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Symmetry of solutions of some semilinear elliptic equations with singular nonlinearities ☆

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ABSTRACT

We consider positive solutions to the singular semilinear elliptic equation $-\Delta u = \frac{1}{u^\gamma} + f(u)$, in bounded smooth domains, with zero Dirichlet boundary conditions.

We provide some weak and strong maximum principles for the $H_0^1(\Omega)$ part of the solution (the solution u generally does not belong to $H_0^1(\Omega)$), that allow to deduce symmetry and monotonicity properties of solutions, via the *Moving Plane Method*.

1. Introduction

In this paper we study symmetry and monotonicity properties of the solutions to the problem

$$\begin{cases} -\Delta u = \frac{1}{u^\gamma} + f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

where $\gamma > 0$, Ω is a bounded smooth domain and $u \in C(\overline{\Omega}) \cap C^2(\Omega)$.

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Our main results will be proved under the following assumption

(H_p) $f(\cdot)$ is locally Lipschitz continuous, non-decreasing, $f(s) > 0$ for $s > 0$ and $f(0) \geqslant 0$.

As a model problem we may consider solutions to $-\Delta u = \frac{1}{u^\gamma} + u^q$ with $q > 0$.

Since the pioneer results in [11] and [24] singular semilinear elliptic equations have been considered by several authors. We refer to [2,3,5–7,12,15–18,23].

The variational characterization of problem (1) is not trivial. In fact, already in the case $f \equiv 0$, the condition $\gamma < 3$ is necessary to have solutions in $H_0^1(\Omega)$ and to have the associated energy functional $I \neq +\infty$, see [18]. A first attempt in this direction can be found in [15] in the case $\gamma \leqslant 1$.

Later in [6] a general approach was developed for any $\gamma > 0$. The main idea in [6], that will be a key ingredient in the present paper, is a translation of the energy functional and of the functions space used, based on the decomposition of the solutions as

$$u = u_0 + w \tag{2}$$

where $w \in H_0^1(\Omega)$ and $u_0 \in C(\overline{\Omega}) \cap C^2(\Omega)$ is the solution to the problem:

$$\begin{cases} -\Delta u_0 = \frac{1}{{u_0}^\gamma} & \text{in } \Omega, \\ u_0 > 0 & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

The solution u_0 is unique (see Lemma 2.8 in [6]) and can be found via a sub- super-solution method like in [6] or via a truncation argument as in [3]. It follows by the comparison argument used in the proof of [6] that the solution u_0 is continuous up to the boundary and is bounded away from zero in the interior of Ω. This latter information also follows by [3] where the solution u_0 is obtained as the limit of an increasing sequence of positive solutions to a regularized problem.

The equation $-\Delta u_0 = \frac{1}{{u_0}^\gamma}$ consequently can be understood in the weak distributional sense with test functions with compact support in Ω, that is

$$\int_\Omega (Du_0, D\varphi)\,dx = \int_\Omega \frac{\varphi}{{u_0}^\gamma}\,dx \quad \forall \varphi \in C_c^1(\Omega). \tag{4}$$

Actually the solution is fulfilled in the classical sense in the interior of Ω by standard regularity results, since u_0 is strictly positive in the interior of the domain.

In any case, taking into account [18], for $\gamma \geqslant 3$ u_0 does not belong to $H_0^1(\Omega)$ and, consequently, u does not belong to $H_0^1(\Omega)$ too.

The proof of our symmetry result is based on the well known *Moving Plane Method* (see [22]), that was used in a clever way in the celebrated paper [13] in the semilinear nondegenerate case. Actually our proof is more similar to the one of [1] and is based on the weak comparison principle in small domains.

Let us mention that the symmetry (and monotonicity) results in [13] hold also in the case when the domain is the whole space $\mathbb{R}^N$ provided that some a-priori assumptions on the solutions are imposed, or considering the case of nonlinearities decreasing at zero.

The same symmetry results in $\mathbb{R}^N$ have been obtained in [4,9] (see also the related paper [20]) without any a-priori assumptions.

We refer the reader to [8,19,21] for results in the case of fully nonlinear elliptic equations.

Finally, let us mention that symmetry results can be obtained in many other contexts, e.g. we refer the reader to [10] for the case of equations in integral form.

In our case, because of the singular nature of our problem, we have to take care of two difficulties, namely:

– u does not belong to $H_0^1(\Omega)$,
– $\frac{1}{s^\gamma} + f(s)$ is not Lipschitz continuous at zero.

This causes that a straightforward modification of the moving plane technique is not possible in our setting and for this reason we need a new technique based on the decomposition in (2).

Let us state our symmetry result:

Theorem 1. *Let $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ be a solution to (1) with $f(\cdot)$ satisfying (H_p). Assume that the domain Ω is strictly convex w.r.t. the ν-direction ($\nu \in S^{N-1}$) and symmetric w.r.t. T_0^ν, where*

$$T_0^\nu = \{x \in \mathbb{R}^N \colon x \cdot \nu = 0\}.$$

Then u is symmetric w.r.t. T_0^ν and non-decreasing w.r.t. the ν-direction in Ω_0^ν, where

$$\Omega_0^\nu = \{x \in \Omega \colon x \cdot \nu < 0\}.$$

Moreover, if Ω is a ball, then u is radially symmetric with $\frac{\partial u}{\partial r}(r) < 0$ for $r \neq 0$.

For the reader's convenience, we describe here below the scheme of the proof.

(i) Since, by [3], u_0 is the limit of a sequence u_n of solutions to a regularized problem (15), we deduce symmetry and monotonicity properties of u_n, and consequently of u_0, applying the moving plane procedure in a standard way to the regularized problem (15).
(ii) By (i), recalling the decomposition in (2): $u = u_0 + w$, we are reduced to prove symmetry and monotonicity properties of w. To do this, in Section 4, we prove some comparison principles for w needed in the application of the moving plane procedure.
(iii) In Section 5, we carry out the adaptation of the moving plane procedure to the study of the monotonicity and symmetry of w. It is worth emphasizing that the moving plane procedure is applied in our approach only to the $H_0^1(\Omega)$ part of u.
Note also that Theorem 1 is proved in Section 6, exploiting the more general result Proposition 9.

2. Notations

To state the next results we need some notations. Let ν be a direction in $\mathbb{R}^N$ with $|\nu| = 1$. Given a real number λ we set

$$T_\lambda^\nu = \{x \in \mathbb{R}^N \colon x \cdot \nu = \lambda\}, \tag{5}$$

$$\Omega_\lambda^\nu = \{x \in \Omega \colon x \cdot \nu < \lambda\} \tag{6}$$

and

$$x_\lambda^\nu = R_\lambda^\nu(x) = x + 2(\lambda - x \cdot \nu)\nu, \tag{7}$$

that is the reflection trough the hyperplane T_λ^ν. Moreover we set

$$(\Omega_\lambda^\nu)' = R_\lambda^\nu(\Omega_\lambda^\nu) \tag{8}$$

and observe that $(\Omega_\lambda^\nu)'$ may be not contained in Ω. Also we take

$$a(\nu) = \inf_{x \in \Omega} x \cdot \nu. \tag{9}$$

When $\lambda > a(\nu)$, since Ω_λ^ν is nonempty, we set

$$\Lambda_1(\nu) = \left\{\lambda \colon \left(\Omega_t^\nu\right)' \subset \Omega \text{ for any } a(\nu) < t \leqslant \lambda\right\}, \tag{10}$$

and

$$\lambda_1(\nu) = \sup \Lambda_1(\nu). \tag{11}$$

Finally we set

$$u_\lambda^\nu(x) = u\left(x_\lambda^\nu\right), \tag{12}$$

for any $a(\nu) < \lambda \leqslant \lambda_1(\nu)$.

3. Symmetry properties of u_0

Basing on the construction of the solution u_0 of (3) we prove in this section some useful symmetry and monotonicity result for u_0.

Proposition 2. *Let $u_0 \in C(\overline{\Omega}) \cap C^2(\Omega)$ be the solution to (3). Then, for any*

$$a(\nu) < \lambda < \lambda_1(\nu)$$

we have

$$u_0(x) < {u_0}_\lambda^\nu(x), \quad \forall x \in \Omega_\lambda^\nu \tag{13}$$

and

$$\frac{\partial u_0}{\partial \nu}(x) > 0, \quad \forall x \in \Omega_{\lambda_1(\nu)}^\nu. \tag{14}$$

Proof. Let $u_n \in H_0^1(\Omega) \cap C(\overline{\Omega})$ be the unique solution to

$$\begin{cases} -\Delta u_n = \frac{1}{(u_n + \frac{1}{n})^\gamma} & \text{for } x \in \Omega, \\ u_n > 0 & \text{for } x \in \Omega, \\ u_n = 0 & \text{for } x \in \partial\Omega. \end{cases} \tag{15}$$

The existence of u_n was proved in [3] and the uniqueness follows by [6]. Since the problem is no more singular, by standard elliptic estimates it follows that $u_n \in C^2(\overline{\Omega})$. Therefore we can use the moving plane technique exactly as in [1,13,22] to deduce that the statement of our proposition holds true for each u_n. By [3] u_n converges to u_0 a.e. as n tends to infinity and therefore (13) follows passing to the limit. Finally in the same way

$$\frac{\partial u_0}{\partial \nu}(x) \geqslant 0, \quad \forall x \in \Omega_{\lambda_1(\nu)}^\nu,$$

and therefore (14) follows via the strong maximum principle. □

As a consequence of Proposition 2, we get

Proposition 3. *Let $u_0 \in C(\overline{\Omega}) \cap C^2(\Omega)$ be the solution of (3) and assume that the domain Ω is strictly convex w.r.t. the ν-direction ($\nu \in S^{N-1}$) and symmetric w.r.t. T_0^ν. Then u_0 is symmetric w.r.t. T_0^ν and non-decreasing w.r.t. the ν-direction in Ω_0^ν. Moreover, if Ω is a ball, then u_0 is radially symmetric with $\frac{\partial u_0}{\partial r}(r) < 0$ for $r \neq 0$.*

4. Comparison principles

Let us start with the following

Lemma 4. *Let $\gamma > 0$ and consider the function*

$$g_\gamma(x, y, z, h) := x^\gamma (x+y)^\gamma (z+h)^\gamma + x^\gamma z^\gamma (z+h)^\gamma - z^\gamma (x+y)^\gamma (z+h)^\gamma - x^\gamma z^\gamma (x+y)^\gamma$$

and the domain $D \subset \mathbb{R}^4$ defined by

$$D := \big\{(x, y, z, h) \,\big|\, 0 \leqslant x \leqslant z;\ 0 \leqslant h \leqslant y\big\}.$$

Then it follows that $g_\gamma \leqslant 0$ in D.

Proof. Since $x \leqslant z$, by a direct calculation we get

$$\frac{\partial g_\gamma}{\partial y}(x, y, z, h) = \gamma x^\gamma (x+y)^{\gamma-1} (z+h)^\gamma - \gamma z^\gamma (x+y)^{\gamma-1} (z+h)^\gamma - \gamma x^\gamma z^\gamma (x+y)^{\gamma-1} \leqslant 0.$$

Therefore we are reduced to prove that $g_\gamma \leqslant 0$ in $D \cap \{h = y\}$, that is

$$g_\gamma(x, y, z, y) = x^\gamma (x+y)^\gamma (z+y)^\gamma + x^\gamma z^\gamma (z+y)^\gamma - z^\gamma (x+y)^\gamma (z+y)^\gamma - x^\gamma z^\gamma (x+y)^\gamma \leqslant 0.$$

For $x = 0$ the thesis follows at once. For $x > 0$ we note that

$$g_\gamma(x, y, z, y) = -\left(\frac{1}{x^\gamma} - \frac{1}{z^\gamma} + \frac{1}{(z+y)^\gamma} - \frac{1}{(x+y)^\gamma}\right)\big(x^\gamma z^\gamma (z+y)^\gamma (x+y)^\gamma\big)$$

and the conclusion follows exploiting the fact that, for $0 < x \leqslant z$ fixed, the function

$$\tilde{g}_\gamma(t) := x^{-\gamma} - z^{-\gamma} + (z+t)^{-\gamma} - (x+t)^{-\gamma}$$

is increasing in $[0, \infty)$ and $\tilde{g}_\gamma(0) = 0$. □

Lemma 5. *Let $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ be a solution to problem (1) with $\gamma > 0$. Assume that Ω is a bounded smooth domain and that $f(\cdot)$ is locally Lipschitz continuous, $f(s) > 0$ for $s > 0$ and $f(0) \geqslant 0$. Let w be given by (2).*

Then it follows

$$w > 0 \quad \text{in } \Omega.$$

Proof. Since $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ and $u_0 \in C(\overline{\Omega}) \cap C^2(\Omega)$, then $w \in H_0^1(\Omega) \cap C(\overline{\Omega}) \cap C^2(\Omega)$.

By hypothesis on $f(\cdot)$, it follows that u is a super-solution (following Definition 2.5 of [6]) to the equation

$$-\Delta v = \frac{1}{v^\gamma}.$$

Therefore, by Lemma 2.8 in [6] we get that

$$u \geqslant u_0 \quad \text{in } \Omega \quad \text{and therefore} \quad w \geqslant 0 \quad \text{in } \Omega.$$

Now let us show that $w > 0$ in the interior of Ω via the maximum principle exploited in regions where the problem is not singular. More precisely let us assume by contradiction that there exists a point $x_0 \in \Omega$ such that $w(x_0) = 0$ and let $r = r(x_0) > 0$ such that $B_r(x_0) \Subset \Omega$. We have, in the classical sense, in $B_r(x_0)$

$$-\Delta w = -\Delta u + \Delta u_0 = \frac{1}{(u_0 + w)^\gamma} + f(u) - \frac{1}{u_0^\gamma} \geqslant \frac{1}{(u_0 + w)^\gamma} - \frac{1}{u_0^\gamma}.$$

Since $u_0(x_0) > 0$ we can assume that u_0 is positive in $B_r(x_0)$. Therefore we get that

$$\frac{1}{(u_0 + w)^\gamma} - \frac{1}{u_0^\gamma} = c(x)(u_0 + w - u_0) = c(x)w$$

for some bounded coefficient $c(x)$. Thus there exists $\Lambda > 0$ such that $\frac{1}{(u_0+w)^\gamma} - \frac{1}{u_0^\gamma} + \Lambda w \geqslant 0$ in $B_r(x_0)$, so that

$$-\Delta w + \Lambda w \geqslant 0 \quad \text{in } B_r(x_0).$$

By the strong maximum principle we get $w \equiv 0$ in $B_r(x_0)$ and by a covering argument that $w \equiv 0$ in Ω. But $w \equiv 0$ in Ω implies $f(\cdot) = 0$ and we get a contradiction. □

Proposition 6 *(A strong maximum principle). Let $a(\nu) < \lambda < \lambda_1(\nu)$ and let Ω' be a sub-domain of Ω_λ^ν. Assume that $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ is a solution to (1) with $f(\cdot)$ satisfying (H_p).*

Let w be given by (2) and assume that

$$\frac{\partial w}{\partial \nu} \geqslant 0 \quad \text{in } \Omega'.$$

Then it holds the alternative

$$\frac{\partial w}{\partial \nu} > 0 \quad \text{in } \Omega' \quad \text{or} \quad \frac{\partial w}{\partial \nu} = 0 \quad \text{in } \Omega'.$$

Proof. Let us use the short hand notation $w_\nu := \frac{\partial w}{\partial \nu}$ and $u_{0\nu} := \frac{\partial u_0}{\partial \nu}$. Since $f'(\cdot) \geqslant 0$ a.e.[1] by assumption (H_p), $u_{0\nu} \geqslant 0$ in Ω' by Proposition 2, $u \geqslant u_0$ by Lemma 5 and finally $w_\nu \geqslant 0$ in Ω' by assumption, differentiating the equation in (1) we get that w_ν solves

$$\begin{aligned} -\Delta w_\nu &= -\frac{\gamma}{u^{\gamma+1}} w_\nu + f'(u)(w_\nu + u_{0\nu}) + \gamma \left(\frac{1}{u_0^{\gamma+1}} - \frac{1}{u^{\gamma+1}} \right) u_{0\nu} \\ &\geqslant -\frac{\gamma}{u^{\gamma+1}} w_\nu. \end{aligned}$$

[1] Note that, even if f' exists a.e., the term $f'(u)(w_\nu + u_{0\nu})$ makes sense in the weak Sobolev meaning thanks to Stampacchia's theorem.

We recall now that u is bounded away from zero in Ω', and therefore we find $\Lambda > 0$ such that

$$-\Delta w_\nu \geqslant -\frac{\gamma}{u^{\gamma+1}} w_\nu \geqslant -\Lambda w_\nu,$$

so that the conclusion follows by the standard strong maximum principle [14]. □

Proposition 7 *(Weak Comparison Principle in small domains). Let $a(\nu) < \lambda < \lambda_1(\nu)$ and $\Omega' \subseteq \Omega_\lambda^\nu$. Assume that $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ is a solution to (1) with $f(\cdot)$ satisfying (H_p).*

Let w be given by (2) and assume that

$$w \leqslant w_\lambda^\nu \quad \text{on } \partial\Omega'.$$

Then there exists a positive constant $\delta = \delta(u, f)$ such that, if $\mathscr{L}(\Omega') \leqslant \delta$, then

$$w \leqslant w_\lambda^\nu \quad \text{in } \Omega'.$$

Proof. We have

$$-\Delta(u_0 + w) = \frac{1}{(u_0 + w)^\gamma} + f(u_0 + w) \quad \text{in } \Omega, \tag{16}$$

$$-\Delta\big({u_0}_\lambda^\nu + w_\lambda^\nu\big) = \frac{1}{({u_0}_\lambda^\nu + w_\lambda^\nu)^\gamma} + f\big({u_0}_\lambda^\nu + w_\lambda^\nu\big) \quad \text{in } \Omega. \tag{17}$$

Since $(w - w_\lambda^\nu)^+ \in H_0^1(\Omega')$ we can consider a sequence of positive functions ψ_n such that

$$\psi_n \in C_c^\infty\big(\Omega'\big) \quad \text{and} \quad \psi_n \xrightarrow{H_0^1(\Omega')} \big(w - w_\lambda^\nu\big)^+.$$

We can also assume that $supp\psi_n \subseteq supp(w - w_\lambda^\nu)^+$. We plug ψ_n into the weak formulation of (16) and (17) and subtracting we get

$$\int_{\Omega'} \big(D(u_0 + w) - D\big({u_0}_\lambda^\nu + w_\lambda^\nu\big), D\psi_n\big)\, dx$$
$$= \int_{\Omega'} \left(\frac{1}{(u_0 + w)^\gamma} + f(u_0 + w) - \frac{1}{({u_0}_\lambda^\nu + w_\lambda^\nu)^\gamma} - f\big({u_0}_\lambda^\nu + w_\lambda^\nu\big)\right)\psi_n\, dx. \tag{18}$$

Since u_0 and ${u_0}_\lambda^\nu$ solve (3) we deduce

$$\int_{\Omega'} \big(D\big(w - w_\lambda^\nu\big), D\psi_n\big)\, dx = \int_{\Omega'} \left(\frac{1}{({u_0}_\lambda^\nu)^\gamma} - \frac{1}{(u_0)^\gamma} + \frac{1}{(u_0 + w)^\gamma} - \frac{1}{({u_0}_\lambda^\nu + w_\lambda^\nu)^\gamma}\right)\psi_n\, dx$$
$$+ \int_{\Omega'} \big(f(u_0 + w) - f\big({u_0}_\lambda^\nu + w_\lambda^\nu\big)\big)\psi_n\, dx. \tag{19}$$

Since $u_0 \leqslant {u_0}_\lambda^\nu$ in Ω_λ^ν and $w \geqslant w_\lambda^\nu$ on the support of ψ_n, by applying Lemma 4 with $u_0 = x$, $w = y$, ${u_0}_\lambda^\nu = z$ and $w_\lambda^\nu = h$ we get

$$(u_0)^\gamma (u_0+w)^\gamma \big({u_0}_\lambda^\nu + w_\lambda^\nu\big)^\gamma + (u_0)^\gamma \big({u_0}_\lambda^\nu\big)^\gamma \big({u_0}_\lambda^\nu + w_\lambda^\nu\big)^\gamma$$
$$- \big({u_0}_\lambda^\nu\big)^\gamma (u_0+w)^\gamma \big({u_0}_\lambda^\nu + w_\lambda^\nu\big)^\gamma - (u_0)^\gamma \big({u_0}_\lambda^\nu\big)^\gamma (u_0+w)^\gamma \leqslant 0$$

and then $\big(\frac{1}{({u_0}_\lambda^\nu)^\gamma} - \frac{1}{(u_0)^\gamma} + \frac{1}{(u_0+w)^\gamma} - \frac{1}{({u_0}_\lambda^\nu + w_\lambda^\nu)^\gamma}\big) \leqslant 0$.

Therefore, by assumption (H_p), we find a constant $C > 0$ such that

$$\int_{\Omega'} \big(D\big(w - w_\lambda^\nu\big), D\psi_n\big)\,dx \leqslant \int_{\Omega'} \big(f(u_0+w) - f\big({u_0}_\lambda^\nu + w_\lambda^\nu\big)\big)\psi_n\,dx$$
$$\leqslant \int_{\Omega'} \big(f\big({u_0}_\lambda^\nu + w\big) - f\big({u_0}_\lambda^\nu + w_\lambda^\nu\big)\big)\psi_n\,dx \leqslant C \int_{\Omega'} \big(w - w_\lambda^\nu\big)\psi_n\,dx. \quad (20)$$

We now pass to the limit for $n \to \infty$ and get

$$\int_{\Omega'} \big|D\big(w - w_\lambda^\nu\big)^+\big|^2\,dx \leqslant C \int_{\Omega'} \big|\big(w - w_\lambda^\nu\big)^+\big|^2\,dx$$

and by the Poincaré inequality

$$\int_{\Omega'} \big|D\big(w - w_\lambda^\nu\big)^+\big|^2\,dx \leqslant C C_p\big(\Omega'\big) \int_{\Omega'} \big|D\big(w - w_\lambda^\nu\big)^+\big|^2\,dx.$$

For δ small it follows that $CC_p(\Omega') < 1$ which shows that actually $(w - w_\lambda^\nu)^+ = 0$ and the thesis follows. □

Lemma 8 *(Strong Comparison Principle). Let $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ be a solution to problem (1), with $f(\cdot)$ satisfying (H_p). Let w be given by (2) and assume that, for some $a(\nu) < \lambda \leqslant \lambda_1(\Omega)$, we have*

$$w \leqslant w_\lambda^\nu \quad \text{in } \Omega_\lambda^\nu.$$

Then $w < w_\lambda^\nu$ in Ω_λ^ν unless $w \equiv w_\lambda^\nu$ in Ω_λ^ν.

Proof. Let us assume that there exists a point $x_0 \in \Omega_\lambda^\nu$ such that $w(x_0) = w_\lambda^\nu(x_0)$ and let $r = r(x_0) > 0$ such that $B_r(x_0) \Subset \Omega_\lambda^\nu$. We have, in the classical sense, in $B_r(x_0)$

$$-\Delta\big(w_\lambda^\nu - w\big) = -\Delta\big(u_\lambda^\nu - {u_0}_\lambda^\nu\big) + \Delta(u - u_0)$$
$$= \bigg(\frac{1}{u_0^\gamma} - \frac{1}{({u_0}_\lambda^\nu)^\gamma} + \frac{1}{({u_0}_\lambda^\nu + w)^\gamma} - \frac{1}{(u_0+w)^\gamma}\bigg) + \big(f\big({u_0}_\lambda^\nu + w_\lambda^\nu\big) - f(u_0+w)\big)$$
$$+ \frac{1}{({u_0}_\lambda^\nu + w_\lambda^\nu)^\gamma} - \frac{1}{({u_0}_\lambda^\nu + w)^\gamma}. \quad (21)$$

Since $f(\cdot)$ is non-decreasing by assumption, $u_0 \leqslant {u_0}_\lambda^\nu$ in Ω_λ^ν by Proposition 2 and $w \leqslant w_\lambda^\nu$ in Ω_λ^ν, we get

$$f\big({u_0}_\lambda^\nu + w_\lambda^\nu\big) - f(u_0+w) \geqslant 0.$$

Moreover, since for $0 < a \leqslant b$ the function $g(t) := a^{-\gamma} - b^{-\gamma} + (b+t)^{-\gamma} - (a+t)^{-\gamma}$ is increasing in $[0, \infty)$, we also have

$$\left(\frac{1}{u_0^\gamma} - \frac{1}{(u_{0\lambda}^\nu)^\gamma} + \frac{1}{(u_{0\lambda}^\nu + w)^\gamma} - \frac{1}{(u_0 + w)^\gamma}\right) \geqslant 0$$

and by (21) we get

$$-\Delta\left(w_\lambda^\nu - w\right) \geqslant \frac{1}{(u_{0\lambda}^\nu + w_\lambda^\nu)^\gamma} - \frac{1}{(u_{0\lambda}^\nu + w)^\gamma}.$$

Since $u_{0\lambda}^\nu(x_0) > 0$, arguing as in Lemma 5, we find $\Lambda > 0$ such that, eventually reducing r, it results $\frac{1}{(u_{0\lambda}^\nu + w_\lambda^\nu)^\gamma} - \frac{1}{(u_{0\lambda}^\nu + w)^\gamma} + \Lambda(w_\lambda^\nu - w) \geqslant 0$ in $B_r(x_0)$, so that

$$-\Delta\left(w_\lambda^\nu - w\right) + \Lambda\left(w_\lambda^\nu - w\right) \geqslant 0 \quad \text{in } B_r(x_0).$$

By the strong maximum principle [14] it follows $(w_\lambda^\nu - w) \equiv 0$ in $B_r(x_0)$, and by a covering argument $(w_\lambda^\nu - w) \equiv 0$ in Ω_λ^ν, proving the result. □

5. Symmetry

Proposition 9. *Let $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ be a solution to (1). Let w be given by (2).*

Then, for any

$$a(\nu) < \lambda < \lambda_1(\nu)$$

we have

$$w(x) < w_\lambda^\nu(x), \quad \forall x \in \Omega_\lambda^\nu. \tag{22}$$

Moreover

$$\frac{\partial w}{\partial \nu}(x) > 0, \quad \forall x \in \Omega_{\lambda_1(\nu)}^\nu. \tag{23}$$

Finally, (22) and (23) hold true replacing w by u.

Proof. Let $\lambda > a(\nu)$. Since $w > 0$ in Ω by Lemma 5 we have:

$$w \leqslant w_\lambda^\nu \quad \text{on } \partial\Omega_\lambda^\nu.$$

Therefore, assuming that $\mathscr{L}(\Omega_\lambda^\nu)$ is sufficiently small (say for $\lambda - a(\nu)$ sufficiently small) so that Proposition 7 applies, we get

$$w \leqslant w_\lambda^\nu \quad \text{in } \Omega_\lambda^\nu, \tag{24}$$

and actually $w < w_\lambda^\nu$ in Ω_λ^ν by the Strong Comparison Principle (Lemma 8).

Let us define

$$\Lambda_0 = \left\{\lambda > a(\nu)\colon\ w \leqslant w_t^\nu \text{ in } \Omega_t^\nu \text{ for all } t \in (a(\nu), \lambda]\right\}$$

which is not empty thanks to (24). Also set

$$\lambda_0 = \sup \Lambda_0.$$

By the definition of $\lambda_1(\nu)$, to prove our result we have to show that actually $\lambda_0 = \lambda_1(\nu)$.

Assume otherwise that $\lambda_0 < \lambda_1(\nu)$ and note that, by continuity, we obtain $w \leqslant w_{\lambda_0}^\nu$ in $\Omega_{\lambda_0}^\nu$. By the Strong Comparison Principle (Lemma 8), it follows $w < w_{\lambda_0}^\nu$ in $\Omega_{\lambda_0}^\nu$ unless $w = w_{\lambda_0}^\nu$ in $\Omega_{\lambda_0}^\nu$. Because of the zero Dirichlet boundary condition and the fact that $w > 0$ in the interior of the domain, the case $w \equiv w_{\lambda_0}^\nu$ in $\Omega_{\lambda_0}^\nu$ is not possible. Thus $w < w_{\lambda_0}^\nu$ in $\Omega_{\lambda_0}^\nu$.

We can now consider δ given by Proposition 7, so that the Weak Comparison Principle holds true in any sub-domain Ω' if $\mathscr{L}(\Omega') \leqslant \delta$. Fix a compact set $\mathscr{K} \subset \Omega_{\lambda_0}^\nu$ so that $\mathscr{L}(\Omega_{\lambda_0}^\nu \setminus \mathscr{K}) \leqslant \frac{\delta}{2}$. By compactness we find $\sigma > 0$ such that

$$w_{\lambda_0}^\nu - w \geqslant 2\sigma > 0 \quad \text{in } \mathscr{K}.$$

Take now $\bar{\varepsilon} > 0$ sufficiently small so that $\lambda_0 + \bar{\varepsilon} < \lambda_1(\nu)$ and, for any $0 < \varepsilon \leqslant \bar{\varepsilon}$

a) $w_{\lambda_0+\varepsilon}^\nu - w \geqslant \sigma > 0$ in $\mathscr{K}$,
b) $\mathscr{L}(\Omega_{\lambda_0+\varepsilon}^\nu \setminus \mathscr{K}) \leqslant \delta$.

Taking into account *a*) it is now easy to check that, for any $0 < \varepsilon \leqslant \bar{\varepsilon}$, we have that $w \leqslant w_{\lambda_0+\varepsilon}^\nu$ on the boundary of $\Omega_{\lambda_0+\varepsilon}^\nu \setminus \mathscr{K}$. Consequently, by *b*), we can apply the Weak Comparison Principle (Proposition 7) and deduce that

$$w \leqslant w_{\lambda_0+\varepsilon}^\nu \quad \text{in } \Omega_{\lambda_0+\varepsilon}^\nu \setminus \mathscr{K}.$$

Thus $w \leqslant w_{\lambda_0+\varepsilon}^\nu$ in $\Omega_{\lambda_0+\varepsilon}^\nu$ and by applying the Strong Comparison Principle (Lemma 8) we have $w < w_{\lambda_0+\varepsilon}^\nu$ in $\Omega_{\lambda_0+\varepsilon}^\nu$. We get a contradiction with the definition of λ_0 and conclude that actually $\lambda_0 = \lambda_1(\nu)$. Then (22) is proved.

It follows now directly from simple geometric considerations and by (22) that w is monotone non-decreasing in $\Omega_{\lambda_1(\nu)}^\nu$ in the ν-direction. This gives

$$\frac{\partial w}{\partial \nu}(x) \geqslant 0 \quad \text{in } \Omega_{\lambda_1(\nu)}^\nu,$$

so it is standard to deduce (23) from Proposition 6.

To prove that (22) and (23) hold true replacing w with u, just recall that

$$u = u_0 + w,$$

and exploit Proposition 2. □

6. Proof of Theorem 1

The proof of Theorem 1 is now a direct consequence of Proposition 9. Observing that by assumption

$$\lambda_1(\nu) = 0,$$

we can apply Proposition 9 in the ν-direction to get

$$u(x) \leqslant u_{\lambda_1(\nu)}^\nu(x), \quad \forall x \in \Omega_0^\nu$$

and in the $(-\nu)$-direction to get

$$u(x) \geqslant u^{\nu}_{\lambda_1(\nu)}(x), \quad \forall x \in \Omega^{\nu}_0.$$

Therefore $u(x) \equiv u^{\nu}_{\lambda_1(\nu)}(x)$ in Ω. The monotonicity of u follows by (23).

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