# MONOTONICITY OF THE SOLUTIONS OF SOME QUASILINEAR ELLIPTIC EQUATIONS IN THE HALF-PLANE, AND APPLICATIONS.

L. DAMASCELLI AND B. SCIUNZI

Abstract. We consider weak positive solutions of the equation $-\Delta_m u = f(u)$ in the half-plane with zero Dirichlet boundary conditions. Assuming that the nonlinearity $f$ is locally Lipschitz continuous and $f(s) > 0$ for $s > 0$, we prove that any solution is monotone. Some Liouville type theorems follow in the case of Lane-Emden-Fowler type equations. Assuming also that $|\nabla u|$ is globally bounded, our result implies that solutions are one-dimensional, and the level sets are flat.

## 1. Introduction and statement of the main results

We consider the problem

$$\begin{cases} -\Delta_m u = f(u), & \text{in } D \equiv \{(x,y) \in \mathbb{R}^2 \mid y > 0\} \\ u(x,y) \geqslant 0, & \text{in } D \\ u(x,0) = 0, & \forall x \in \mathbb{R} \end{cases} \tag{1}$$

where $1 < m < \infty$ and $\Delta_m u \equiv div(|\nabla u|^{m-2}\nabla u)$ is the $m$-Laplace operator. It is well known that solutions of $m$-Laplace equations are generally of class $C^{1,\alpha}$ and the equation has to be understood in the weak distributional sense. To be more precise, by the results in [Di, Lie, Tol], it follows that if (H1) below holds then

$$\text{given a compact set } \mathcal{K} \subset \mathbb{R}^2, \text{ we have } u \in C^{1,\alpha}(\mathcal{K} \cap \overline{D})$$

We will assume that the nonlinearity $f$ satisfies

- ($H_1$) $f$ is locally Lipschitz continuous in $[0\,,\,\infty)$, i.e. $f$ is Lipschitz continuous in $[0\,,\,b]$ for any $b \in \mathbb{R}^+$.
- ($H_2$) For any given $\tau \in \mathbb{R}^+$, there exists a positive constant $K$ such that $f(s) + Ks^q \geq 0$ for some $q \geq m-1$ and for any $s \in [0, \tau]$. Observe that this implies $f(0) \geqslant 0$
- ($H_3$) If $m \neq 2$, we assume that $f$ is positive in $(0\,,\infty)$. That is $f(s) > 0$ if $s > 0$.

The main result in this paper is the following

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Address: *LD* – Universitá di Roma "Tor Vergata" – (Dipartimento di Matematica) – V. della Ricerca Scientifica 1 – Roma, Italy. Email: `damascel@mat.uniroma2.it`. *BS* – Università della Calabria – (Dipartimento di Matematica) – V. P. Bucci – Arcavacata di Rende (CS), Italy. Email: `sciunzi@mat.unical.it`.
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**Theorem 1.1.** *Let $u$ be a nontrivial weak $C^{1,\alpha}_{loc}$ solution of (1). Assume that $f$ satisfies hypotheses $(H_1)$,$(H_2)$ and $(H_3)$[1] above, and $\frac{3}{2} < m < \infty$[2]. Then $u > 0$ in $D$ and*

$$\frac{\partial u}{\partial y}(x,y) > 0, \quad \forall (x,y) \in \overline{D}.$$

Results of this kind have been studied in the literature in the semilinear case $m = 2$. We refer in particular to a series of papers by Berestycki, Caffarelli and Nirenberg (see [BCN1, BCN2, BCN3]). In particular our techniques are very much related to [BCN2]. Also many interesting results have been obtained by Dancer, starting from [Dan].

When considering the case $m \neq 2$ there are no general results corresponding to the semilinear case. Some interesting results for the case $m \neq 2$ have been obtained in [DG], where the case of some special nonlinearities is considered.

We prove Theorem 1.1 exploiting a refined version of the Alexandrov-Serrin moving plane method. In particular we improve a geometrical technique that allows us to use only a weak comparison principle in small domains.

Once we prove Theorem 1.1, we also get interesting consequences such as some Liouville type theorems. Let us remark that Liouville type theorems for solutions of $m$-Laplace equations in the whole space $\mathbb{R}^N$ are known in some cases (see the celebrated paper [SZ]) whereas to our knowledge the corresponding result is an open problem in half spaces. Moreover, it is well known that the problem in the half-space is related to a famous conjecture of De Giorgi [Deg], see [BCN2] and [GG]. This link will be also exploited in our Theorem 1.4 below.

**Remark 1.2.** *In Theorem 1.1, when the case $m \neq 2$ is considered, it is necessary to assume that $f$ is positive in order to exploit the results in* [DS1, DS2]. *We refer in particular to weak and strong maximum and comparison principles. If we restrict our attention to the semilinear case $m = 2$, our techniques apply assuming only that the nonlinearity $f$ satisfies $(H_1)$ and $(H_2)$, since maximum and comparison principles are standard in this case. We remark that we do not assume that the solution $u$ is bounded and we do not assume that the nonlinearity $f$ is globally Lipschitz continuous.*

We point out that from Theorem 1.1 it follows that the critical set $Z_u$ is empty. That is

$$Z_u \equiv \left\{ x \in \Omega \,|\, \nabla u(x) = 0 \right\} = \emptyset\,.$$

As a consequence, the solution $u$ is regular (say of class $C^2$) since the $m$-Laplace operator is nondegenerate outside $Z_u$. Also, we point out here some remarkable consequences of Theorem 1.1.

As was shown in [DFSV, FSV]), once we know that the solution $u$ is monotone increasing, with $\frac{\partial u}{\partial y}(x,y) > 0$, it follows that $u$ is also *stable*, following definition 2.7 below. Therefore, if we consider the case of a power nonlinearity, by [DFSV](see also [Far]) we get the following

[1]We remark that ($H_3$) is not needed if $m = 2$.

[2]The assumption $m > \frac{3}{2}$ correspond in dimension 2 to the assumption $m > \frac{2N+2}{N+2}$, that appears in some maximum and comparison principles recalled in Section 2.

**Theorem 1.3.** *Let $u$ be a $C^{1,\alpha}_{loc}$ solution of* (1) *with $m \geqslant 2$ and*

$$f(u) = u^p$$

*with $(m-1) < p < \infty$. Then $u \equiv 0$.*

Once again we remark that in Theorem 1.1 (and in Theorem 1.3), we do not need to assume that the solution $u$ is bounded.

In the following application, instead we will assume that

$$|\nabla u| \text{ is bounded.}$$

Generally, by standard regularity theory any bounded solution has bounded gradient and therefore this is a weaker assumption than the assumption that $u$ is bounded. We can in this case exploit the results in [FSV] and get the following

**Theorem 1.4.** *Under the assumption of Theorem 1.1, assuming also that $|\nabla u|$ is bounded and that $f(0) = 0$, it follows that $u$ is monotone and stable (see definition 2.7 below) and has one-dimensional symmetry, in the sense that there exists $\bar{u} : \mathbb{R} \to \mathbb{R}$ such that*

$$u(x, y) \equiv \bar{u}(y).$$

Theorem 1.4 allows us to reduce our problem to a simpler one dimensional problem. Therefore it allows many applications. As an example we give the following

**Corollary 1.5.** *Let $u$ be a bounded (with bounded $|\nabla u|$) weak $C^{1,\alpha}_{loc}$ solution of* (1) *with $\frac{3}{2} < m < \infty$. Assume that $f$ satisfies hypothesis $(H_1)$,$(H_2)$. Assume also that $f(0) = 0$ and $f(s) > 0$ for $s > 0$*[3]. *Then $u \equiv 0$ .*

The paper is organized as follows. In Section 2 we recall some known results for the reader' s convenience. In Section 3 we state Theorem 3.1, which easily leads to the proofs of Theorem 1.1, Theorem 1.3, Theorem 1.4 and Corollary 1.5. In Section 4 we prove Theorem 3.1, which is the core of the paper since it introduces the techniques that allow the moving-plane procedure to work in our setting.

## 2. Preliminaries

In this section we recall some known results on $m$-Laplace equations, in particular maximum and comparison principles.

The famous Lemma of H. Hopf was improved and extended to the case of $m$-Laplace equations by J.L.Vazquez [Vas], and to a broad class of quasilinear elliptic operators by P. Pucci, J. Serrin and H. Zou [PSZ, PS1]. In particular the interested reader may find very useful the recent book by P. Pucci and J. Serrin [PS3] and the references therein.

**Theorem 2.1.** *(Strong Maximum Principle and Hopf's Lemma). Let $\Omega$ be a domain in $\mathbb{R}^N$ and suppose that $u \in C^1(\Omega)$, $u \geqslant 0$ in $\Omega$, weakly solves*

$$-\Delta_m u + c u^q = g \geqslant 0 \quad in \quad \Omega$$

*with $1 < m < \infty$, $q \geqslant m - 1$, $c \geqslant 0$ and $g \in L^\infty_{loc}(\Omega)$. If $u \neq 0$ then $u > 0$ in $\Omega$. Moreover for any point $x_0 \in \partial\Omega$ where the interior sphere condition is satisfied, and such*

[3] Observe that here we require ($H_3$) also when $m = 2$

*that $u \in C^1(\Omega \cup \{x_0\})$ and $u(x_0) = 0$ we have that $\frac{\partial u}{\partial s} > 0$ for any inward directional derivative (this means that if $y$ approaches $x_0$ in a ball $B \subseteq \Omega$ that has $x_0$ on its boundary, then $\lim_{y \to x_0} \frac{u(y)-u(x_0)}{|y-x_0|} > 0$).*

Theorem 2.1 is a useful tool when dealing with $m$-Laplace equation. As an example we note that by Theorem 2.1 we immediately get the following:

**Corollary 2.2.** *Let $\Omega$ be a smooth domain in $\mathbb{R}^N$ and let $u \in C^1(\overline{\Omega})$ be a weak solution of*

$$\begin{cases} -\Delta_m u &= f(u) & in\,\Omega \\ \quad u &\geq 0 & in\,\Omega \\ \quad u &= 0 & on\,\partial\Omega \end{cases} \tag{2}$$

*and suppose that $f$ satisfies $(H_2)$. Then*

$$u > 0 \qquad in \quad \Omega \qquad or \qquad u = 0 \qquad in \quad \Omega$$

*and*

$$Z_u \equiv \{\nabla u = 0\} \cap \partial\Omega = \emptyset \tag{3}$$

*Proof.* It is sufficient to observe that locally

$$-\Delta_m(u) + Ku^q = f(u) + Ku^q \geq 0 \tag{4}$$

and therefore Theorem 2.1 applies in this case. □

**Theorem 2.3** (Weak Comparison Principle [DP, DS1])**.** *Suppose that $u$, $v$ weakly solve*

$$-\Delta_m(u) - f(u) \leqslant -\Delta_m(v) - f(v) \ in\ \Omega \tag{5}$$

*Assume that either $1 < m < 2$ and $u, v \in W^{1,\infty}(\Omega)$ and $f(\cdot)$ is locally Lipschitz continuous or that or $m \geqslant 2$, $u, v \in W^{1,m}(\Omega) \cap L^\infty(\Omega)$ and $f(\cdot)$ is positive and locally Lipschitz continuous. Assume also that either $u$ or $v$ weakly solves the equation $-\Delta_m(w) = f(w)$.*

*Let $\Omega' \subseteq \Omega$ be open and suppose $u \leqslant v$ on $\partial\Omega'$, then there exists $\delta > 0$, depending on the Lipschitz constant of $f$ in $[0, \|u\|_{L^\infty(\Omega)}]$, such that, if $|\Omega'| \leqslant \delta$, then $u \leqslant v$ in $\Omega'$.*

**Theorem 2.4** (Strong Comparison Principle [DS2])**.** *Let $u, v \in C^1(\overline{\Omega})$ where $\Omega$ is a bounded smooth domain of $\mathbb{R}^N$ with $\frac{2N+2}{N+2} < m < \infty$. Suppose that either $u$ or $v$ is a weak solution of $-\Delta_m(w) = f(w)$ with $f$ positive and locally Lipschitz continuous. Assume*

$$-\Delta_m(u) - f(u) \leqslant -\Delta_m(v) - f(v) \qquad u \leqslant v \quad in \quad \Omega \tag{6}$$

*Then $u \equiv v$ in $\Omega$ or $u < v$ in $\Omega$.*

Let us recall that the linearized operator $L_u(v, \varphi)$ at a fixed solution $u$ of $-\Delta_m(u) = f(u)$ is well defined, for every $v$, $\varphi \in H^{1,2}_\rho(\Omega)$ with $\rho \equiv |\nabla u|^{m-2}$(see [DS1] for details), by

$$L_u(v, \varphi) \equiv \int_\Omega [|\nabla u|^{m-2}(\nabla v, \nabla\varphi) + (m-2)|\nabla u|^{m-4}(\nabla u, \nabla v)(\nabla u, \nabla\varphi) - f'(u)v\varphi]dx$$

Moreover, $v \in H^{1,2}_\rho(\Omega)$ is a weak solution of the linearized equation if

$$L_u(v, \varphi) = 0 \tag{7}$$

for any $\varphi \in H^{1,2}_{0,\rho}(\Omega)$. More generally, $v \in H^{1,2}_{\rho}(\Omega)$ is a weak supersolution (subsolution) of (7) if $L_u(v,\varphi) \geqslant 0 (\leqslant 0)$ for any nonnegative $\varphi \in H^{1,2}_{0,\rho}(\Omega)$.

By [DS1] we have $u_{x_i} \in H^{1,2}_{\rho}(\Omega)$ for $i = 1,\ldots,N$, and $L_u(u_{x_i},\varphi)$ is well defined for every $\varphi \in H^{1,2}_{0,\rho}(\Omega)$, with

$$L_u(u_{x_i},\varphi) = 0 \qquad \forall \varphi \in H^{1,2}_{0,\rho}(\Omega). \tag{8}$$

In other words, the derivatives of $u$ are weak solutions of the linearized equation.

**Theorem 2.5** (Strong Maximum Principle for the Linearized Operator)**.** *Let $v \in H^{1,2}_{\rho}(\Omega) \cap C^0(\overline{\Omega})$ be weak supersolution of (7) in a bounded smooth domain $\Omega$ of $\mathbb{R}^N$, $N \geqslant 2$ with $\frac{2N+2}{N+2} < m < \infty$ with $f$ positive and locally Lipschitz continuous. Then, for any domain $\Omega' \subset \Omega$ with $v \geqslant 0$ in $\Omega'$, we have $v \equiv 0$ in $\Omega'$ or $v > 0$ in $\Omega'$.*

Since $u_{x_i}$ weakly solves (7), by Theorem 2.5 we obtain

**Theorem 2.6.** *Let $u \in C^1(\overline{\Omega})$ be a weak solution of $-\Delta_m(u) = f(u)$ in a bounded smooth domain $\Omega$ of $\mathbb{R}^N$ with $\frac{2N+2}{N+2} < m < \infty$, and $f$ positive and locally Lipschitz continuous. Then, for any $i \in \{1,\ldots,N\}$ and any domain $\Omega' \subset \Omega$ with $u_{x_i} \geqslant 0$ in $\Omega'$, we have either $u_{x_i} \equiv 0$ in $\Omega'$ or $u_{x_i} > 0$ in $\Omega'$.*

We also recall the following definition

**Definition 2.7.** We say that a weak solution $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$ of $-\Delta_m(u) = f(u)$

- ◆ is **stable** if $L_u(\varphi,\varphi) \geqslant 0$, for every $\varphi \in C^1_c(\Omega)$.

By density arguments, it is enough to consider the case $\varphi \in H^{1,2}_{\rho,\mathrm{loc}}(\Omega)$. We refer the reader to [DFSV] for more details.

## 3. Proof of Theorem 1.1

We prove here Theorem 1.1 via Theorem 3.1, stated below and proved later. We set

$$\Sigma_\lambda \equiv \{(x,y) \in \mathbb{R}^2 \,|\, 0 < y < \lambda\}$$

and $u_\lambda$ defined in $\Sigma_\lambda$ by

$$u_\lambda(x,y) = u(x, 2\lambda - y)$$

**Theorem 3.1.** *Let $u$ be a weak $C^{1,\alpha}_{loc}$ solution of (1). Assume that $f$ satisfies $(H_1)$, $(H_2)$ and $(H_3)$ and $\frac{3}{2} < m < \infty$. Let $x_0 \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ fixed, and assume that*

- a) $\frac{\partial u}{\partial y}(x_0\,,\,y) > 0$ *for every* $y \in [0,\lambda]$
- b) *For every $0 < \lambda' \leqslant \lambda$ we have $u(x_0,y) < u(x_0\,,\,2\lambda' - y)$(that is $u < u_{\lambda'}$) provided that $y \in [0\,,\,\lambda')$.*

*Then, for every $0 < \lambda' \leqslant \lambda$ and $(x,y) \in \Sigma_{\lambda'}$, we have*

$$u(x\,,\,y) < u(x\,,\,2\lambda' - y)$$

Theorem 3.1 essentially says that if the moving plane procedure works on a vertical segment, then it works in the corresponding strip. Using it we can prove our main result, Theorem 1.1 and its consequences, as follows.

**Proof of Theorem 1.1**

Since $u$ does not coincide with zero, by $(H_2)$ we can exploit the strong maximum principle (see Theorem 2.1 and Corollary 2.2) and get $u > 0$ in $D$. Also, given any $x \in \mathbb{R}$, by the Hopf boundary Lemma, (see [Vas] and Corollary 2.2 above), it follows that

$$u_y(x,0) = \frac{\partial u}{\partial y}(x,0) > 0;$$

obviously, $u_y(x,0)$ possibly goes to 0 if $x \to \pm\infty$. Let $x_0$ be fixed and let $r > 0$ be such that

$$\frac{\partial u}{\partial y}(x,y) \geqslant \gamma > 0 \quad \forall (x,y) \in B_{2r}(x_0), \tag{9}$$

where $B_{2r}(x_0)$ is the ball of radius $2r$ centered at $x_0$ Note that such values $\gamma > 0$ and $r > 0$ exist since $u \in C^{1,\alpha}(\overline{B_{2r}(x_0) \cap D})$. Now, it follows that for $\lambda \leqslant r$ fixed we have $\frac{\partial u}{\partial y}(x_0, y) > 0$ provided $0 \leqslant y \leqslant \lambda$. Also by (9)

$$\text{for every } 0 < \lambda' \leqslant \lambda \text{ we have } u(x_0, y) < u(x_0\,,\,2\lambda' - y) \text{ provided that } y \in [0\,,\,\lambda')$$

Therefore we can exploit Theorem 3.1 and obtain

$$\text{for every } 0 < \lambda' \leqslant \lambda \text{ we have } u(x_0, y) < u(x_0\,,\,2\lambda' - y) \text{ in } \Sigma_{\lambda'} \equiv \{(x,y)\,|\,0 < y < \lambda'\},$$

that is $u < u_{\lambda'}$ in $\Sigma_{\lambda'}$.

Let us set

$$\Lambda = \{\lambda \in \mathbb{R}^+ \,:\, u < u_{\lambda'} \text{ in } \Sigma_{\lambda'} \ \ \forall \lambda' \leqslant \lambda\}$$

which is a nonempty set as shown above. Define

$$\bar{\lambda} = \sup \Lambda\,.$$

To prove the theorem, we have to show that actually $\bar{\lambda} = \infty$.

Note that by continuity $u \leqslant u_{\bar{\lambda}}$ in $\Sigma_{\bar{\lambda}}$ and also $u < u_{\bar{\lambda}}$ by the strong comparison principle (see theorem 2.4). Moreover $u$ is strictly increasing in the $e_2$-direction in $\Sigma_{\bar{\lambda}}$. In fact, given $(x, y_1)$ and $(x, y_2)$ in $\Sigma_{\bar{\lambda}}$ (say $0 \leqslant y_1 < y_2 \leqslant \bar{\lambda}$), we have by construction that $u(x,y_1) < u_{\frac{y_1+y_2}{2}}(x,y_1)$ which gives exactly

$$u(x,y_1) < u(x,y_2).$$

This immediately gives $\frac{\partial u}{\partial y}(x,y) \geqslant 0$ in $\Sigma_{\bar{\lambda}}$. But actually, by the strong maximum principle for the linearized operator $L_u$ (see [DS2] and Theorem 2.6 above), it follows that

$$\frac{\partial u}{\partial y}(x,y) > 0$$

in $\Sigma_{\bar{\lambda}}$. To prove that actually $\bar{\lambda} = \infty$, let us argue by contradiction, and assume $\bar{\lambda} < \infty$. First of all let us show that there exists some $x_0 \in \mathbb{R}$ such that

$$\frac{\partial u}{\partial y}(x_0\,,\,\bar{\lambda}) > 0.$$

Observe that in the case $m = 2$ (or more generally when dealing with strictly elliptic operators) by Hopf's Lemma it follows easily that $\frac{\partial u}{\partial y}(x_0\,,\,\bar{\lambda}) > 0$ for every $x_0 \in \mathbb{R}$. In the general case, we argue as follows.

Note that since $\frac{\partial u}{\partial y}(x, y) > 0$ in $\Sigma_{\bar{\lambda}}$, by continuity ($u \in C^{1,\alpha}$), it follows that $\frac{\partial u}{\partial y}(x, \bar{\lambda}) \geqslant 0$. We argue by contradiction, and assume that the thesis fails, that is

$$\frac{\partial u}{\partial y}(x\,,\,\bar{\lambda}) = 0$$

for every $x \in \mathbb{R}$.
Now consider the function $u^\star(x, y)$ defined in $\Sigma_{2\bar{\lambda}}$ by

$$u_\star(x, y) \equiv \begin{cases} u(x, y) & \text{if} \quad 0 \leqslant y \leqslant \bar{\lambda} \\ u(x, 2\bar{\lambda} - y) & \text{if} \quad \bar{\lambda} \leqslant y \leqslant 2\bar{\lambda} \end{cases}$$

and the function $u_\star(x, y)$ defined in $\Sigma_{2\bar{\lambda}}$ by

$$u^\star(x, y) \equiv \begin{cases} u(x, 2\bar{\lambda} - y) & \text{if} \quad 0 \leqslant y \leqslant \bar{\lambda} \\ u(x, y) & \text{if} \quad \bar{\lambda} \leqslant y \leqslant 2\bar{\lambda}. \end{cases}$$

Note that $u_\star$ is the even reflection of $u|_{\Sigma_{\bar{\lambda}}}$ and $u^\star$ the even reflection of $u|_{\Sigma_{2\bar{\lambda}} \setminus \Sigma_{\bar{\lambda}}}$. Since we are assuming that $\frac{\partial u}{\partial y}(x\,,\,\bar{\lambda}) = 0$ for every $x \in \mathbb{R}$, it follows that $u^\star$ and $u_\star$ are $C^1$ solutions of $-\Delta_m w = f(w)$. Since by definition $u < u_{\bar{\lambda}}$ in $\Sigma_{\bar{\lambda}}$, we have

$$u_\star \leqslant u^\star$$

in $\Sigma_{2\bar{\lambda}}$ and $u_\star$ does not coincide with $u^\star$ because of the strict inequality $u < u_{\bar{\lambda}}$ in $\Sigma_{\bar{\lambda}}$. Since $u_\star(x, \bar{\lambda}) = u^\star(x, \bar{\lambda})$ for any $x \in \mathbb{R}$, by Theorem 2.4 it would follow that $u_\star \equiv u^\star$ in $\Sigma_{2\bar{\lambda}}$. This contradiction actually proves that there exists some $x_0 \in \mathbb{R}$ such that $\frac{\partial u}{\partial y}(x_0\,,\,\bar{\lambda}) > 0$.

Let us now consider the segment $\{(x_0, y)\,|\,0 \leqslant y \leqslant \bar{\lambda}\}$, where $\frac{\partial u}{\partial y}(x_0\,,\,\bar{\lambda}) > 0$. Since $\frac{\partial u}{\partial y}(x_0\,,\,y) > 0$ for every $y \in [0, \bar{\lambda}]$, it follows that we can find $\varepsilon > 0$ such that

a) $\frac{\partial u}{\partial y}(x_0\,,\,y) > 0$ for every $y \in [0, \bar{\lambda} + \varepsilon]$
b) For every $0 < \lambda' \leqslant \bar{\lambda} + \varepsilon$ we get $u(x_0, y) < u(x_0\,,\,2\lambda' - y)$(that is $u < u_{\lambda'}$) provided that $y \in [0\,,\,\lambda')$.

Note that $a)$ follows easily by the continuity of the derivatives. The proof of $b)$ is standard in the moving plane technique (and it is in any case also contained in the proof of claim-1 in the proof of Theorem 3.1).
By Theorem 3.1 we now get that $u < u_{\lambda'}$ for every $0 < \lambda' < \bar{\lambda} + \varepsilon$ which implies $\sup \Lambda > \bar{\lambda}$, a contradiction. Therefore $\bar{\lambda} = \infty$.

**Proof of Theorem 1.3**

Let $u$ be a $C^{1,\alpha}$ solution of (1) with $m \geqslant 2$ and $f(u) = u^p$. It was shown in [DFSV](see Theorem 1.8) that any nonnegative solution is actually the trivial solution $u = 0$, in the case when the domain is a coercive epigraph. This assumption was only used in order to get the monotonicity of the solution and consequently the stability of the solution. In fact the monotonicity in one direction was obtained via the Alexandrov-Serrin moving plane method.[4] Here the monotonicity of the solution is obtained in Theorem 1.1 and therefore the proof of Theorem 1.8 of [DFSV] can be carried out as well as in [DFSV] obtaining Theorem 1.3.

**Proof of Theorem 1.4**

For any $(x, y) \in \mathbb{R}^2$ and $t \in \mathbb{R}$, we define

$$u^\star(x, y) := \begin{cases} u(x, y) & \text{if } y \geq 0, \\ -u(x, -y) & \text{if } y \leq 0 \end{cases}$$

and

$$f^\star(t) := \begin{cases} f(t) & \text{if } t \geq 0, \\ -f(-t) & \text{if } t \leq 0. \end{cases}$$

It follows easily that

$$-\Delta_m u^\star = f^\star(u^\star) \tag{10}$$

By construction, since $u$ is monotone and $u_y > 0$ in $\mathbb{R}^2 \cap \{y \geqslant 0\}$, it follows that

$$\frac{\partial u^\star}{\partial y} = u^\star_y > 0 \tag{11}$$

in $\mathbb{R}^2$. Moreover, $f^\star$ is locally Lipschitz continuous, since $f(0) = 0$. We note now that the fact that $u^\star_y > 0$ implies that $u$ is stable. This fact is classic for the semilinear case $m = 2$. For the general case $m \neq 2$ we refer the reader to [DFSV, FSV]. Also since the gradient of $u$ is bounded, the gradient of $u^\star$ is bounded too. We can therefore exploit[5] Theorem 1.1 in [FSV] to get that $u$ has one-dimensional symmetry in the sense that there exists $\bar{u} : \mathbb{R} \to \mathbb{R}$ and $\omega \in S^1$ in such a way that $u^\star(z) = \bar{u}(\omega \cdot z)$, for any $z \in \mathbb{R}^2$. Since in this case the level sets of our solution are parallel hyperplanes, and since the zero level set is $\{u = 0\} = \{y = 0\}$, it follows that necessarily $\omega = e_2$ and the thesis.

**Proof of Corollary 1.5**

Let us first note that, we have the right assumption to exploit Theorem 1.4 and get that $u(x, y) = \bar{u}(y)$. Therefore, if $u$ is not the trivial solution, then $\bar{u}$ is a solution of the one dimensional problem

$$\begin{cases} -\left((\bar{u}')^{(m-1)}\right)' = -(m-1)(\bar{u}')^{(m-2)}\bar{u}'' = f(\bar{u}), & \text{in } \mathbb{R}^+ \\ \bar{u} > 0, & \text{in } \mathbb{R}^+ \\ \bar{u}(0) = 0, & \forall x \in \mathbb{R} \\ \bar{u}' > 0, & \text{in } \mathbb{R}^+ \cup \{0\} \end{cases} \tag{12}$$

Since $\bar{u}$ is bounded and monotone, we have that

$$\lim_{y \to \infty} \bar{u}(y) = c > 0$$

[4] We refer to [BCN1, BCN2, BCN3, Dan, EL] for the semilinear case $m = 2$, while we refer to [DP, DS1] for the generalization of the moving plane technique to the case of the $m$-Laplace operator.

[5] Note that, thanks to (11), we have that the set $\{\nabla u = 0\}$ is empty. Therefore all the assumptions of Theorem 1.1 in [FSV] are fulfilled.

Now, since $f$ is positive and $\bar{u}'$ also is positive, by $-(m-1)(\bar{u}')^{(m-2)}\bar{u}'' = f(\bar{u})$ it follows that $\bar{u}'' < 0$. Therefore $\bar{u}'$ decreases at 0 at infinity as well as the term $(\bar{u}')^{(m-1)}$ .
It then follows that also $\left((\bar{u}')^{(m-1)}\right)'$ goes to 0 at infinity, at least on a sequences of points $y_n \underset{n\to\infty}{\longrightarrow} \infty$. This would imply that $\lim\limits_{n\to\infty} f(\bar{u}(y_n)) = 0$, a contradiction because $\lim\limits_{y\to\infty} f(\bar{u}(y)) = f(c) > 0$ since we assumed that $f$ is positive. We have therefore that

$$u(x,y) = \bar{u}(y) = 0$$

## 4. Proof of Theorem 3.1

Let $L_\theta$ be the vector $(\cos(\theta)\,,\, \sin(\theta))$ and $V_\theta$ the vector orthogonal to $L_\theta$ such that $(V_\theta, e_2) \geqslant 0$.

We define $L_{x_0,s,\theta}$ the line parallel to $L_\theta$ passing through $(x_0, s)$.

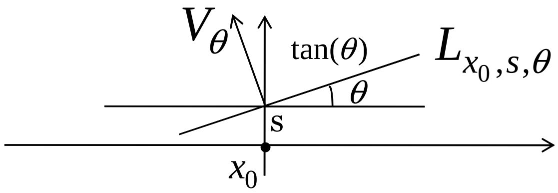


Figure 1

We define $\mathcal{T}_{x_0,s,\theta}$ as the triangle delimited by $L_{x_0,s,\theta}$, $\{y = 0\}$ and $\{x = x_0\}$ (see figure 2), and we set

$$u_{x_0,s,\theta}(x) = u(T_{x_0,s,\theta}(x)),$$

where $T_{x_0,s,\theta}(x)$ is the point symmetric to $x$, w.r.t. $L_{x_0,s,\theta}$ (see figure 2). And

$$w_{x_0,s,\theta} = u - u_{x_0,s,\theta}$$

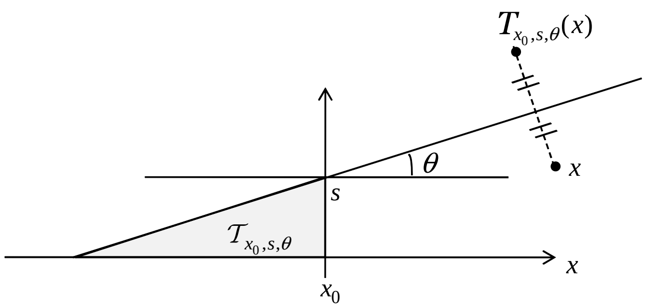


Figure 2

It is well known that $u_{x_0,s,\theta}$ still satisfies

$$-\Delta_m u_{x_0,s,\theta} = f(u_{x_0,s,\theta})$$

Also for simplicity we set

$$u_{x_0,s,0} = u_s \text{ and } u_s(x,y) = u(x, 2s - y)$$

Let us now consider $x_0 \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ fixed and assume that

a) $\frac{\partial u}{\partial y}(x_0\,,\,y) > 0$ for every $y \in [0, \lambda]$.
b) For every $0 < \lambda' \leqslant \lambda$ we get $u(x_0, y) < u(x_0\,,\,2\lambda' - y)$(that is $u < u_{\lambda'}$) provided that $y \in [0\,,\,\lambda')$.

as in the statement of Theorem 3.1. We have the following

**Claim-1**

Let $x_0 \in \mathbb{R}$ and $\lambda$ as above and as in the statement of Theorem 3.1. Then there exists $\delta > 0$ such that, for any $-\delta \leqslant \theta \leqslant \delta$ and for any $0 < \lambda' \leqslant \lambda + \delta$ we have

$$u(x_0, y) < u_{x_0,\lambda',\theta}(x_0, y) \qquad \text{for every} \qquad 0 \leqslant y < \lambda'.$$

*Proof.* We argue by contradiction. If the claim were false, we could find a sequence of values $\delta_n$, $\lambda_n$, $y_n$, $\theta_n$ such that

$\delta_n \to 0$, $-\delta_n \leqslant \theta_n \leqslant \delta_n$, $0 < \lambda_n \leqslant \lambda + \delta_n$, $0 \leqslant y_n < \lambda_n$ with $u(x_0, y_n) \geqslant u_{x_0,\lambda_n,\theta_n}(x_0, y_n)$.

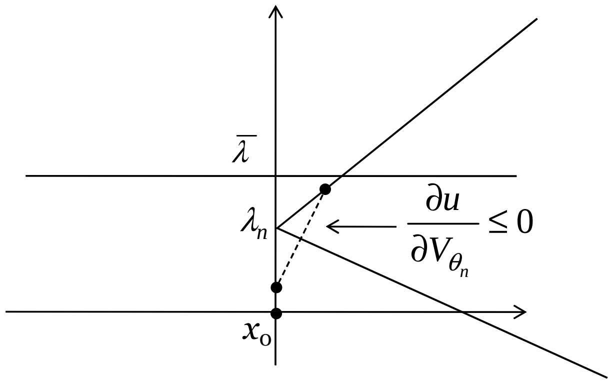


FIGURE 3

Passing to the limit (of a subsequence) we get $\lambda_n \to \tilde{\lambda} \leqslant \lambda$ and $y_n \to \tilde{y} \leqslant \tilde{\lambda}$. Let us show that $\tilde{y} = \tilde{\lambda}$.
In fact if $\tilde{\lambda} = 0$ it also follows $\tilde{y} = \tilde{\lambda} = 0$ since $0 \leqslant y_n < \lambda_n$. If instead $\tilde{\lambda} > 0$, by continuity it follows that $u(x_0, \tilde{y}) \geqslant u_{\tilde{\lambda}}(x_0, \tilde{y})$. Consequently $y_n \to \tilde{\lambda} = \tilde{y}$ since we know that $u < u_{\lambda'}\, \forall \lambda' \leqslant \bar{\lambda}$ in $\Sigma_{\lambda'}$.

By the mean value theorem since $u(x_0, y_n) \geqslant u_{x_0,\lambda_n,\theta_n}(x_0, y_n)$ (see figure 3), it follows that

$$\frac{\partial u}{\partial V_{\theta_n}}(\tilde{x}_n, \tilde{y}_n) \leqslant 0$$

at some point $\xi_n \equiv (\tilde{x}_n, \tilde{y}_n)$ lying on the line from $(x_0, y_n)$ to $T_{x_0,\lambda_n,\theta_n}(x_0, y_n)$. We recall that the vector $V_{\theta_n}$ is orthogonal to the line $L_{x_0,\lambda_n,\theta_n}$ and $V_{\theta_n} \to e_2$ since $\theta_n \to 0$. Passing to the limit it follows that

$$\frac{\partial u}{\partial y}(x_0, \tilde{\lambda}) \leqslant 0$$

which is impossible by the assumptions. This contradiction proves claim-1. $\square$

**Claim-2**

Consider $\delta$ given by claim-1. Then we find $\rho = \rho(\delta)$ such that for every $0 < s \leqslant \rho$ we have

$$u < u_{x_0,s,\delta} \text{ in } \mathcal{T}_{x_0,s,\delta} \text{ (and } u \leqslant u_{x_0,s,\delta} \text{ on } \partial(\mathcal{T}_{x_0,s,\delta}))$$
$$u < u_{x_0,s,-\delta} \text{ in } \mathcal{T}_{x_0,s,-\delta} \text{ (and } u \leqslant u_{x_0,s,-\delta} \text{ on } \partial(\mathcal{T}_{x_0,s,-\delta}))$$

*Proof.* We prove that we can find $\rho = \rho(\delta)$ such that, for every $0 < s \leqslant \rho$ we have

$u < u_{x_0,s,\delta}$ in $\mathcal{T}_{x_0,s,\delta}$ (and $u \leqslant u_{x_0,s,\delta}$ on $\partial(\mathcal{T}_{x_0,s,\delta})$).
If we replace $\delta$ by $-\delta$ the proof is exactly the same.

To prove this, note that $\delta$ is fixed and we can fix $\rho$ in such a way that

- $\rho < \lambda$, where $\lambda$ is given in the statement of Theorem 3.1 that we are proving.
- For every $0 < s \leqslant \rho$ we have $u \leqslant u_{x_0,s,\delta}$ on $\partial(\mathcal{T}_{x_0,s,\delta})$. In fact we have $u < u_{x_0,s,\delta}$ on the line $(x_0, y)$ for $0 \leqslant y < s$, by claim-1.
  Also $u \leqslant u_{x_0,s,\delta}$ if $y = 0$ by the Dirichlet assumption, and the fact that $u$ is positive in the interior of the domain. And finally $u \equiv u_{x_0,s,\delta}$ on $L_{x_0,s,\delta}$.
- Taking $\rho$ sufficiently small, since $\delta$ is fixed, we assume that for every $0 < s \leqslant \rho$ the Lebesgue measure $\mathcal{L}(\mathcal{T}_{x_0,s,\delta})$ is sufficiently small in order to exploit the weak comparison principle in small domains (see Theorem 2.3).

**Remark 4.1.** *In order to exploit the weak comparison principle (Theorem 2.3) we need that the functions that we compare are bounded in the domain that we consider. The constant $\delta$ in Theorem 2.3 in fact depends also on the $L^\infty$ norms of these functions. Under our assumption, $u$ is possibly unbounded in $D$ and $f$ is possibly only locally Lipschitz continuous in $[0\,,\,\infty)$.*

*We will always apply Theorem 2.3 in compact sets, so that everything works as well as in the case of bounded domains. In particular we may assume that $u\,,\,u_{x_0,s,\delta} \in C^1(\mathcal{K})$ where $\mathcal{K} \subset D$ is a compact set which contains $\mathcal{T}_{x_0,s,\delta}$ and the reflection of $\mathcal{T}_{x_0,s,\delta}$ w.r.t. $L_{x_0,s,\delta}$.*

Therefore, given any $0 < s \leqslant \rho$, if we consider $w_{x_0,s,\delta} = u - u_{x_0,s,\delta}$, we have $w_{x_0,s,\delta} \leqslant 0$ on $\partial\mathcal{T}_{x_0,s,\delta}$ and therefore, by the weak comparison principle in small domains (see Theorem 2.3) we get

$$w_{x_0,s,\delta} \leqslant 0 \text{ in } \mathcal{T}_{x_0,s,\delta}.$$

Also, by the strong comparison principle (see Theorem 2.4), we get

$$w_{x_0,s,\delta} < 0 \text{ in } \mathcal{T}_{x_0,s,\delta}.$$

since the case $w_{x_0,s,\delta} \equiv 0$ is clearly impossible. This proves claim-2.

□

**Remark 4.2.** *In what follows we will use a refined version of the Alexandrov-Serrin moving plane method(see* [BN, GNN, Ser]*). In particular we refer to* [BN]. *We refer the readers to* [Dam, DP, DS1] *for the case of problems involving the m-Laplace operator.*

Let us consider $(\rho, \delta)$ given by claim-1 and claim-2 and $\lambda$ as in the statement of the theorem. Consider $0 < \lambda' \leqslant \lambda$ and let us fix $0 < \bar{s} < \min\{\rho\,,\,\lambda'\}$ so that by claim-2 we have

$$(\star) \qquad w_{x_0,\bar{s},\delta} \leqslant 0 \text{ on } \partial(\mathcal{T}_{x_0,\bar{s},\delta}) \text{ and } w_{x_0,\bar{s},\delta} < 0 \text{ in } \mathcal{T}_{x_0,\bar{s},\delta}$$

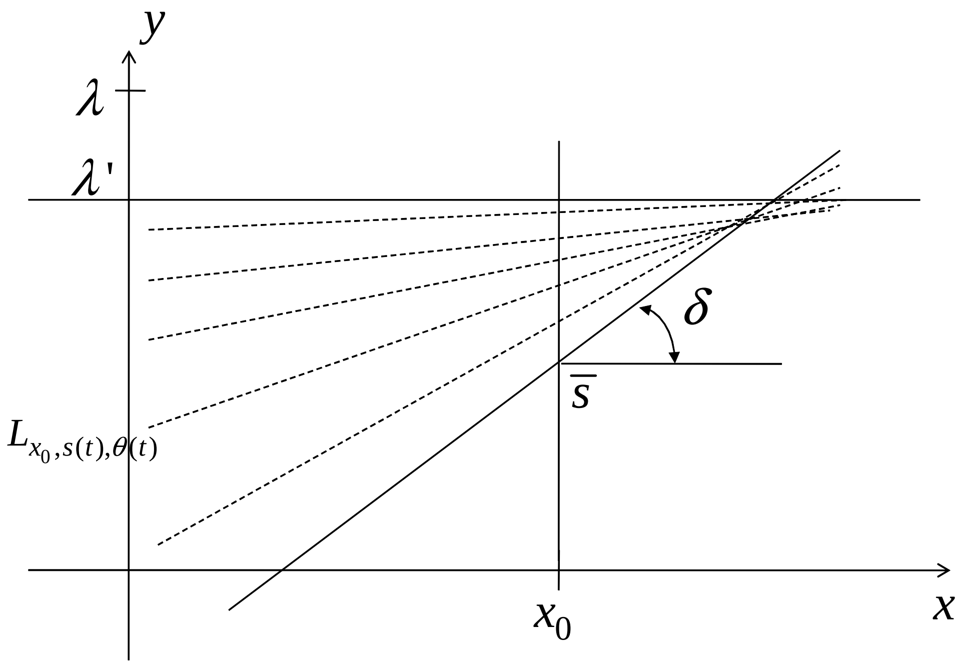


FIGURE 4

We define the continuous function $g(t) = (s(t), \theta(t)) : [0,1] \to \mathbb{R}^2$ (in figure 4 is represented $L_{x_0,s(t),\theta(t)}$), by

$$g(t) = (s(t), \theta(t)) = (t\lambda' + (1-t)\bar{s}, (1-t)\delta)$$

so that, $g(0) = (\bar{s}\,,\,\delta)$, $g(1) = (\lambda', 0)$ and $\theta(t) \neq 0$ for every $t \in [0,1)$.
Moreover, by claim-1, we have

$$w_{x_0,s(t),\theta(t)} \leqslant 0 \quad \text{on} \quad \partial(\mathcal{T}_{x_0,s(t),\theta(t)}) \quad \text{for every } t \in [0,1)$$

and $w_{x_0,s(t),\theta(t)}$ is not identically zero on $\partial(\mathcal{T}_{x_0,s(t),\theta(t)})$ for every $t \in [0,1)$.
We now let

$$\overline{T} \equiv \ \{\tilde{t} \in [0,1] \ s.t. \ w_{x_0,s(t),\theta(t)} < 0 \text{ in } \mathcal{T}_{x_0,s(t),\theta(t)} \text{ for every } 0 \leqslant t \leqslant \tilde{t}\} \quad \text{and} \quad \bar{t} = \sup\ \overline{T}$$

where possibly $\bar{t} = 0$.

**Claim-3**

Given $\bar{t}$ defined here above, we have $\bar{t} = 1$.

*Proof.* To prove this, assume $\bar{t} < 1$ and note that in this case, by continuity

$$w_{x_0,s(\bar{t}),\theta(\bar{t})} \leqslant 0 \quad \text{in} \quad \mathcal{T}_{x_0,s(\bar{t}),\theta(\bar{t})}$$

and, by the strong comparison theorem

$$w_{x_0,s(\bar{t}),\theta(\bar{t})} < 0 \quad \text{in} \quad \mathcal{T}_{x_0,s(\bar{t}),\theta(\bar{t})}.$$

Since $w_{x_0,s(\bar{t}),\theta(\bar{t})} < 0$ in $\mathcal{T}_{x_0,s(\bar{t}),\theta(\bar{t})}$ we can therefore take a compact set $K \subset \mathcal{T}_{x_0,s(\bar{t}),\theta(\bar{t})}$ where $w_{x_0,s(\bar{t}),\theta(\bar{t})} \leqslant \rho < 0$. Considering $\mathcal{T}_{x_0,s(\bar{t}+\varepsilon),\theta(\bar{t}+\varepsilon)} \approx \mathcal{T}_{x_0,s(\bar{t}),\theta(\bar{t})}$, for $\bar{\varepsilon} > 0$ sufficiently small we may assume that for every $0 < \varepsilon \leqslant \bar{\varepsilon}$ we have $K \subset \mathcal{T}_{x_0,s(\bar{t}+\varepsilon),\theta(\bar{t}+\varepsilon)}$ and

$$w_{x_0,s(\bar{t}+\varepsilon),\theta(\bar{t}+\varepsilon)} \leqslant \tfrac{\rho}{2} < 0 \qquad \text{in} \qquad K.$$

We can also take $K$ such that the Lebesgue measure $\mathcal{L}(\mathcal{T}_{x_0,s(\bar{t}+\varepsilon),\theta(\bar{t}+\varepsilon)} \setminus K)$ is small enough in order to exploit the weak comparison principle in small domains (see Theorem 2.3). Therefore, recalling Remark 4.1, since $w_{x_0,s(\bar{t}+\varepsilon),\theta(\bar{t}+\varepsilon)} \leqslant 0$ on $\partial\big(\mathcal{T}_{x_0,s(\bar{t}+\varepsilon),\theta(\bar{t}+\varepsilon)} \setminus K\big)$ (recall that $w_{x_0,s(t),\theta(t)} \leqslant 0$ on $\partial(\mathcal{T}_{x_0,s(t),\theta(t)})$ for every $t \in [0,1)$), we have

$$w_{x_0,s(\bar{t}+\varepsilon),\theta(\bar{t}+\varepsilon)} \leqslant 0 \quad \text{ in } \mathcal{T}_{x_0,s(\bar{t}+\varepsilon),\theta(\bar{t}+\varepsilon)} \setminus K$$

and therefore in $\mathcal{T}_{x_0,s(\bar{t}+\varepsilon),\theta(\bar{t}+\varepsilon)}$. Also, by the strong comparison principle (see Theorem 2.4), we get

$$w_{x_0,s(\bar{t}+\varepsilon),\theta(\bar{t}+\varepsilon)} < 0$$

in $\mathcal{T}_{x_0,s(\bar{t}+\varepsilon),\theta(\bar{t}+\varepsilon)}$.

Concluding, for $\bar{\varepsilon}$ sufficiently small and for every $0 < \varepsilon \leqslant \bar{\varepsilon}$ we get[6]

$$(\star) \quad w_{x_0,s(\bar{t}+\varepsilon),\theta(\bar{t}+\varepsilon)} \leqslant 0 \text{ on } \partial\big(\mathcal{T}_{x_0,s(\bar{t}+\varepsilon),\theta(\bar{t}+\varepsilon)}\big) \text{ and } w_{x_0,s(\bar{t}+\varepsilon),\theta(\bar{t}+\varepsilon)} < 0 \text{ in } \mathcal{T}_{x_0,s(\bar{t}+\varepsilon),\theta(\bar{t}+\varepsilon)}$$

We therefore obtain that $\sup \overline{T} > \bar{t}$ that is a contradiction with the definition of $\bar{t}$, proving therefore that actually

$$\bar{t} = 1$$

and claim-3. □

### Conclusion

By claim-3 it follows that $w_{x_0,s(t),\theta(t)} \leqslant 0$ in $\mathcal{T}_{x_0,s(t),\theta(t)}$ for every $t \in [0,1)$. Therefore by continuity we get $w_{\lambda'} = w_{x_0,\lambda',0} = w_{x_0,s(1),\theta(1)} \leqslant 0$ in $\mathcal{T}_{x_0,s(1),\theta(1)} \equiv \big(\Sigma_{\lambda'} \cap \{x \leqslant x_0\}\big)$. By the strong comparison principle $u < u_{\lambda'}$ in $\Sigma_{\lambda'} \cap \{x \leqslant x_0\}$, that is

$$u(x,y) < u_{\lambda'}(x,y) = u(x, 2\lambda' - y) \text{ in } \Sigma_{\lambda'} \cap \{x \leqslant x_0\}. \tag{13}$$

If now we consider

$$g(t) = (s(t), \theta(t)) = (t\lambda' + (1-t)\bar{s}, (1-t)(-\delta)) = (t\lambda' + (1-t)\bar{s}, (t-1)\delta)$$

we can argue exactly as in claim-3 above. It is easy to see that, with the same procedure, we get

$$u(x,y) < u_{\lambda'}(x,y) = u(x, 2\lambda' - y) \text{ in } \Sigma_{\lambda'} \cap \{x \geqslant x_0\}. \tag{14}$$

Finally, by (13) and (14), we get

$$u(x,y) < u_{\lambda'}(x,y) = u(x, 2\lambda' - y) \text{ in } \Sigma_{\lambda'}$$

for every $0 < \lambda' \leqslant \lambda$, and the theorem is proved.

[6]That is, once we have the right conditions on the boundary by claim-1, we can make small movements (translations and rotations) and recover the same situation $(\star)$.

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