frac{\mathcal{E}}{p}\right)^{\frac{p-1}{p+1}} \varepsilon_p^{-2} \\
&= 64R^{-4} + C\|\tilde{\varphi}_{1,p}\|^2_{L^\infty(\{|x|\geq R\})} \, \mathcal{E}^{\frac{p-1}{p+1}} p^{\frac{2}{p+1}}.
\end{aligned}$$

Using that $\|\tilde{\varphi}_{1,p}^2\|_{L^\infty(\{|x|\geq R\})} \to 0$ as $R \to \infty$ uniformly in p, see Remark 3.3, and *ii*) of Proposition 2.4, we may fix R large enough so that

$$\int_{R<|x|<\varepsilon_p^{-1}} |e^{z^*} - \tilde{V}_p| \tilde{\varphi}_{1,p}^2 \leq \epsilon/2 \tag{3.8}$$

for all $p > 1$. By (2.21) we get that $\tilde{V}_p = (1 + \frac{z_p}{p})^{p-1}$ converges uniformly to e^{z^*} on compact sets. In this way, for R fixed as above and p sufficiently large

$$\int_{|x|\leq R} |e^{z^*} - \tilde{V}_p| \tilde{\varphi}_{1,p}^2 \leq \epsilon/2. \tag{3.9}$$

Step 1 then follows from (3.7), (3.8) and (3.9).

Step 2*: Given $\epsilon > 0$, we have that*

$$\tilde{\lambda}_1(p) \leq \lambda_1^* + \epsilon \quad \textit{for } p \textit{ sufficiently large.}$$

To prove this, let us consider for $R > 0$ a cut-off regular function $\psi_R(x) = \psi_R(r)$ such that

- $0 \le \psi_R \le 1$ with $\psi_R = 1$ for $r \le R$ and $\psi_R = 0$ for $r \ge 2R$,
- $|\nabla \psi_R| \le 2/R$

and set

$$w_R = \frac{\psi_R\, \varphi_1^*}{\|\psi_R\, \varphi_1^*\|_{L^2(\mathbb{R}^2)}}\,.$$

We take R such that the ball of radius $2R$ is contained in Ω_{ε_p}. Since Ω_{ε_p} converges to the whole space as p tends to infinity, we can assume that R is arbitrarily large for p large enough.
From the variational characterization of $\tilde{\lambda}_1(p)$ we deduce that

$$\begin{aligned} \tilde{\lambda}_1(p) &\le \int_{\mathbb{R}^2} |\nabla w_R|^2 - \tilde{V}_p w_R^2 \\ &= \int_{\mathbb{R}^2} |\nabla w_R|^2 - e^{z^*} w_R^2 + \int_{\mathbb{R}^2} (e^{z^*} - \tilde{V}_p) w_R^2 \end{aligned} \tag{3.10}$$

for all $p > 1$. It is easy to see that $w_R \to \varphi_1^*$ in $H^1(\mathbb{R}^2)$ as $R \to \infty$. Therefore, given $\epsilon > 0$ we can fix $R > 0$ such that

$$\int_{\mathbb{R}^2} |\nabla w_R|^2 - e^{z^*} w_R^2 \le \lambda_1^* + \epsilon\,. \tag{3.11}$$

For such a fixed value of R, we can argue as in Step 1 to obtain that

$$\int_{\mathbb{R}^2} (e^{z^*} - \tilde{V}_p) w_R^2 \le \epsilon \tag{3.12}$$

for p large enough. Now (3.10), (3.11) and (3.12) yield Step 2.
Assertion (3.6) follows from Step 1 and Step 2.

□

We may now prove the convergence of the eigenfunctions $\tilde{\varphi}_{1,p}$.

Corollary 3.5. *$\tilde{\varphi}_{1,p}$ strongly converges to φ_1^* in $L^2(\mathbb{R}^2)$.*

Proof. Theorem 3.4 shows that $\tilde{\varphi}_{1,p}$ is a minimizing sequence for (3.1), and so the result follows by ii) and iii) of Proposition 3.1. □

4. Proof of Theorem 1.1

We start with the

Proof of Theorem 1.3. Using $\varphi_{1,p} \in H_0^1(\Omega)$ as a test function in (1.2) gives

$$\int_\Omega \nabla u_p \cdot \nabla \varphi_{1,p} = \int_\Omega |u_p|^{p-1} u_p\, \varphi_{1,p}\,,$$

while using u_p as a test function in (3.2) yields

$$\int_\Omega \nabla u_p \cdot \nabla \varphi_{1,p} = \int_\Omega p|u_p|^{p-1} u_p\, \varphi_{1,p} + \lambda_1(p) \int_\Omega u_p\, \varphi_{1,p}\,.$$

Subtracting the first equation from the second we obtain

$$\frac{p-1}{-\lambda_1(p)}\int_\Omega |u_p|^{p-1}u_p\,\varphi_{1,p} \;=\; \int_\Omega u_p\,\varphi_{1,p}\,.$$

We may therefore study the sign of $\int_\Omega |u_p|^{p-1}u_p\,\varphi_{1,p}$ which is equivalent to studying the sign of

$$\frac{1}{u_p(0)^p\,\varepsilon_p}\int_\Omega |u_p|^{p-1}u_p\,\varphi_{1,p}\,.$$

In order to prove the result, we will show that

$$\frac{1}{u_p(0)^p\,\varepsilon_p}\int_\Omega |u_p|^{p-1}u_p\,\varphi_{1,p} \underset{p\to\infty}{\longrightarrow} \int_{\mathbb{R}^2} e^{z^*}\varphi_1^* > 0 \quad \text{as } p\to\infty. \tag{4.1}$$

To do so, we take $\epsilon>0$ and choose $R>0$ such that

$$\int_{|x|\ge R} e^{z^*}\,\varphi_1^* \le \epsilon\,. \tag{4.2}$$

We then write

$$\begin{aligned}
&\frac{1}{u_p(0)^p\,\varepsilon_p}\int_\Omega |u_p|^{p-1}u_p\,\varphi_{1,p} = \frac{1}{u_p(0)^p}\int_{\Omega_{\varepsilon_p}} |u_p(\varepsilon_p\,x)|^{p-1}u_p(\varepsilon_p\,x)\,\tilde\varphi_{1,p}(x)\\
&= \frac{1}{u_p(0)^p}\int_{|x|<R} |u_p(\varepsilon_p\,x)|^{p-1}u_p(\varepsilon_p\,x)\,\tilde\varphi_{1,p}(x)\\
&+ \frac{1}{u_p(0)^p}\int_{R<|x|<\varepsilon_p^{-1}} |u_p(\varepsilon_p\,x)|^{p-1}u_p(\varepsilon_p\,x)\,\tilde\varphi_{1,p}(x).
\end{aligned} \tag{4.3}$$

Using the decay properties of $\tilde\varphi_{1,p}$, see Remark 3.3, we may take R eventually larger so that

$$\begin{aligned}
&\frac{1}{u_p(0)^p}\int_{R<|x|<\varepsilon_p^{-1}} |u_p(\varepsilon_p\,x)|^{p}\,\tilde\varphi_{1,p}\\
&\le C\|\tilde\varphi_{1,p}\|_{L^\infty(\{|x|\ge R\})}\frac{1}{u_p(0)^p}\Big(\int_{R<|x|<\varepsilon_p^{-1}} |u_p(\varepsilon_p\,x)|^{p+1}\Big)^{\frac{p}{p+1}}\varepsilon_p^{-\frac{2}{p+1}}\\
&\le C\|\tilde\varphi_{1,p}\|_{L^\infty(\{|x|\ge R\})}\frac{1}{u_p(0)^p}\Big(\int_{\Omega} |u_p|^{p+1}\Big)^{\frac{p}{p+1}}\varepsilon_p^{-2}\\
&\le C\|\tilde\varphi_{1,p}\|_{L^\infty(\{|x|\ge R\})}\frac{1}{u_p(0)}p^{\frac{1}{p+1}}\mathcal{E}^{\frac{p}{p+1}} \le \epsilon
\end{aligned} \tag{4.4}$$

for all $p>1$, where we have used (2.1), *ii)* of Proposition 2.4, (2.11), Hölder's inequality and a change of variables for the integration.

Moreover, (3.3), (2.21) and Corollary 3.5 yield

$$\begin{aligned}
&\Big|\int_{|x|\le R}\Big(\frac{u_p(\varepsilon_p\,x)}{u_p(0)}\Big)^p\,\tilde\varphi_{1,p} - \int_{|x|\le R} e^{z^*}\,\varphi_1^*\Big|\\
&= \Big|\int_{|x|\le R}\Big(1+\frac{z_p}{p}\Big)^p\,\tilde\varphi_{1,p} - \int_{|x|\le R} e^{z^*}\,\varphi_1^*\Big| \le \epsilon
\end{aligned} \tag{4.5}$$

for p eventually larger. Thus (4.1) is a consequence of (4.2)-(4.5). □

We finish by proving our main result.

Proof of Theorem 1.1. Theorem 1.1 follows immediately from Theorem 1.3 and Proposition 1.2. □

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