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Harnack inequalities, maximum and comparison principles, and regularity of positive solutions of m-laplace equations

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Abstract We consider the Dirichlet problem for positive solutions of the equation $-\Delta_m(u) = f(u)$ in a bounded smooth domain Ω, with f positive and locally Lipschitz continuous. We prove a Harnack type inequality for the solutions of the linearized operator, a Harnack type comparison inequality for the solutions, and exploit them to prove a Strong Comparison Principle for solutions of the equation, as well as a Strong Maximum Principle for the solutions of the linearized operator. We then apply these results, together with monotonicity results recently obtained by the authors, to get regularity results for the solutions. In particular we prove that in convex and symmetric domains, the only point where the gradient of a solution u vanishes is the center of symmetry (i.e. $Z \equiv \{x \in \Omega \mid D(u)(x) = 0\} = \{0\}$ assuming that 0 is the center of symmetry). This is crucial in the study of m-Laplace equations, since Z is exactly the set of points where the m-Laplace operator is degenerate elliptic. As a corollary $u \in C^2(\Omega \setminus \{0\})$.

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1 Introduction and statement of the results

Let us consider weak $C^1(\overline{\Omega})$ solutions of the problem

$$\begin{cases} -\Delta_m(u) & = f(u) & \text{in } \Omega \\ u & > 0 & \text{in } \Omega \\ u & = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

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where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geqslant 2$, $\Delta_m(u) = \operatorname{div}(|Du|^{m-2}Du)$ is the m-Laplace operator, $1 < m < \infty$ and we have the following hypotheses on f:

(*) $f : [0, \infty) \to \mathbb{R}$ is a continuous function which is positive and locally Lipschitz continuous in $(0, \infty)$.

It is well known that, since the m-Laplace operator is singular or degenerate elliptic (respectively if $1 < m < 2$ or $m > 2$) in the critical set

$$Z \equiv \{x \in \Omega \mid D(u)(x) = 0\} \tag{1.2}$$

solutions of (1.1) belong generally to the class $C^{1,\tau}$ with $\tau < 1$, and solve (1.1) only in the weak sense. Moreover there are not general comparison theorems for the solutions of (1.1) and maximum principles for solutions of the corresponding equation involving the linearized operator at a fixed solution, as is the case when strictly elliptic operators are considered.

In a recent paper [11] the authors proved regularity properties of positive solutions u of (1.1) when f satisfies (*), such as summability properties of $\frac{1}{|Du|}$, where Du is the gradient of u, and Sobolev and Poincaré type inequalities in weighted Sobolev spaces with weight $\rho = |Du|^{m-2}$.

In the present paper we first exploit the general Sobolev inequality proved in [11] to prove a Harnack type inequality for solutions v of the linearized equation at a fixed solution u of (1.1), as well as a Harnack type comparison inequality for two solutions of (1.1).

These inequalities allow us to prove a strong maximum principle for solutions of the linearized equation and a strong comparison principle for two solutions of (1.1).

We then use these maximum and comparison principles, together with monotonicity and symmetry results proved in [10] for the case $1 < m < 2$ and by the authors in [11] for the case $1 < m < \infty$ using the well known Alexandrov-Serrin moving plane method. The combination of these results allows us to prove that the critical set of a solution u of (1.1) must be contained in a region which depends on the geometry of the domain through the moving plane method (see Sect. 3 for the details).

As a particular case we get the following striking result: if the domain is convex and symmetric with respect to N orthogonal directions ($\frac{2N+2}{N+2} < m < 2$ or $m > 2$), then

$$Z \equiv \{0\} \tag{1.3}$$

assuming that 0 is the center of symmetry. This is crucial in the study of m-Laplace equations, since Z is exactly the set of points where the m-Laplace operator is degenerate elliptic. Moreover as a corollary we get that a solution u of (1.1) belongs to the space $C^2(\Omega \setminus \{0\})$.

Previously this result was known only for radial solutions in a ball, or general solutions again in a ball, once radial symmetry results are available (see e.g. [4], [5], [10]).

Let us now state some of the results proved in the sequel, referring to the relevant sections for more general statements and results.

Let us recall that the linearized operator at a fixed solution u of (1.1), $L_u(v, \varphi)$, is well defined, for every v and φ in the weighted Sobolev space $H^{1,2}_{\rho}(\Omega)$ (see Sect. 2 for details) with $\rho \equiv |Du|^{m-2}$, by

$$L_u(v, \varphi) \equiv$$

$$\int_{\Omega} [|Du|^{m-2}(Dv, D\varphi) + (m-2)|Du|^{m-4}(Du, Dv)(Du, D\varphi) - f'(u)v\varphi]dx$$

Moreover, $v \in H^{1,2}_{\rho}(\Omega)$ is a weak solution of the linearized operator if

$$L_u(v, \varphi) = 0 \tag{1.4}$$

for any $\varphi \in H^{1,2}_{0,\rho}(\Omega)$.

More generally, $v \in H^{1,2}_{\rho}(\Omega)$ is a weak supersolution (subsolution) of (1.4) if $L_u(v, \varphi) \geqslant 0 (\leqslant 0)$ for any nonnegative $\varphi \in H^{1,2}_{0,\rho}(\Omega)$.

According to these definitions we will prove the following weak Harnack inequality for the linearized operator (see Sect. 3 for the case when v is a weak subsolution and the case when v is a weak solution).

Theorem 1.1 *Let $u \in C^1(\overline{\Omega})$ be a weak solution of (1.1), where $m > 2$, Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geqslant 2$, and f satisfies (*), and define $\rho \equiv |Du|^{m-2}$. Suppose that $\overline{B(x, 5\delta)} \subset \Omega$, and put*

$$\frac{1}{\overline{2}^*} = \frac{1}{2} - \frac{1}{N} + \frac{1}{N}\left(\frac{m-2}{m-1}\right)$$

(consequently $\overline{2}^* > 2$ for $m > 2$) and let 2^* be any real number such that $2 < 2^* < \overline{2}^*$.

If $v \in H^{1,2}_{\rho}(\Omega) \cap L^{\infty}(\Omega)$ is a nonnegative weak supersolution of (1.4), then for every $0 < s < \chi$, $\chi \equiv \frac{2^*}{2}$, there exists $C > 0$ such that

$$\|v\|_{L^s(B(x,2\delta))} \leqslant C \inf_{B(x,\delta)} v \tag{1.5}$$

where C is a constant depending on x, s, N, u, m, f.

If $\frac{2N+2}{N+2} < m < 2$ the same result holds with χ replaced by $\chi' \equiv \frac{2^\sharp}{s^\sharp}$ where $2^\sharp$ is the classical Sobolev exponent, $\frac{2}{s^\sharp} \equiv 1 - \frac{1}{s}$ and $s < \frac{m-1}{2-m}$.

The iterative technique we use to prove Theorem 1.1 is due to J. K. Moser [20] and was first used to prove Holder continuity properties of solutions of some strictly elliptic linear operators (this problem had been previously studied by E. De Giorgi [12] and J. Nash [22] in their famous papers). It was then used to study also the case of degenerate operators in [24] and [30].

In [31] N. S. Trudinger considers the case of degenerate operators which satisfy some a-priori assumptions on the matrix of the coefficients (see [31]). The work of N. S. Trudinger stemmed originally from the paper of J. K. Moser, but it made no use of (a variant of) the famous John-Nirenberg Lemma (see [20]), exploiting in the proof only weighted Sobolev inequalities and a clever use of test-functions techniques.

Our main contribution in Theorem 1.1 consists in showing that the techniques proposed by N. S. Trudinger, in our context, can be exploited using Sobolev inequalities and summability properties of $\frac{1}{|Du|}$ we have obtained in [11].

As a consequence of the previous inequality we get a Strong Maximum Principle for solutions v of (1.4) (see Sect. 3), and since any derivative u_{x_i}, $1 \leqslant i \leqslant N$, satisfies (1.4) (see [11]) we get the following

Theorem 1.2 *Let $u \in C^1(\overline{\Omega})$ a weak solution of (1.1) in a bounded smooth domain Ω of $\mathbb{R}^N$ with $m > 2$ or $\frac{2N+2}{N+2} < m < 2$ and f satisfying (*). Then, for any $i \in \{1, \dots, N\}$ and any domain $\Omega' \subset \Omega$ with $u_{x_i} \geqslant 0$ in Ω', we have that either $u_{x_i} \equiv 0$ in Ω' or $u_{x_i} > 0$ in Ω'.*

Using this maximum principle together with monotonicity and symmetry results known for the solutions of (1.1) (see [11] and the references therein) we prove regularity results for solutions in general domains. In particular we prove the following striking result (see Sect. 3 for a more general statement):

Theorem 1.3 *Let Ω be a bounded smooth domain in $\mathbb{R}^N$, $N \geqslant 2$, $\frac{2N+2}{N+2} < m < 2$ or $m > 2$, and $u \in C^1(\overline{\Omega})$ a weak solution of (1.1) with $f : [0, \infty) \to \mathbb{R}$ positive ($f(s) > 0$ for $s > 0$) and locally Lipschitz continuous. Then, if for N orthogonal directions e_i, $i = 1, \dots, N$, the domain Ω is convex in the e_i direction and symmetric with respect to the hyperplanes $T_0^{e_i} = \{x_i = 0\}$ for every $i = 1, \dots, N$, it follows that*

$$Z \equiv \{x \in \Omega \mid D(u)(x) = 0\} = \{0\} \tag{1.6}$$

Consequently $u \in C^2(\Omega \setminus \{0\})$.

Using the same arguments we also prove a weak Harnack type comparison inequality for the difference of two solutions of (1.1) (see Theorem 3.3). This implies in turn the following

Theorem 1.4 (Strong Comparison Principle) *Let $u, v \in C^1(\overline{\Omega})$ where Ω is a bounded smooth domain of $\mathbb{R}^N$ with $\frac{2N+2}{N+2} < m < 2$ or $m > 2$. Suppose that either u or v is a weak solution of (1.1) with f satisfying $(*)$. Assume*

$$-\Delta_m(u) + \Lambda u \leqslant -\Delta_m(v) + \Lambda v \qquad u \leqslant v \quad in \quad \Omega \tag{1.7}$$

where $\Lambda \in \mathbb{R}$. Then $u \equiv v$ in Ω unless

$$u < v \qquad in \qquad \Omega \tag{1.8}$$

The same result holds (see Remark 3.2) if u and v are weak solutions of (1.1) or more generally if

$$-\Delta_m(u) - f(u) \leqslant -\Delta_m(v) - f(v) \qquad u \leqslant v \quad in \quad \Omega \tag{1.9}$$

with u or v weakly solving (1.1).

Theorem 1.4 improves previous similar results. In particular we refer to [15] for the case of strictly elliptic operators or for the case of degenerate operators with $f = 0$ (see also [8]). As shown in [9] the arguments of [15] can be applied to the case of m-Laplace equations with locally Lipschitz continuous nonlinearities in $\Omega \setminus Z$ proving a Strong Comparison Principle which holds in any connected component of $\Omega \setminus Z$. In [6] the authors consider m-Laplace equations with non-linearities which are slightly more general than $f(u) = \lambda u^{m-1}$ and λ is strictly less than the first eigenvalue of the m-Laplace operator and prove a Strong Comparison Principle for solutions which vanishes on the boundary (see also [7] for a more general class of operators). Here, assuming only that f is a positive and locally Lipschitz continuous function, we prove a Strong comparison Principle for the solutions in the entire region Ω without assumptions on their sign on the boundary and without any a priori assumption on the critical sets.

2 Preliminaries

We start by recalling some qualitative properties of the solutions of (1.1) proved by the authors in [11].

In the sequel, as in [21], if $\rho \in L^1(\Omega)$ $1 \leqslant p < \infty$, the space $H^{1,p}_\rho(\Omega)$ is defined as the completion of $C^1(\overline{\Omega})$ (or $C^\infty(\overline{\Omega})$) under the norm

$$\|v\|_{H^{1,p}_\rho} = \|v\|_{L^p(\Omega)} + \|Dv\|_{L^p(\Omega,\rho)} \tag{2.1}$$

and $\|Dv\|^p_{L^p(\Omega,\rho)} = \int_\Omega |Dv|^p \rho \, dx$. In this way $H^{1,p}_\rho(\Omega)$ is a Banach space and $H^{1,2}_\rho(\Omega)$ is a Hilbert space. Moreover we define $H^{1,p}_{o,\rho}(\Omega)$ as the closure of $C^1_c(\overline{\Omega})$ (or $C^\infty_c(\overline{\Omega})$) in $H^{1,p}_\rho(\Omega)$. We also recall that in [31] $H^{1,p}_{o,\rho}$ is defined as the space of functions having distributional derivatives represented by a function for which the norm defined in (2.1) is bounded. These two definitions are equivalent if the domain has piecewise regular boundary (as is in our case).

From now on, given a fixed $C^1(\overline{\Omega})$ solution of (1.1), we will consider

$$\rho \equiv |Du|^{m-2} \tag{2.2}$$

Therefore $\rho \in L^\infty(\Omega)$ if $m > 2$ since $u \in C^1(\overline{\Omega})$. If instead $\frac{2N+2}{N+2} < m < 2$, then $\rho \in L^1(\Omega)$ as easily follows by the results of [11] (see Theorem 2.1 below).

If u is a $C^1(\overline{\Omega})$ solution of (1.1) with f locally Lipschitz continuous, then $u_{x_i} \in H^{1,2}_\rho(\Omega)$ for $i = 1, \ldots, N$ as proved in [11], and we can show that $L_u(u_{x_i}, \varphi)$ is well defined by

$$\begin{aligned} L_u(u_{x_i}, \varphi) \equiv \\ \int_\Omega [|Du|^{m-2}(Du_{x_i}, D\varphi) + (m-2)|Du|^{m-4}(Du, Du_{x_i})(Du, D\varphi)]\, dx + \\ - \int_\Omega f'(u) u_{x_i} \varphi \, dx \end{aligned} \tag{2.3}$$

for every $\varphi \in C^1_0(\Omega)$. Moreover the following equation holds

$$L_u(u_{x_i}, \varphi) = 0 \qquad \forall \varphi \in C^1_0(\Omega). \tag{2.4}$$

Using (2.4) and (1.1) and some test functions techniques, in [11] the following result is proved:

Theorem 2.1 *Let Ω be a smooth domain in $\mathbb{R}^N$, $N \geqslant 2$ and $u \in C^1(\overline{\Omega})$ a weak solution of (1.1) with f satisfying (*), $1 < m < \infty$ and let $Z \equiv \{x \in \Omega \mid D(u)(x) = 0\}$. Then $|Z| = 0$ and, for any $x \in \Omega$ and for every $r < 1$, $\gamma < N - 2$ if $N \geqslant 3$ and $\gamma = 0$ if $N = 2$, we have*

$$\int_\Omega \frac{1}{|Du|^{(m-1)r}} \frac{1}{|x-y|^\gamma}\, dy \leqslant C$$

where C does not depend on x.

Exploiting these results in [11] a weighted Sobolev (and Poincaré) type inequality is obtained:

Theorem 2.2 *Let $u \in C^1(\overline{\Omega})$ be a weak solution of (1.1) with f satisfying (*), $m > 2$. Define $\overline{2}^*$ by*

$$\frac{1}{\overline{2}^*} = \frac{1}{2} - \frac{1}{N} + \frac{1}{N}\left(\frac{m-2}{m-1}\right)$$

(consequently $\overline{2}^* > 2$ for $m > 2$).
Then we get that there exists a positive constant $c_0 = c_0(N, p, \rho, t, \gamma)$ such that the following weighted Sobolev inequality holds:

$$\|v\|_{L^{2^*}} \leqslant c_0 \|Dv\|_{L^2(\Omega,\rho)}, \tag{2.5}$$

for any $v \in H^{1,2}_{0,\rho}(\Omega)$, $2^* < \overline{2}^*$ and $\rho \equiv |Du|^{m-2}$.
Moreover for $v \in H^{1,2}_{0,\rho}(\Omega)$ we have the following weighted Poincaré inequality

$$\|v\|_{L^2(\Omega)} \leqslant C(|\Omega|)\|Dv\|_{L^2(\Omega,\rho)} \tag{2.6}$$

where $C(|\Omega|) \to 0$ if $|\Omega| \to 0$.

Remark 2.1 *The proof of Theorem 2.2 (see [11]) is based on Theorem 2.1 and on potential estimates. Since potential estimates are also available for functions with zero mean (see [15]), then we can prove weighted Sobolev inequality and weighted Poincaré inequality, also for functions with zero mean.*

These results have been exploited in [11] to prove some monotonicity and symmetry results for the solutions of (1.1) which extend to the case $1 < m < \infty$ previous results proved in [10] in the case $1 < m < 2$. These results rely on some weak comparison principles (proved there) and on the Alexandrov-Serrin moving plane method [23]. To state these results we need some notations.
Let ν be a direction in $\mathbb{R}^N$. For a real number λ we define

$$T^\nu_\lambda = \{x \in \mathbb{R} : x \cdot \nu = \lambda\} \tag{2.7}$$
$$\Omega^\nu_\lambda = \{x \in \Omega : x \cdot \nu < \lambda\} \tag{2.8}$$
$$x^\nu_\lambda = R^\nu_\lambda(x) = x + 2(\lambda - x\cdot\nu)\nu, \quad x \in \mathbb{R}^N \tag{2.9}$$

and

$$a(\nu) = \inf_{x\in\Omega} x \cdot \nu. \tag{2.10}$$

If $\lambda > a(\nu)$ then Ω_λ^ν is nonempty, thus we set

$$\left(\Omega_\lambda^\nu\right)' = R_\lambda^\nu\left(\Omega_\lambda^\nu\right). \tag{2.11}$$

Following [14, 23] we observe that for $\lambda - a(\nu)$ small then $(\Omega_\lambda^\nu)'$ is contained in Ω and will remain in it, at least until one of the following occurs:

(i) $(\Omega_\lambda^\nu)'$ becomes internally tangent to $\partial\Omega$.
(ii) T_λ^ν is orthogonal to $\partial\Omega$.

Let $\Lambda_1(\nu)$ be the set of those $\lambda > a(\nu)$ such that for each $\mu < \lambda$ none of the conditions (i) and (ii) holds and define

$$\lambda_1(\nu) = \sup\ \Lambda_1(\nu). \tag{2.12}$$

Moreover let

$$\Lambda_2(\nu) = \left\{\lambda > a(\nu) : \left(\Omega_\mu^\nu\right)' \subseteq \Omega \quad \forall \mu \in (a(\nu), \lambda]\right\} \tag{2.13}$$

and

$$\lambda_2(\nu) = \sup\ \Lambda_2(\nu). \tag{2.14}$$

Note that since Ω is supposed to be smooth neither $\Lambda_1(\nu)$ nor $\Lambda_2(\nu)$ are empty, and $\Lambda_1(\nu) \subseteq \Lambda_2(\nu)$ so that $\lambda_1(\nu) \leqslant \lambda_2(\nu)$ (in the terminology of [14] $\Omega_{\lambda_1(\nu)}^\nu$ and $\Omega_{\lambda_2(\nu)}^\nu$ correspond to the 'maximal cap', respectively to the 'optimal cap'). Finally define

$$\Lambda_0(\nu) = \{\lambda > a(\nu) : u \leqslant u_\lambda^\nu \quad \forall \mu \in (a(\nu), \lambda]\} \tag{2.15}$$

and

$$\lambda_0(\nu) = \sup\ \Lambda_0(\nu). \tag{2.16}$$

Theorem 2.3 ([11]) *Let Ω be a bounded smooth domain in $\mathbb{R}^N$, $N \geqslant 2$, $1 < m < \infty$, and $u \in C^1(\overline{\Omega})$ a weak solution of (1.1) with f satisfying (*).*
For any direction ν and for λ in the interval $(a(\nu), \lambda_1(\nu)]$ we have

$$u(x) \leqslant u\left(x_\lambda^\nu\right) \quad \forall x \in \Omega_\lambda^\nu. \tag{2.17}$$

Moreover, for any λ with $a(\nu) < \lambda < \lambda_1(\nu)$ we have

$$u(x) < u\left(x_\lambda^\nu\right) \quad \forall x \in \Omega_\lambda^\nu \setminus Z_\lambda^\nu \tag{2.18}$$

where $Z_\lambda^\nu \equiv \{x \in \Omega_\lambda^\nu\ :\ Du(x) = Du_\lambda^\nu(x) = 0\}$. Finally

$$\frac{\partial u}{\partial \nu}(x) > 0 \quad \forall x \in \Omega_{\lambda_1(\nu)}^\nu \setminus Z \tag{2.19}$$

where $Z = \{x \in \Omega : Du(x) = 0\}$.
If moreover f is locally Lipschitz continuous in the closed interval $[0, \infty)$ then (2.17) and (2.18) hold for any λ in the interval $(a(\nu), \lambda_2(\nu))$ and (2.19) holds for any $x \in \Omega_{\lambda_2(\nu)}^\nu \setminus Z$.

Corollary 2.1 *If f satisfies (*) and the domain Ω is symmetric with respect to the hyperplane $T_0^\nu = \{x \in \mathbb{R}^N : x \cdot \nu = 0\}$ and $\lambda_1(\nu) = \lambda_1(-\nu) = 0$, then u is symmetric, i. e. $u(x) = u(x_0^\nu)$, and nondecreasing in the ν–direction in Ω_0^ν with $\frac{\partial u}{\partial \nu}(x) > 0$ in $\Omega_0^\nu \setminus Z$.*
In particular if Ω is a ball then u is radially symmetric and $\frac{\partial u}{\partial r} < 0$, where $\frac{\partial u}{\partial r}$ is the derivative in the radial direction.
If f is locally Lipschitz continuous in the closed interval $[0, \infty)$ then the same result follows assuming only that the domain Ω is convex and symmetric in the ν-direction($\lambda_2(\nu) = \lambda_2(-\nu) = 0$).

Let us introduce some notations which we will then use in the proof of the Harnack inequality. For a fixed $x \in \Omega$ and v positive we put

$$\phi(p, R, v) = \left(\int_{B(x,R)} |v|^p \, dx\right)^{\frac{1}{p}} \tag{2.20}$$

for $p \neq 0$, so that for $p \geqslant 1$

$$\phi(p, R, v) = \|v\|_{L^p(B(x,R))} \tag{2.21}$$

The following well known properties of the functional ϕ will be used in the proofs of our results.

$$\lim_{p\to\infty} \phi(p, R, v) = \phi(\infty, R, v) = \sup_{B(x,R)} |v| \tag{2.22}$$

$$\lim_{p\to-\infty} \phi(p, R, v) = \phi(-\infty, R, v) = \inf_{B(x,R)} |v| \tag{2.23}$$

3 Weak Harnack Inequalities and applications

In this section we state a weak Harnack inequality for the solutions of the linearized operator and a weak Harnack comparison inequality for two solutions of (1.1). We then exploit these results showing the main consequences and applications, postponing the proofs of weak Harnack inequalities to the appendix.

Theorem 3.1 *Let $u \in C^1(\overline{\Omega})$ be a weak solution of (1.1), where $m > 2$, Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geqslant 2$, and f satisfies (*), and define $\rho \equiv |Du|^{m-2}$. Suppose that $\overline{B(x, 5\delta)} \subset \Omega$, and put*

$$\frac{1}{\overline{2}^*} = \frac{1}{2} - \frac{1}{N} + \frac{1}{N}\left(\frac{m-2}{m-1}\right)$$

(consequently $\overline{2}^* > 2$ for $m > 2$) and let 2^* be any real number such that $2 < 2^* < \overline{2}^*$.
If $v \in H^{1,2}_\rho(\Omega) \cap L^\infty(\Omega)$ is a nonnegative weak supersolution of (1.4), then for every $0 < s < \chi$, $\chi \equiv \frac{2^*}{2}$, there exists $C > 0$ such that

$$\|v\|_{L^s(B(x,2\delta))} \leqslant C \inf_{B(x,\delta)} v \tag{3.1}$$

where C is a constant depending on x, s, N, u, m, f.
If $\frac{2N+2}{N+2} < m < 2$ the same result holds with χ replaced by $\chi' \equiv \frac{2^\sharp}{s^\sharp}$ where $2^\sharp$ is the classical Sobolev exponent, $\frac{2}{s^\sharp} \equiv 1 - \frac{1}{s}$ and $s < \frac{m-1}{2-m}$.

If we consider a subsolution v of (1.4), the iteration technique, which is simpler in this case (see [15, 31]), allows to prove the following

Theorem 3.2 *Let $v \in H^{1,2}_\rho(\Omega) \cap L^\infty(\Omega)$ be a nonnegative weak subsolution of (1.4) in a bounded smooth domain Ω of $\mathbb{R}^N$ with $m > 2$ and f satisfying (*). Suppose that $\overline{B(x, 5\delta)} \subset \Omega$. Then there exists $C > 0$ such that*

$$\sup_{B(x,\delta)} v \leqslant C\|v\|_{L^p(B(x,2\delta))} \tag{3.2}$$

for any $p > 1$, and C is a constant depending on x, s, N, u, m, f.
If $\frac{2N+2}{N+2} < m < 2$ the same result holds for any $p > \frac{s^\sharp}{2}$ where $\frac{2}{s^\sharp} \equiv 1 - \frac{1}{s}$ and $s < \frac{m-1}{2-m}$.

If v is a solution of (1.4), then we can use Theorem 3.1 and Theorem 3.2 together, obtaining the following

Corollary 3.1 *Let $v \in H^{1,2}_\rho(\Omega) \cap L^\infty(\Omega)$ be a nonnegative weak solution of (1.4) in a bounded smooth domain Ω of $\mathbb{R}^N$. Suppose that $\overline{B(x, 5\delta)} \subset \Omega$. Then there exists $C > 0$, depending on x, N, u, m, f, such that*

$$\sup_{B(x,\delta)} v \leqslant C \inf_{B(x,2\delta)} v \tag{3.3}$$

Remark 3.1 To prove the Harnack inequality for the solutions of the linearized operator we exploit the technique introduced by N.S. Trudinger [31], which is based on Sobolev weighted inequality.
When dealing with the linearized operator L_u of (1.1) at u, the natural weight is $\rho \equiv |Du|^{m-2}$ and we use only the summability properties of $\frac{1}{|Du|}$ that we have proved in [11], where the fact that u is a solution of (1.1) is crucial.
However, Sobolev and Poincarè weighted inequalities are proved in [11] also for an abstract weight ρ on which we make the following a priori assumption:

$$\int_\Omega \frac{1}{\rho^t} \frac{1}{|x-y|^\gamma}\, dy \leqslant C \tag{3.4}$$

where C does not depend on $x \in \Omega$, $\gamma < N$, $t > \frac{N-\gamma}{p}$ and $p > 1 + \frac{1}{t}$.
Therefore we could also consider a more general class of linear degenerate elliptic operators with ellipticity coefficient $\rho(x)$ as in [31], using (3.4) as an abstract a priori assumption on $\rho(x)$.
Since our main purpose here is to use the Harnack inequality together with symmetry and monotonicity results to prove remarkable properties of the critical set Z of any solution u of (1.1), we limit ourselves to consider Eq. (1.4) using the summability properties of $\frac{1}{|Du|}$ that we proved in [11].

The techniques used to get the weak Harnack inequality for solutions of the linearized operator can also be used to prove the following weak Harnack comparison inequality for the difference of two solutions of (1.1) (see the proof in the appendix).

Theorem 3.3 *Let $u, v \in C^1(\overline{\Omega})$ where Ω is a bounded smooth domain of $\mathbb{R}^N$, $N \geqslant 2$ and $m > 2$. Suppose that either u or v is a weak solution of (1.1) with f satisfying $(*)$. Assume that*

$$-\Delta_m(u) + \Lambda u \leqslant -\Delta_m(v) + \Lambda v \qquad u \leqslant v \quad \text{in} \quad B(x, 5\delta) \tag{3.5}$$

where $\Lambda \in \mathbb{R}$. Suppose that $\overline{B(x, 5\delta)} \subset \Omega$ and define $\overline{2}^*$ by

$$\frac{1}{\overline{2}^*} = \frac{1}{2} - \frac{1}{N} + \frac{1}{N}\left(\frac{m-2}{m-1}\right)$$

Then for every $0 < s < \chi$, with $\chi \equiv \frac{2^*}{2}$ and 2^* is any real number such that $2 < 2^* < \overline{2}^*$($\chi < \frac{\overline{2}^*}{2}$), there exists $C > 0$ such that

$$\|(v-u)\|_{L^s(B(x,2\delta))} \leqslant C \inf_{B(x,\delta)} (v-u) \tag{3.6}$$

where C is a constant depending on $x, s, N, u, v, m, \Lambda$.
If $\frac{2N+2}{N+2} < m < 2$ the same result holds with χ replaced by $\chi' \equiv \frac{2^\sharp}{s^\sharp}$ where $2^\sharp$ is the classical Sobolev exponent, $\frac{2}{s^\sharp} \equiv 1 - \frac{1}{s}$ and $s < \frac{m-1}{2-m}$.

Remark 3.2 Note that a function $f : I \longrightarrow \mathbb{R}$ is locally Lipschitz continuous in the interval I if and only if, for each compact subinterval $[a, b] \subset I$, there exist two positive costants C_1 and C_2 such that

i) $f_1(s) = f(s) - C_1 s$ is nonincreasing in $[a, b]$.
ii) $f_2(s) = f(s) + C_2 s$ is nondecreasing in $[a, b]$.

Therefore we get that, if

$$-\Delta_m(u) - f(u) \leqslant -\Delta_m(v) - f(v) \qquad u \leqslant v \quad \text{in} \quad B(x, 5\delta) \tag{3.7}$$

then

$$-\Delta_m(u) + \Lambda u \leqslant -\Delta_m(v) + \Lambda v \qquad u \leqslant v \quad \text{in} \quad B(x, 5\delta) \tag{3.8}$$

for $\Lambda \in \mathbb{R}$ sufficiently large, and the previous result applies also in this case.

Arguing exactly as for the case of the linearized operator, we can also prove the following

Corollary 3.2 *Let $u, v \in C^1(\overline{\Omega})$ be weak solutions of (1.1) in a bounded smooth domain Ω of $\mathbb{R}^N$, $N \geqslant 2$ with f positive satisfying $(*)$. Suppose that $\overline{B(x, 5\delta)} \subset \Omega$ and $u \leqslant v$ in $B(x, 5\delta)$. Then there exists $C > 0$, depending on x, N, m, u, v, f, such that*

$$\sup_{B(x,\delta)} (v-u) \leqslant C \inf_{B(x,2\delta)} (v-u) \tag{3.9}$$

We prove now some remarkable consequences of weak Harnack inequalities which give information about the critical set Z of solutions of (1.1). This is particularly interesting since Z is also the set of points where the operator is degenerate elliptic. Let us start with a general result:

Theorem 3.4 (Strong Maximum Principle) *Let $v \in H^{1,2}_\rho(\Omega) \cap C^0(\overline{\Omega})$ be weak supersolution of (1.4) in a bounded smooth domain Ω of $\mathbb{R}^N$, $N \geqslant 2$ with $\frac{2N+2}{N+2} < m < 2$ or $m > 2$ with f satisfying $(*)$. Then, for any domain $\Omega' \subset \Omega$ with $v \geqslant 0$ in Ω', we have $v \equiv 0$ in Ω' or $v > 0$ in Ω'.*

Proof Let us define $K_v = \{x \in \Omega' \mid v(x) = 0\}$. By the continuity of v, then K_v is closed in Ω'. Moreover by Theorem 3.1 K_v is also open in Ω' and the thesis follows. □

Since by [11] u_{x_i} weakly solves (1.4) then Theorem 1.2 follows immediately by Theorem 3.4. Moreover we can prove the following

Theorem 3.5 *Let Ω be a bounded smooth domain in $\mathbb{R}^N$, $N \geqslant 2$, $\frac{2N+2}{N+2} < m <$ or $m > 2$, $f : [0, \infty) \to \mathbb{R}$ a continuous function which is strictly positive and locally Lipschitz continuous in $(0, \infty)$ (i.e. f satisfies $(*)$), and $u \in C^1(\overline{\Omega})$ a weak solution of (1.1).*
For any direction ν and for λ in the interval $(a(\nu), \lambda_1(\nu)]$ we have

$$u(x) \leqslant u(x^\nu_\lambda) \quad \forall x \in \Omega^\nu_\lambda. \tag{3.10}$$

Moreover, for any λ with $a(\nu) < \lambda < \lambda_1(\nu)$ we have

$$\frac{\partial u}{\partial \nu}(x) > 0 \quad \forall x \in \Omega^\nu_{\lambda_1(\nu)} \tag{3.11}$$

If f is locally Lipschitz continuous in the closed interval $[0, \infty)$ then (3.10) holds for any λ in the interval $(a(\nu), \lambda_2(\nu))$ and (3.11) holds for any $x \in \Omega^\nu_{\lambda_2(\nu)}$.

Remark 3.3 *Previously (3.11) was proved only for $x \in \Omega^\nu_{\lambda_1(\nu)} \setminus Z$ (see Theorem 2.3)*

Proof Let us consider for simplicity the case $\nu \equiv e_i$ and $\frac{\partial u}{\partial \nu} \equiv u_{x_i}$. The general case may be easily treated using the fact that $\frac{\partial u}{\partial \nu}$ is a linear combination of u_{x_i} $i = 1, \dots, N$ and the linearized operator is linear. Equation (3.10) follows by Theorem 2.3. To prove (3.11) let us define $K^{e_i}_{\lambda_1(e_i)} \equiv \{x \in \Omega^{e_i}_{\lambda_1(e_i)} \mid u_{x_i}(x) = 0\}$. Since, as proved in [11], u_{x_i} is a weak solution of (1.4), and by Theorem 2.3 u_{x_i} is nonnegative in $\Omega^{e_i}_{\lambda_1(e_i)}$, then Theorem 3.1 applies and shows that $K^{e_i}_{\lambda_1(e_i)} \cap C^{e_i}_{\lambda_1(e_i)}$ is open in $C^{e_i}_{\lambda_1(e_i)}$ for every connected component $C^{e_i}_{\lambda_1(e_i)}$ of $\Omega^{e_i}_{\lambda_1(e_i)}$ (here we use the fact that u_{x_i} is continuous and therefore $\inf u_{x_i} \equiv \min u_{x_i}$). Since by continuity $K^{e_i}_{\lambda_1(e_i)} \cap C^{e_i}_{\lambda_1(e_i)}$ is also closed in $C^{e_i}_{\lambda_1(e_i)}$, it follows that

$$K^{e_i}_{\lambda_1(e_i)} \cap C^{e_i}_{\lambda_1(e_i)} = C^{e_i}_{\lambda_1(e_i)} \tag{3.12}$$

or

$$K^{e_i}_{\lambda_1(e_i)} \cap C^{e_i}_{\lambda_1(e_i)} = \emptyset \tag{3.13}$$

The thesis follows now by observing that 3.12 would imply that u is constant in the ν direction in $C^{e_i}_{\lambda_1(e_i)}$ which is clearly impossible (note that $C^{e_i}_{\lambda_1(e_i)} \cap \partial\Omega \neq \emptyset$, $u = 0$ on $\partial\Omega$ and $u > 0$ in Ω). □

Corollary 3.3 *If $f : [0, \infty) \to \mathbb{R}$ is a continuous function which is strictly positive and locally Lipschitz continuous in $(0, \infty)$ (i.e. f satisfies (*)), the domain Ω is symmetric with respect to the hyperplane $T_0^\nu = \{x \in \mathbb{R}^N : x \cdot \nu = 0\}$ and strictly convex in the ν direction ($\lambda_1(\nu) = \lambda_1(-\nu) = 0$), then u is symmetric, i. e. $u(x) = u(x_0^\nu)$, and strictly increasing in the ν–direction in Ω_0^ν with $\frac{\partial u}{\partial \nu}(x) > 0$ in Ω_0^ν. In particular the only points where the gradient of u vanishes belong to T_0^ν. Therefore if for N orthogonal directions e_i the domain Ω is symmetric with respect to any hyperplane $T_0^{e_i}$ and $\lambda_1(e_i) = \lambda_1(-e_i) = 0$, then*

$$Z \equiv \{x \in \Omega \mid D(u)(x) = 0\} = \{0\} \tag{3.14}$$

assuming that 0 is the center of symmetry.

If f is locally Lipschitz continuous in the closed interval $[0, +\infty)$ then the same result follows assuming only that the domain Ω is convex and symmetric ($\lambda_2(e_i) = \lambda_2(-e_i) = 0$) with respect to N orthogonal directions e_i.

We now bring to the attention an elementary consequence of our results, which may be of remarkable utility:

Theorem 3.6 *Let Ω be a bounded smooth domain in $\mathbb{R}^N$, $N \geqslant 2$, $\frac{2N+2}{N+2} < m < 2$ or $m > 2$, $f : [0, \infty) \to \mathbb{R}$ a continuous function which is strictly positive and locally Lipschitz continuous in $(0, \infty)$ (i.e. f satisfies (*)), and $u \in C^1(\overline{\Omega})$ a weak solution of (1.1). Then if for N orthogonal directions e_i the domain Ω is symmetric with respect to any hyperplane $T_0^{e_i}$ and $\lambda_1(e_i) = \lambda_1(-e_i) = 0$, it follows $u \in C^2(\Omega \setminus \{0\})$ assuming that 0 is the center of symmetry.*
If f is locally Lipschitz continuous in the closed interval $[0, +\infty)$ then the same result follows assuming only that the domain Ω is convex and symmetric ($\lambda_2(e_i) = \lambda_2(-e_i) = 0$) with respect to N orthogonal directions e_i.

Proof We only have to note that, by Corollary 3.3 it follows $Z \equiv \{0\}$, therefore the m-Laplace operator is not degenerate in $\Omega \setminus \{0\}$ and we can apply standard regularity results for non degenerate operators (see e.g. [15, 19]) □

As a consequence of Theorem 3.3 we also prove the following

Theorem 3.7 (Strong Comparison Principle) *Let $u, v \in C^1(\overline{\Omega})$ where Ω is a bounded smooth domain of $\mathbb{R}^N$, $N \geqslant 2$ with $\frac{2N+2}{N+2} < m < 2$ or $m > 2$. Suppose that either u or v is a weak solution of (1.1) with f positive satisfying (∗). Assume*

$$-\Delta_m(u) + \Lambda u \leqslant -\Delta_m(v) + \Lambda v \qquad u \leqslant v \quad in \quad \Omega \tag{3.15}$$

where $\Lambda \in \mathbb{R}$. Then $u \equiv v$ in Ω unless

$$u < v \qquad in \qquad \Omega \tag{3.16}$$

The same result holds (see Remark 3.2) if u and v are weak solutions of (1.1) or more generally if

$$-\Delta_m(u) - f(u) \leqslant -\Delta_m(v) - f(v) \qquad u \leqslant v \quad in \quad \Omega \tag{3.17}$$

with u or v weakly solving (1.1).

Proof Let us define

$$K_{uv} = \{x \in \Omega \mid u(x) = v(x)\} \tag{3.18}$$

By the continuity of u and v we have that K_{uv} is closed in Ω. Since, by Theorem 3.3, for any $x \in K_{uv}$ there exists a ball $B(x)$ centered in x all contained in K_{uv}, then K_{uv} is also open in Ω and the thesis follows. □

Appendix A

In this appendix we prove Theorem 3.1 and Theorem 3.3. We use Moser's iterative technique [20] as improved by Trudinger in [31] where the case of degenerate elliptic operators in considered. The main novelty here is that we study the linearized operator of (1.1) at a fixed solution u and we do not assume any a priori assumptions on the weight $\rho \equiv |Du|^{m-2}$ using only summability properties of $\rho \equiv |Du|^{m-2}$ proved by the authors in [11].

Proof of Theorem 3.1 **Case 1** ($m > 2$):
Let $v \in H^{1,2}_{\rho}(\Omega) \cap L^{\infty}(\Omega)$ be a nonnegative weak supersolution of (1.4), i.e.

$$L_u(v, \varphi) \geqslant 0 \tag{A.1}$$

where

$$L_u(v, \varphi) \equiv$$
$$\int_\Omega [|Du|^{m-2}(Dv, D\varphi) + (m-2)|Du|^{m-4}(Du, Dv)(Du, D\varphi) - f'(u)v\varphi]dx$$

Let us first note that we can suppose $v \geqslant \tau > 0$ for some $\tau \in \mathbb{R}$. In fact, if this is not true, we can consider $v + \tau$ and then let $\tau \to 0$. Under this assumption we define

$$\phi \equiv \eta^2 v^\beta \qquad \beta < 0 \tag{A.2}$$

with $\eta \in C^1_0(B(x, 5\delta))$ and $\eta \geqslant 0$ in Ω. Later we will make some more assumptions on the cut-off function η. We have

$$D\phi = 2\eta v^\beta D\eta + \beta\eta^2 v^{\beta-1} Dv \tag{A.3}$$

and, since $\phi \in H^{1,2}_{0,\rho}(\Omega)$, by density arguments, we can use it as a test function in (A.1) getting

$$\begin{aligned}
&\int_\Omega [\beta\rho|Dv|^2\eta^2 v^{\beta-1} + \beta(m-2)|Du|^{m-4}(Du, Dv)^2\eta^2 v^{\beta-1}]\,dx + \\
&+ \int_\Omega [2\eta v^\beta \rho(Dv, D\eta) + 2\eta(m-2)v^\beta |Du|^{m-4}(Du, D\eta)(Du, Dv)]\,dx \geqslant \\
&\geqslant \int_\Omega f'(u)\eta^2 v^{\beta+1}\,dx
\end{aligned} \tag{A.4}$$

where $\rho \equiv |Du|^{m-2}$.
For $\beta < 0$, since the term $\beta(m-2)|Du|^{m-4}(Du, Dv)^2\eta^2 v^{\beta-1}$ is negative for $m > 2$, we get

$$\begin{aligned}
|\beta| \int_\Omega \rho|Dv|^2\eta^2 v^{\beta-1} \leqslant & 2(m-1)\int_\Omega \rho\eta v^\beta |Dv||D\eta|\,dx + \\
& + C_1 \int_\Omega |f'(u)|\eta^2 v^{\beta+1}\,dx
\end{aligned} \tag{A.5}$$

Let us remark for future use that if $1 < m < 2$ we can use the fact that $\beta(m-2) > 0$ and therefore $(m-1)\int_\Omega [\beta\rho|Dv|^2\eta^2 v^{\beta-1} \geqslant \int_\Omega [\beta\rho|Dv|^2\eta^2 v^{\beta-1}$ $+\beta(m-2)|Du|^{m-4}(Du, Dv)^2\eta^2 v^{\beta-1}]\,dx$.

Applying Young's inequality ($ab \leqslant |\beta| a^2 \backslash 2 + 2b^2 \backslash |\beta|$) we obtain

$$\begin{aligned} \frac{|\beta|}{2} \int_\Omega \rho |Dv|^2 \eta^2 v^{\beta-1} \leqslant & C_2 \frac{1}{|\beta|} \int_\Omega \rho v^{\beta+1} |D\eta|^2 \, dx + \\ & + C_1 \int_\Omega |f'(u)| \eta^2 v^{\beta+1} \, dx \end{aligned} \tag{A.6}$$

Since, by the regularity of f and u, we know that $\sup_{x \in B(x,5\delta)} |f'(u)| < \infty$, we find a constant $C > 0$ such that

$$\int_\Omega \rho |Dv|^2 \eta^2 v^{\beta-1} \, dx \leqslant \frac{C}{|\beta|} \left(1 + \frac{1}{|\beta|}\right) \int_\Omega v^{\beta+1} [\eta^2 + \rho |D\eta|^2] \, dx \tag{A.7}$$

Let us now define

$$w \equiv \begin{cases} v^{\frac{\beta+1}{2}} & \text{if} \quad \beta \neq -1 \\ \log(v) & \text{if} \quad \beta = -1 \end{cases} \tag{A.8}$$

and set

$$r \equiv \beta + 1$$

With these definitions we can write (A.7) as follows

$$\int_\Omega \rho \eta^2 |Dw|^2 \, dx \leqslant \begin{cases} C \left(1 + \frac{1}{|\beta|}\right)^2 \int_\Omega w^2 [\eta^2 + \rho |D\eta|^2] \, dx & \beta \neq -1 \\ C_0 \int_\Omega [\eta^2 + \rho |D\eta|^2] \, dx & \beta = -1 \end{cases} \tag{A.9}$$

Now, if $m > 2$, then ρ is bounded in $\overline{\Omega}$ and by Theorem 2.2 [11], since $\overline{2}^* > 2$, then for every $2 < 2^* < \overline{2}^*$ a Sobolev weighted inequality holds. Therefore, for $\beta \neq -1$,

$$\|\eta w\|^2_{L^{2^*}(\Omega)} \leqslant C \int_\Omega \rho [\eta^2 |Dw|^2 + w^2 |D\eta|^2] \, dx \tag{A.10}$$

Therefore, by (A.9), we get

$$\begin{aligned} \|\eta w\|^2_{L^{2^*}(\Omega)} & \leqslant C r^2 \int_\Omega w^2 (\eta^2 + |D\eta|^2) \, dx \\ & \leqslant C r^2 \|w(\eta + |D\eta|)\|^2_{L^2(\Omega)} \end{aligned} \tag{A.11}$$

where the constant C depends on β but is bounded if $|\beta| \geqslant c > 0$.
From now on we will make some more specifications on η. Let $\delta \leqslant h' < h'' \leqslant 5\delta$ and suppose $\eta \equiv 1$ in $B(x, h')$ while $\eta \equiv 0$ outside $B(x, h'')$. Moreover suppose $|D\eta| \leqslant \frac{2}{h''-h'}$. With these assumptions, by (A.11), it follows

$$\|w\|_{L^{2^*}(B(x,h'))} \leqslant \frac{C|r|}{h'' - h'} \|w\|_{L^2(B(x,h''))} \tag{A.12}$$

We define now $\chi \equiv \frac{2^*}{2}$ (note that $\chi > 1$ since $2^* > 2$). By the definition of w, if $0 < r < 1$ ($-1 < \beta < 0$), taking the $\frac{2}{r}$ power, we have

$$\begin{aligned} \left[\int_{B(x,h')} \left(v^{\frac{\beta+1}{2}}\right)^{2^*} dx\right]^{\frac{1}{2^*}\frac{2}{r}} = & \left[\int_{B(x,h')} v^{\chi r} \, dx\right]^{\frac{1}{\chi r}} \leqslant \\ \leqslant \frac{(Cr)^{\frac{2}{r}}}{(h'' - h')^{\frac{2}{r}}} & \left[\int_{B(x,h'')} v^r \, dx\right]^{\frac{1}{r}} \end{aligned} \tag{A.13}$$

i.e.

$$\phi(\chi r, h', v) \leqslant \frac{(C|r|)^{\frac{2}{r}}}{(h'' - h')^{\frac{2}{r}}} \phi(r, h'', v) \tag{A.14}$$

where ϕ is defined as in Sect. 2.

If instead $r < 0$ ($\beta < -1$) we have

$$\phi(\chi r, h', v) \geqslant \frac{(C|r|)^{\frac{2}{r}}}{(h''-h')^{\frac{2}{r}}}\phi(r, h'', v) \tag{A.15}$$

Let us now exploits Moser's iterative technique [20] (see also [29]). For $r_0 > 0$ given, we define $r_k = (-r_0)\chi^k$ and $h_k = \delta[1 + \frac{3}{2}(\frac{1}{2}^k)]$. In this case it follows that $r_k \to -\infty$, and $\beta_k = r_k - 1 \to -\infty$. Moreover $h_0 = \frac{5\delta}{2}$, $h_k \to \delta$ and $h_k - h_{k+1} = \frac{3\delta}{2}\frac{1}{2^{k+1}}$. Using these definitions we can iterate (A.15) as follows

$$\begin{aligned} &\phi(r_{k+1}, h_{k+1}, v) \geqslant \\ &\geqslant \left(C_7^{-\frac{2}{r_0}}\right)^{\frac{1}{\chi^k}} \left[(|r_0|\chi^k)^{-\frac{2}{r_0}}\right]^{\frac{1}{\chi^k}} \left[\left(\frac{3\delta}{2}\frac{1}{2^{k+1}}\right)^{\frac{2}{r_0}}\right]^{\frac{1}{\chi^k}} \phi(r_k, h_k, v) \\ &\geqslant C_8^{\sum_{k\geqslant 0}\frac{1}{\chi^k}}(2\chi)^{\sum_{k\geqslant 0}\frac{k}{\chi^k}}(\delta^{\frac{2}{r_0}})^{\sum_{0\leqslant j\leqslant k}\frac{1}{\chi^j}}\phi\left(-r_0, \frac{5\delta}{2}, v\right) \end{aligned} \tag{A.16}$$

Since by definition $\chi > 1$, the series converge and we find a constant $C > 0$ such that

$$\phi(-\infty, \delta, v) \geqslant C\phi\left(-r_0, \frac{5\delta}{2}, v\right) \tag{A.17}$$

We will now suppose (we will prove it later) that there exist $r_0 > 0$ and a constant $C > 0$ such that

$$\phi\left(r_0, \frac{5\delta}{2}, v\right) \leqslant C\phi\left(-r_0, \frac{5\delta}{2}, v\right) \tag{A.18}$$

Since for $0 < s \leqslant r_0$

$$\phi(s, 2\delta, v) \leqslant \phi\left(s, \frac{5\delta}{2}, v\right) \leqslant \phi\left(r_0, \frac{5\delta}{2}, v\right) \tag{A.19}$$

by (A.17) and by (A.18) we get

$$\phi(-\infty, \delta, v) \geqslant C\phi(s, 2\delta, v) \tag{A.20}$$

Therefore, since $\phi(-\infty, \delta, v) = \inf_{B(x,\delta)} v$, (3.1) follows for $0 < s \leqslant r_0$.
If instead $r_0 < s < \chi$, we take a finite number of iterations of (A.14), and then we reduce once again to the arguments already developed. More precisely, in this case, $r_1 = \frac{s}{\chi^{k_0+1}} \leqslant r_0$ for a natural number k_0 sufficiently large. Consider, for $k = 0, \ldots, k_0 + 1$, the values $r_k = r_1\chi^k$ and $h_0 = \frac{5\delta}{2} > h_1 > \ldots > h_{k_0+1} = 2\delta$. With these assumptions we can use (A.14) and, after k_0 iterations, we obtain

$$\phi(s, 2\delta, v) \leqslant C\phi\left(r_1, \frac{5\delta}{2}, v\right) \tag{A.21}$$

Since $r_1 \leqslant r_0$, then (A.18) is certainly true with r_0 replaced by r_1. Moreover also (A.17) can be deduced with r_0 replaced by r_1. Using now (A.21) together with (A.17), we prove the theorem also for $r_0 < s < \chi$. Note that, to iterate (A.14) (considering for example the first step), we need to set $r_{k_0+1} = r'\chi$ with $r' < 1$ ($\beta' < 1$). Therefore we need $\frac{s}{\chi} < 1$ and the condition $s < \chi$ is necessary (this condition is also sufficient for the other steps).

The second part of the proof is devoted to show that there exists $r_0 > 0$ for which (A.18) holds. We follow closely the technique introduced by N.S. Trudinger in [31] and define

$$w \equiv \log(v) \tag{A.22}$$

Using (A.9) with $\eta = 1$ in $B(x, 5\delta)$, we can suppose

$$\int_{B(x,5\delta)} \rho |Dw|^2\, dx \leqslant C \tag{A.23}$$

where C does not depend on w (we could also suppose $C \leqslant C'\delta^{N-2}$). Replacing v with $\frac{v}{k}$ with k defined by $k = e^{\frac{1}{|B(x,5\delta)|}\int_{B(x,5\delta)} \log(v)\, dx}$, we can also suppose that w has zero mean on $B(x, 5\delta)$. Therefore, by weighted Sobolev inequality, available for functions with zero mean (see Remark 2.1), we get

$$\|w\|_{L^{2^*}(B(x,5\delta))} \leqslant C \tag{A.24}$$

where C is a constant not depending on w.
Note that, when $\tau \to 0$, the fact that (A.24) does not depend on w is crucial and guarantees that the constants do not blow up. Moreover the constant k that we introduce does not modificate the following calculations and can be cancelled in the conclusive inequality.

Let us now set

$$\Phi = \eta^2 \frac{1}{v}(|w|^\beta + (2\beta)^\beta) \qquad \beta \geqslant 1 \tag{A.25}$$

with $\eta \geqslant 0$ and $\eta \in C_0^1(B(x, 5\delta))$. Therefore

$$D\Phi = 2\frac{\eta}{v}(|w|^\beta + (2\beta)^\beta)D\eta + \eta^2 \frac{1}{v^2}(\beta sign(w)|w|^{\beta-1} - |w|^\beta - (2\beta)^\beta)Dv \tag{A.26}$$

and, by (A.1), we get

$$\begin{aligned}
&\int_\Omega \rho \frac{\eta^2}{v^2}(\beta sign(w)|w|^{\beta-1} - |w|^\beta - (2\beta)^\beta)|Dv|^2\, dx \\
&+ (m-2)\int_\Omega |Du|^{m-4}\frac{\eta^2}{v^2}(\beta sign(w)|w|^{\beta-1} - |w|^\beta - (2\beta)^\beta)(Du, Dv)^2 dx \\
&+ \int_\Omega 2\rho\eta\frac{1}{v}(|w|^\beta + (2\beta)^\beta)(Dv, D\eta)\, dx \\
&+ 2(m-2)\int_\Omega |Du|^{m-4}\eta\frac{1}{v}(|w|^\beta + (2\beta)^\beta)(Du, Dv)(Du, D\eta)\, dx \\
&\geqslant \int_\Omega f'(u)\eta^2(|w|^\beta + (2\beta)^\beta)\, dx
\end{aligned} \tag{A.27}$$

In the following, if $\beta \geqslant 1$, we will make repeated use of the following simple inequality

$$2\beta|w|^{\beta-1} \leqslant \frac{\beta-1}{\beta}|w|^\beta + \frac{1}{\beta}(2\beta)^\beta \leqslant |w|^\beta + (2\beta)^\beta \tag{A.28}$$

or

$$-\beta sign(w)|w|^{\beta-1} + |w|^\beta + (2\beta)^\beta \geqslant \beta|w|^{\beta-1} \tag{A.29}$$

Using (A.28) and (A.29) and estimating the above integrals, we get

$$\begin{aligned}
\beta\int_\Omega \rho\eta^2|w|^{\beta-1}|Dw|^2\, dx \leqslant &\int_\Omega \rho\frac{\eta}{v}(|w|^\beta + (2\beta)^\beta)|D\eta||Dv|\, dx \\
&+ \int_\Omega |f'(u)|\eta^2(|w|^\beta + (2\beta)^\beta)\, dx
\end{aligned} \tag{A.30}$$

Note that, since in this case $\beta \geqslant 1$, the constants are bounded independently from β. Moreover, since $m > 2$, ρ is bounded in $\overline{\Omega}$ and $|f'(u)|$ is bounded in $B(x, 5\delta)$ by the assumptions on the nonlinearity f. Applying Young's inequality we therefore obtain

$$\begin{aligned}\int_\Omega \rho\eta^2|w|^{\beta-1}|Dw|^2\,dx \leqslant \frac{C_1}{4\sigma}\int_\Omega \rho|w|^{\beta+1}|D\eta|^2\,dx\\ + \frac{C_1}{2}\int_\Omega \rho(2\beta)^\beta|D\eta|^2\,dx + \frac{C_1}{2}\int_\Omega \rho\frac{\eta^2}{v^2}(2\beta)^\beta|Dv|^2\,dx\\ + C_2\int_\Omega \eta^2(|w|^\beta+(2\beta)^\beta)\,dx\end{aligned} \tag{A.31}$$

i.e.

$$\begin{aligned}\int_\Omega \rho\eta^2|w|^{\beta-1}|Dw|^2\,dx \leqslant C_3\int_\Omega \rho(|w|^{\beta+1}+(2\beta)^\beta|)D\eta|^2\,dx\\ + C_4(2\beta)^\beta + C_2\int_\Omega \eta^2(|w|^\beta+(2\beta)^\beta)\,dx\\ \leqslant C_3\int_\Omega \rho(|w|^{\beta+1}+(2\beta)^\beta)|D\eta|^2\,dx + C_5\int_\Omega \eta^2(|w|^\beta+(2\beta)^\beta)\,dx\end{aligned} \tag{A.32}$$

where, including the term $C_4(2\beta)^\beta$ in the constant C_5, since η has a support which depends on δ, therefore the constant C_5 will also depend on δ. Since by (A.28) we get $\int_\Omega \eta^2(|w|^\beta + (2\beta)^\beta)\,dx \leqslant c\int_\Omega \eta^2(|w|^{\beta+1}+(2\beta)^\beta)\,dx$, we obtain

$$\int_\Omega \rho\eta^2|w|^{\beta-1}|Dw|^2\,dx \leqslant C\int_\Omega (|w|^{\beta+1}+(2\beta)^\beta)[\eta^2+|D\eta|^2]\,dx \tag{A.33}$$

where the constant C depends on δ. Equation (A.33) resembles (A.7), except for the extra term $(2\beta)^\beta$. Applying the iterative technique as above we get, for $\beta \geqslant 1$, the following inequality

$$\phi(\chi r, h', w) \leqslant \left(\frac{Cr}{h''-h'}\right)^{\frac{2}{r}}\left[\phi(r,h'',w)+\gamma r\right] \tag{A.34}$$

where γ is a constant.

We claim now that there exists a constant C such that

$$\phi\left(p,\frac{5\delta}{2},w\right) \leqslant C(\phi(2^*,5\delta,w)+p) \qquad \forall p \geqslant 2^* \tag{A.35}$$

To prove this let us choose $h_k \equiv \frac{5\delta}{2}[1+\frac{1}{2^k}]$. Moreover let us consider $\chi^k 2^* \equiv \chi r \equiv \chi(\beta+1)$ (which is always solvable for $\beta > 1$) and put it in (A.34) obtaining

$$\phi(\chi^k 2^*, h_k, w) \leqslant \left(\frac{2C\chi^k}{h_{k-1}-h_k}\right)^{\frac{1}{\chi^k}}(\phi(\chi^{k-1}2^*, h_{k-1}, w)+\gamma\chi^{k-1}2^*) \tag{A.36}$$

iterating we get

$$\begin{aligned}\phi(\chi^k 2^*, h_k, w) \leqslant (2C)^{\sum_{k=1}^{+\infty}\frac{1}{\chi^k}}\frac{\prod_{k=0}(\chi^k)^{\frac{1}{\chi^k}}}{\prod_{k=0}(h_{k-1}-h_k)^{\frac{1}{\chi^k}}}\phi(2^*,h_0,w)\\ + \sum_{h=0}^{k-1}\gamma\chi^{h-k}2^*\left[\prod_{\tau=h}^{k+1}\left(\frac{2C\chi^\tau}{h_\tau-h_{\tau+1}}\right)^{\frac{1}{\chi^\tau}}\right]\chi^k 2^*\end{aligned} \tag{A.37}$$

Estimating the products above (using e.g. the logarithm function), we obtain $C > 0$ such that

$$\phi(\chi^k 2^*, h_k, w) \leqslant C(\phi(2^*,5\delta,w)+\gamma\chi^k 2^*) \tag{A.38}$$

where C does not depend on k and we have used the fact that $h_0 \equiv 5\delta$. Setting $k_p \equiv \inf_{h\in\mathbb{N}}\{h|\chi^k 2^* \geqslant p\}$, we have

$$\begin{aligned}\phi\left(p, \frac{5\delta}{2}, w\right) &\leqslant C_1\phi(\chi^{k_p}2^*, h_{k_p}, w) \leqslant C_2[\phi(2^*, 5\delta, w) + \chi^{k_p}2^*] \\ &\leqslant C_3[\phi(2^*, 5\delta, w) + \chi p] \leqslant C[\phi(2^*, 5\delta, w) + p]\end{aligned} \tag{A.39}$$

proving (A.35).

We will now apply (A.35) to prove that there exists $r_0 > 0$ for which (A.18) holds. For $r_0 > 0$ given, considering the power series expansion of $e^{r_0|w|}$, we get

$$\begin{aligned}\int_{B\left(x,\frac{5\delta}{2}\right)} e^{r_0|w|}\,dx &\leqslant \sum_{k=0} \int_{B\left(x,\frac{5\delta}{2}\right)} \frac{(r_0|w|)^k}{k!}\,dx \leqslant \sum_{k=0} \frac{\left(r_0\phi(k, \frac{5\delta}{2}, w)\right)^k}{k!} \\ &\leqslant \sum_{k=0} \frac{(Cr_0)^k(\phi(2^*, 5\delta, w)^k + k^k)}{k!}\end{aligned} \tag{A.40}$$

Using the ratio test, we can easily prove that, if $r_0 > 0$ is sufficiently small, then the last series is convergent. Therefore, for such values of r_0 it follows that

$$\int_{B\left(x,\frac{5\delta}{2}\right)} e^{r_0|w|}\,dx \leqslant C \tag{A.41}$$

Therefore

$$\int_{B\left(x,\frac{5\delta}{2}\right)} e^{r_0 w}\,dx \int_{B\left(x,\frac{5\delta}{2}\right)} e^{-r_0 w}\,dx \leqslant \left[\int_{B\left(x,\frac{5\delta}{2}\right)} e^{r_0|w|}\,dx\right]^2 \leqslant C^2 \tag{A.42}$$

Taking the $\frac{1}{r_0}$ power of (A.42), recalling that $w \equiv \log(v)$ and $\phi(2^*, 5\delta, w)$ is bounded above by a constant which does not depend on w (and does not blow-up if $\tau \to 0$ by (A.24)), we prove that for this choice of r_0, (A.18) holds.

Case 2 $\left(\frac{2N+2}{N+2} < m < 2\right)$:
As already observed (A.5) holds in this case and proceeding as before we get (A.9). Moreover, since $u \in C^1(\overline{\Omega})$, then $\rho \equiv |Du|^{m-2} \geqslant \lambda > 0$ in this case, so that, by classic Sobolev inequality, we get

$$\|\eta w\|^2_{L^{2^\sharp}(\Omega)} \leqslant C\int_\Omega |D(\eta w)|^2\,dx \leqslant C_0\int_\Omega \rho|D(\eta w)|^2\,dx \tag{A.43}$$

Therefore, arguing as above, we get

$$\|\eta w\|^2_{L^{2^\sharp}(\Omega)} \leqslant Cr^2\int_\Omega w^2(\eta^2 + \rho|D\eta|^2)\,dx \tag{A.44}$$

Now, if $\rho \in L^s(\Omega)$, applying Holder's inequality with exponents s and s', we deduce

$$\|\eta w\|^2_{L^{2^\sharp}(\Omega)} \leqslant cr^2\|\rho\|_{L^s(\Omega)}\|w(\eta + |D\eta|)\|^2_{L^{s^\sharp}(\Omega)}\,dx \tag{A.45}$$

where $\frac{2}{s^\sharp} \equiv 1 - \frac{1}{s}$. Now, setting $\chi' \equiv \frac{2^\sharp}{s^\sharp}$, in order to run again the above arguments, we only need $\chi' > 1$ (in this case we will define $r \equiv (\beta+1)\frac{2^\sharp}{2}$). This condition is obviously satisfied if $s > \frac{N}{2}$. By Theorem 2.1 (see also [11]), if $m < 2$, we know that $\rho \in L^{(\frac{m-1}{2-m})\theta}(\Omega)$ for every $0 < \theta < 1$. Therefore, if $\frac{2N+2}{N+2} < m < 2$, then $\rho \in L^s(\Omega)$ with $s > \frac{N}{2}$ and the thesis follows. □

Theorem 3.3 can be proved using the same techniques and standard estimates for the m-Laplace operator. For the readers convenience we give some details:

Proof of Theorem 3.3 The proof is very similar the proof of Theorem 3.1 following in the same way Moser's iteration technique, as improved by Trudinger [31].
We consider only the case $m > 2$. The case $\frac{2N+2}{N+2} < m < 2$ will then follow with the same modifications as in Theorem 3.1.
Let us suppose that

$$-\Delta_m(u) + \Lambda u \leqslant -\Delta_m(v) + \Lambda v \qquad u \leqslant v \quad \text{in} \quad B(x, 5\delta) \tag{3.46}$$

where $\Lambda \in \mathbb{R}$, and suppose that the other hypothesis of Theorem 3.3 are satisfied.
As above we can suppose $v - u \geqslant \tau > 0$ for some $\tau \in \mathbb{R}$. In fact, if this is not true, we can consider $(v - u) + \tau$ and then let $\tau \to 0$. Under this assumption we define

$$\phi \equiv \eta^2 (v - u)^{\beta} \qquad \beta < 0 \tag{3.47}$$

with $\eta \in C_0^1(B(x, 5\delta))$ and $\eta \geqslant 0$ in Ω. Later we will make some more assumptions on the cut-off function η.
Using ϕ as test function in (3.46), since

$$D\phi = 2\eta(v - u)^{\beta} D\eta + \beta\eta^2 (v - u)^{\beta-1} D(v - u) \tag{3.48}$$

we get

$$\begin{aligned}
&\beta \int_{\Omega} \eta^2 (v - u)^{\beta-1} [|Du|^{m-2} Du - |Dv|^{m-2} Dv](Dv - Du)\, dx \\
&+ \int_{\Omega} [2\eta(v - u)^{\beta} [|Du|^{m-2} Du - |Dv|^{m-2} Dv] D\eta\, dx \\
&\geqslant \Lambda \int_{\Omega} \eta^2 (v - u)^{\beta+1}\, dx
\end{aligned} \tag{3.49}$$

In what follows, we will use the following standard estimates for the m-Laplace operator (see e.g. Lemma 2.1 of [9]):

$$||\eta|^{m-2}\eta - |\eta'|^{m-2}\eta'| \leqslant c_1 (|\eta| + |\eta'|)^{m-2} |\eta - \eta'| \tag{3.50}$$

$$[|\eta|^{m-2}\eta - |\eta'|^{m-2}\eta'][\eta - \eta'] \geqslant c_2 (|\eta| + |\eta'|)^{m-2} |\eta - \eta'|^2 \tag{3.51}$$

Using these estimates, by (3.49), it follows

$$\begin{aligned}
&|\beta| \int_{\Omega} (v - u)^{\beta-1} |D(v - u)|^2 \rho\, dx \\
&\leqslant C_1 \int_{\Omega} \eta(v - u)^{\beta} |D(v - u)||D\eta| \rho\, dx + \Lambda \int_{\Omega} \eta^2 (v - u)^{\beta+1}\, dx
\end{aligned} \tag{3.52}$$

where in this case $\rho \equiv (|Du| + |Dv|)^{m-2}$.
Note that, since $\frac{1}{(|Du|+|Dv|)^{m-2}} \leqslant \frac{1}{|Du|^{m-2}}$ and $\frac{1}{(|Du|+|Dv|)^{m-2}} \leqslant \frac{1}{|Dv|^{m-2}}$, then the weight ρ satisfies the same properties of $|Du|^{m-2}$ and of $|Dv|^{m-2}$ (see Theorem 2.1 and Theorem 2.2). Therefore, having assumed that either u or v is a weak solution of (1.1) with f positive, a weighted Sobolev type inequality is also available in this case and then we can use it following closely the proof of Theorem 3.1.
Using Young's inequality (as in Theorem3.1) we get

$$\begin{aligned}
&\int_{\Omega} \eta^2 (v - u)^{\beta-1} |D(v - u)|^2 \rho\, dx \\
&\leqslant \frac{C}{|\beta|} \left(1 + \frac{1}{|\beta|}\right) \int_{\Omega} (v - u)^{\beta+1} [\eta^2 + \rho |D\eta|^2]\, dx
\end{aligned} \tag{3.53}$$

Equation (3.53) gives an estimate for $v-u$ which is of the same type of the estimate given by (A.7) for the solutions of the linearized operator. Therefore, arguing exactly as above, we reduce the proof of our result to proving that there exists $r_0 > 0$ such that

$$\phi\left(r_0, \frac{5\delta}{2}, (v-u)\right) \leqslant C\phi\left(-r_0, \frac{5\delta}{2}, (v-u)\right) \tag{3.54}$$

To prove this, let us define

$$w \equiv \log(v-u)$$

Proceeding as in Theorem 3.1 and using (3.50) and (3.51) we get

$$\begin{aligned} \beta \int_\Omega \rho\eta^2 |w|^{\beta-1}|Dw|^2\,dx \leqslant &\int_\Omega \rho\frac{\eta}{v}(|w|^\beta + (2\beta)^\beta)|D\eta||D(v-u)|\,dx \\ &+ \Lambda \int_\Omega \eta^2(|w|^\beta + (2\beta)^\beta)\,dx \end{aligned} \tag{3.55}$$

Exploiting Eq. (3.55) as done with Eq. (A.30), we can conclude the proof exactly as in Theorem 3.1. □

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