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Monotonicity of solutions of Fully nonlinear uniformly elliptic equations in the half-plane ☆

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ABSTRACT

In this paper we study the monotonicity of positive (or non-negative) viscosity solutions to uniformly elliptic equations $F(\nabla u, D^2u) = f(u)$ in the half plane, where f is locally Lipschitz continuous (with $f(0) \geqslant 0$) and zero Dirichlet boundary conditions are imposed. The result is obtained without assuming the u or $|\nabla u|$ are bounded.

1. Introduction

We consider the following problem:

$$\begin{cases} F\big(\nabla u, D^2u\big) = f(u), & \text{in } D \equiv \big\{(x, y) \in \mathbb{R}^2\colon\ y > 0\big\}, \\ u(x, 0) = 0, & \forall x \in \mathbb{R}, \end{cases} \tag{1}$$

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with $F:\mathbb{R}^2\times S^2\to\mathbb{R}$ uniformly elliptic, f locally Lipschitz continuous with $f(0)\geqslant 0$, and we study the monotonicity of positive (or non-negative) solutions. More precisely we consider $F:\mathbb{R}^2\times S^2\to\mathbb{R}$ satisfying the following structural hypotheses,

$(F1)$ *Uniform ellipticity*: There exist constants $0<\theta\leqslant\Theta$ such that for all $X,Y\in S^2$ with $Y\geqslant 0$,

$$-\Theta\,\mathrm{trace}(Y)\leqslant F(\xi,X+Y)-F(\xi,X)\leqslant -\theta\,\mathrm{trace}(Y),$$

for every $\xi\in\mathbb{R}^2$.

$(F2)$ *Homogeneity*: $F(t\xi,tX)=t\cdot F(\xi,X)$ for all $t>0$. We further assume $F(0,0)=0$.

$(F3)$ *Structure condition*: There exists $\gamma>0$ such that for all $X,Y\in S^2$, and $\xi_1,\xi_2\in\mathbb{R}^2$, we have,

$$\mathcal{P}^-_{\theta,\Theta}(X-Y)-\gamma|\xi_1-\xi_2|\leqslant F(\xi_1,X)-F(\xi_2,Y)\leqslant \mathcal{P}^+_{\theta,\Theta}(X-Y)+\gamma|\xi_1-\xi_2|,$$

where $\mathcal{P}^\pm_{\theta,\Theta}$ are the extremal *Pucci* operators, defined as

$$\begin{aligned}\mathcal{P}^+_{\theta,\Theta}(X)&=-\theta\sum_{\lambda_i>0}\lambda_i(X)-\Theta\sum_{\lambda_i<0}\lambda_i(X),\\ \mathcal{P}^-_{\theta,\Theta}(X)&=-\Theta\sum_{\lambda_i>0}\lambda_i(X)-\theta\sum_{\lambda_i<0}\lambda_i(X),\end{aligned}\tag{2}$$

with $\lambda_i(X)$, $i=1,\dots,n$, the eigenvalues of X.

$(F4)$ *Symmetry*: $F(\xi^tQ,Q^tXQ)=F(\xi,X)$ where $Q\in O(n)=\{Q\in S^2\colon\ Q\cdot Q^t=Id\}$.

Some comments about the hypothesis: it can be checked that,

$$\mathcal{P}^-_{\theta,\Theta}(X)=\inf_{A\in\mathcal{A}_{\theta,\Theta}}\bigl\{-\mathrm{trace}(AX)\bigr\},\qquad \mathcal{P}^+_{\theta,\Theta}(X)=\sup_{A\in\mathcal{A}_{\theta,\Theta}}\bigl\{-\mathrm{trace}(AX)\bigr\}$$

for $\mathcal{A}_{\theta,\Theta}=\{A\in S^2\colon\ \theta|\xi|^2\leqslant\langle A\xi,\xi\rangle\leqslant\Theta|\xi|^2,\ \forall\xi\in\mathbb{R}^2\}$. Notice that when $\Theta=\theta=1$ we have $\mathcal{P}^+_{\theta,\Theta}=\mathcal{P}^-_{\theta,\Theta}=-\Delta$. Also we point out that nonlinear degenerate operators, such as the p-Laplacian operator, are not included because of the uniform ellipticity assumption above. We also remark that $(F3)$ is equivalent to uniform ellipticity when $\xi_1=\xi_2$ and that hypothesis $(F4)$ is naturally satisfied by Pucci's operators. Just mention that in [8], Pucci's operators are defined with a different sign convention. Both definitions are related through the following expressions

$$\mathcal{M}^-(M,\theta,\Theta)=-\mathcal{P}^+_{\theta,\Theta}(M),\qquad \mathcal{M}^+(M,\theta,\Theta)=-\mathcal{P}^-_{\theta,\Theta}(M),$$

where $\mathcal{M}^\pm(M,\theta,\Theta)$ is the notation used in [8].

The main result in this paper is the following:

Theorem 1. *Let u be a non-negative (nontrivial) viscosity solution of (1), with $F:\mathbb{R}^2\times S^2\to\mathbb{R}$ satisfying $(F1)$–$(F4)$. Assume f locally Lipschitz continuous and $f(0)\geqslant 0$. Then, u is positive in the interior of the domain and monotone in the e_2-direction. Moreover,*

$$\frac{\partial u}{\partial y}(x,y)>0,\quad \forall(x,y)\in\overline{D}.$$

The monotonicity of solutions in half-spaces is an important issue that arises naturally in many applications such as blow-up analysis, a priori estimates and Liouville-type theorems.

The study of monotonicity in the semilinear nondegenerate case is mostly based on the moving-plane method that goes back to Alexandrov [1] and Serrin [22]. A clever use of the moving-plane method was shown in the celebrated papers [6,19]. We also refer the reader to the series of papers [3–5,14–16].

Considering quasilinear and, more generally, fully nonlinear elliptic equations, one of the main difficulties is the fact that comparison principles are not equivalent to Maximum Principles, as in the semilinear case. Moreover, the application of the moving plane technique to problems posed in half-spaces, is usually more delicate, since comparison results, in domains of small measure, have to be replaced by comparison results in narrow unbounded domains such as narrow strips. Generally, this is a demanding task.

The results in this paper are closely related to a geometric approach which goes back to [4] and was successfully exploited in [13], in the p-Laplacian case, to the study of monotonicity of positive (or non-negative solutions) in the *two-dimensional* half space. There, the use of weak comparison principles in narrow (unbounded) domains is avoided by means of a geometrical argument in the spirit of [4] that allows one to use only a weak comparison principle in domains of small measure. The main advantage of this argument is that there is no need to assume that either the solution u, or the gradient $|\nabla u|$ are bounded, a usual hypothesis in the literature.

In this paper, in order to prove Theorem 1, we shall bring the geometric ideas in [4,13] to the context of uniformly elliptic fully nonlinear problems under the conditions $(F1)$–$(F4)$, and we shall provide the necessary tools. Recall that, the notion of viscosity solution, is the natural notion of solution in this context, and adapting the devices in [4,13] to the viscosity setting carries a number of technical complications.

Monotonicity results for non-negative solutions of fully nonlinear uniformly elliptic operators are known, see [10,21] under the assumption that u or $|\nabla u|$ are bounded, that we are able to remove in dimension two. We point out that, already in the case of $F(\nabla u, D^2u) \equiv \Delta u$, there exist unbounded monotone solutions u whose gradient $|\nabla u|$ is also unbounded (think for example to $u(x, y) = e^x y$). These solutions satisfy the hypothesis of our monotonicity result. This motivates our analysis.

We think that the geometric ideas in the sequel can be adapted to many other situations and operators; however, it seems impossible to attack the higher-dimensional case with these arguments. In a recent paper [17], the authors prove a monotonicity result for positive weak solutions to $\Delta_p u + f(u) = 0$ in n-dimensional half spaces. We believe that the techniques developed in [17] might also be useful in the fully nonlinear case, but this would in any case require u or $|\nabla u|$ be bounded.

The rest of the paper is organized as follows: in Section 2 we give some preliminaries regarding fully nonlinear operators, the notion of viscosity solution and the Maximum Principle for fully nonlinear operators, and in Section 3 we prove Theorem 1.

2. Preliminaries

In this section, we provide some preliminaries on fully nonlinear equations, and some results that will be needed in the sequel.

2.1. Notion of viscosity solution

Let us recall here the definition of viscosity solution, which will be used in the sequel.

Definition 2. A function $u \in \mathscr{C}(\Omega)$ is a *viscosity subsolution* of (1) if for all $\varphi \in \mathscr{C}^2$ and $\hat{x} \in \Omega$ such that $u - \varphi$ attains a local maximum at $\hat{x}$, we have that

$$F\big(\nabla\varphi(\hat{x}), D^2\varphi(\hat{x})\big) \leqslant f\big(u(\hat{x})\big).$$

Analogously, $u \in \mathscr{C}(\Omega)$ is a *viscosity supersolution* of (1) if for all $\varphi \in \mathscr{C}^2$ and $\hat{x} \in \Omega$ such that $u - \varphi$

attains a local minimum in $\hat{x}$, we have that

$$F\big(\nabla\varphi(\hat{x}), D^2\varphi(\hat{x})\big) \geqslant f\big(u(\hat{x})\big).$$

Finally, $u \in \mathscr{C}(\Omega)$ is a *viscosity solution* of (1) in Ω if it is both a viscosity subsolution and supersolution.

Remark 3. Viscosity solutions to the above problem are of class $C^{1,\alpha}$, that is, $u \in C^{1,\alpha}(\mathscr{K} \cap \overline{D})$ for any compact set $\mathscr{K} \subset \mathbb{R}^2$ (see [8]), the proof being based on the Aleksandrov–Bakelman–Pucci estimate which is available in our problem by hypotheses (F1) and (F3), see for instance [9].

Since we are dealing with elliptic equations of second order, all the relevant information concerning the test functions $\varphi \in \mathscr{C}^2$ in Definition 2 is codified in the first two derivatives of φ at the contact point with the solution u. Hence, we can give an equivalent definition of viscosity sub- and supersolution in terms of the *upper and lower semijets* (of degree 2) of u at the point $\hat{x}$, which are the sets of quadratic polynomials touching u respectively from above and below at the contact point $\hat{x}$.

Definition 4. Given $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$, $\hat{x} \in \Omega$, we can define second order *semijets* as

$$J^{2,+}_{\Omega} u(\hat{x}) = \bigg\{(p, X) \in \mathbb{R}^2 \times S^2\colon\ \varphi(x) = u(\hat{x}) + \big\langle p, (x-\hat{x})\big\rangle + \frac{1}{2}\big\langle X(x-\hat{x}), (x-\hat{x})\big\rangle$$
$$\text{touches } u \text{ from above at } \hat{x},\ \forall x \in B_r(\hat{x}) \cap \Omega \text{ with } r > 0 \text{ small enough}\bigg\},$$
$$J^{2,-}_{\Omega} u(\hat{x}) = \bigg\{(p, X) \in \mathbb{R}^2 \times S^2\colon\ \varphi(x) = u(\hat{x}) + \big\langle p, (x-\hat{x})\big\rangle + \frac{1}{2}\big\langle X(x-\hat{x}), (x-\hat{x})\big\rangle$$
$$\text{touches } u \text{ from below at } \hat{x},\ \forall x \in B_r(\hat{x}) \cap \Omega \text{ with } r > 0 \text{ small enough}\bigg\},$$

and their closures,

$$\overline{J}^{2,+}_{\Omega} u(\hat{x}) = \big\{(p, X) \in \mathbb{R}^2 \times S^2\colon\ \exists x_n \in B_r(\hat{x}),\ (p_n, X_n) \in J^{2,+}u(x_n)$$
$$\text{such that } (x_n, p_n, X_n) \to (\hat{x}, p, X) \text{ when } n \to \infty\big\},$$

$$\overline{J}^{2,-}_{\Omega} u(\hat{x}) = \big\{(p, X) \in \mathbb{R}^2 \times S^2\colon\ \exists x_n \in B_r(\hat{x}),\ (p_n, X_n) \in J^{2,-}u(x_n)$$
$$\text{such that } (x_n, p_n, X_n) \to (\hat{x}, p, X) \text{ when } n \to \infty\big\}.$$

Next, we rewrite Definition 2 in terms of semijets.

Definition 5.

1. We say that $u \in \mathscr{C}(\Omega)$ is a *viscosity subsolution* of (1) in Ω if for all $\hat{x} \in \Omega$ such that $J^{2,+}u(\hat{x}) \neq \emptyset$ we have

$$F(p, X) \leqslant f\big(u(\hat{x})\big), \quad \forall (p, X) \in J^{2,+}u(\hat{x}).$$

2. Analogously, a *viscosity supersolution* of (1) in Ω is a function $u \in \mathscr{C}(\Omega)$ such that for all $\hat{x} \in \Omega$ such that $J^{2,-}u(\hat{x}) \neq \emptyset$ we have

$$F(p, X) \geqslant f\big(u(\hat{x})\big), \quad \forall (p, X) \in J^{2,-}u(\hat{x}).$$

3. Finally, u *is a viscosity supersolution* of (1) in Ω if it is both a viscosity subsolution and supersolution.

Remark 6. In the previous definition we can also consider $\overline{J}^{2\pm}u(x)$ instead of $J^{2\pm}u(x)$.

2.2. Maximum Principles

Let us recall some results (see also [7] and the references therein) concerning the ABP estimate and the Maximum Principle which will be needed later.

First, we recall the basic ABP estimate, for which proof, the interested reader is referred for instance to [9, Proposition 2.12].

Proposition 7 *(Aleksandrov–Bakelman–Pucci estimate). Consider a bounded domain $\Omega \subset \mathbb{R}^n$ and let $f \in L^n(\Omega) \cap \mathscr{C}(\Omega)$, $\gamma \geqslant 0$ and $u \in \mathscr{C}(\overline{\Omega})$ be a viscosity solution of*

$$\mathscr{P}^-_{\theta,\Theta}\big(D^2u\big) - \gamma|\nabla u| \leqslant f(x) \quad \text{in } \{u > 0\}.$$

Then, there exists a constant $C = C(\theta, \gamma, n)$, only depending on the ellipticity constants and the dimension, such that

$$\sup_{\Omega} u^+ \leqslant \sup_{\partial\Omega} u^+ + C \cdot \operatorname{diam}(\Omega) \cdot \big\| f^+ \big\|_{L^n(\Gamma^+(u^+))},$$

where $\Gamma^+(w)$ denotes the upper contact set of a function $w : \Omega \to \mathbb{R}$, that is,

$$\Gamma^+(w) = \big\{x \in \Omega\colon \exists p \in \mathbb{R}^n \text{ such that } w(y) \leqslant w(x) + \langle p, y - x\rangle \text{ for } y \in \Omega\big\}.$$

Then, by means of a simple argument, the same result is true for an equation with zero order terms having positive coefficients.

Proposition 8 *(Full Aleksandrov–Bakelman–Pucci estimate). Consider a bounded domain $\Omega \subset \mathbb{R}^n$ and let $c(x) \geqslant 0$ in Ω, $f \in L^n(\Omega) \cap \mathscr{C}(\Omega)$, $\gamma \geqslant 0$ and $u \in \mathscr{C}(\overline{\Omega})$ be a viscosity solution of*

$$\mathscr{P}^-_{\theta,\Theta}\big(D^2u\big) - \gamma|\nabla u| + c(x)u \leqslant f(x) \quad \text{in } \{u > 0\}.$$

Then, there exists a constant $C = C(\theta, \gamma, n)$, only depending on the ellipticity constants and the dimension, such that

$$\sup_{\Omega} u^+ \leqslant \sup_{\partial\Omega} u^+ + C \cdot \operatorname{diam}(\Omega) \cdot \big\| f^+ \big\|_{L^n(\Gamma^+(u^+))}, \tag{3}$$

where $\Gamma^+(w)$ denotes the upper contact set of a function $w : \Omega \to \mathbb{R}$, that is,

$$\Gamma^+(w) = \big\{x \in \Omega\colon \exists p \in \mathbb{R}^n \text{ such that } w(y) \leqslant w(x) + \langle p, y - x\rangle \text{ for } y \in \Omega\big\}.$$

Proof. Clearly, in the set $\{u > 0\}$ we have

$$\mathscr{P}^-_{\theta,\Theta}\big(D^2u\big) - \gamma|\nabla u| \leqslant \mathscr{P}^-_{\theta,\Theta}\big(D^2u\big) - \gamma|\nabla u| + c(x)u \leqslant f(x).$$

Hence, we get (3) from Proposition 7. □

Next, we present the Maximum Principle as an inmediate consequence of the ABP estimate in Proposition 8.

Corollary 9 *(Maximum Principle). Consider a bounded domain $\Omega \subset \mathbb{R}^n$, and $c(x) \geqslant 0$ in Ω, $\gamma \geqslant 0$ and let $u \in \mathscr{C}(\overline{\Omega})$ be a viscosity solution of*

$$\begin{cases} \mathscr{P}^-_{\theta,\Theta}\big(D^2u\big) - \gamma|\nabla u| + c(x)u \leqslant 0 & \text{in } \Omega, \\ u \leqslant 0 & \text{on } \partial\Omega. \end{cases}$$

Then, $u \leqslant 0$ in Ω.

The condition $c \leqslant 0$ in Corollary 9 is quite restrictive for our purposes. Alternatively, we shall use the following Maximum Principle which does not make any assumption on the sign of $c(x)$ but, instead, on the size of both the coefficients and the measure of the domain Ω.

Proposition 10 *(Maximum Principle in domains of small measure). Consider a bounded domain $\Omega \subset \mathbb{R}^n$ and assume $|c(x)| \leqslant b$ in Ω and $\gamma \geqslant 0$. Let $u \in \mathscr{C}(\overline{\Omega})$ be a viscosity solution of*

$$\begin{cases} \mathscr{P}^-_{\theta,\Theta}\big(D^2u\big) - \gamma|\nabla u| + c(x)u \leqslant 0 & \text{in } \Omega, \\ u \leqslant 0 & \text{on } \partial\Omega. \end{cases}$$

Then, there exists a constant $\delta = \delta(\theta, \gamma, n, b, \operatorname{diam}(\Omega)) > 0$ such that, if $|\Omega| < \delta$, then $u \leqslant 0$ in Ω.

Proof. Writing $c = c^+ - c^-$, we can put the equation in the following form

$$\mathscr{P}^-_{\theta,\Theta}\big(D^2u\big) - \gamma|\nabla u| + c^+(x)u \leqslant c^-(x)u \quad \text{in } \Omega.$$

We can apply the ABP estimate (3) to the above expression and get

$$\begin{aligned} \sup_\Omega u^+ &\leqslant C(\theta,\gamma,n)\cdot\operatorname{diam}(\Omega)\cdot\big\|c^-u^+\big\|_{L^n(\Omega)} \\ &\leqslant C(\theta,\gamma,n)\cdot\operatorname{diam}(\Omega)\cdot b\cdot|\Omega|^{1/n}\cdot\sup_\Omega u^+ \\ &\leqslant C\big(\theta,\gamma,n,b,\operatorname{diam}(\Omega)\big)\cdot|\Omega|^{1/n}\cdot\sup_\Omega u^+. \end{aligned}$$

Then, if $C(\theta,\gamma,n,b,\operatorname{diam}(\Omega))\cdot|\Omega|^{1/n} \leqslant 1/2$, we conclude that $u \leqslant 0$ in Ω. □

Finally, we use the ABP estimate to get a Strong Maximum Principle following [18, Chapters 3 and 9]. Here, we do not make any assumption on the sign of c, but instead we suppose that $u \leqslant 0$.

Proposition 11 *(Strong Maximum Principle and Hopf Lemma). Consider a bounded domain $\Omega \subset \mathbb{R}^n$ and assume that $u \in \mathscr{C}(\overline{\Omega})$ is a non-positive viscosity solution of*

$$\mathscr{P}^-_{\theta,\Theta}\big(D^2u\big) - \gamma|\nabla u| + c(x)u \leqslant 0 \quad \text{in } \Omega \tag{4}$$

with $c(x) \in L^\infty$. Then, either $u \equiv 0$ or $u < 0$ in Ω. Furthermore, in the latter case, for any $z \in \partial\Omega$ such that,

(a) *$u(z) > u(x)$ for all $x \in \Omega$, and*
(b) *$\partial\Omega$ satisfies an interior sphere condition at z,*

we have that

$$\lim_{t\to 0^+} \frac{u(z+t\xi)-u(z)}{t} < 0$$

for every non-tangential direction ξ pointing into Ω.

Our proof adapts [18, Sections 3.2 and 9.1]. For further refinements of the Hopf Lemma, see [2,20].

Proof of Proposition 11. We can suppose without loss of generality that $c \geqslant 0$, since otherwise we can proceed by writing

$$\mathscr{P}^-_{\theta,\Theta}\big(D^2u\big) - \gamma|\nabla u| + c^+(x)u \leqslant c^-(x)u \leqslant 0 \quad \text{in } \Omega.$$

1. We start with the proof of the Strong Maximum Principle. Suppose to the contrary that u is not identically 0 and $u(x) = 0$ for some $x \in \Omega$. Then, there must exist concentric balls $B_\rho(y) \subset B_R(y) \subset \Omega$ such that $u < 0$ in $\overline{B}_\rho(y)$ and $u(x_0) = 0$ for some $x_0 \in B_R(y)$.

For $0 < \rho < R$, we consider $A = \{x \in \Omega\colon\ \rho < |x-y| < R\}$ and define

$$v(x) = e^{\frac{-\alpha|x-y|^2}{2}} - e^{\frac{-\alpha R^2}{2}}, \tag{5}$$

and

$$w(x) = u(x) + \varepsilon v(x),$$

for $x \in A$, where $\alpha, \varepsilon > 0$ are constants yet to be determined. Then,

(i) $\mathscr{P}^-_{\theta,\Theta}(D^2w) - \gamma|\nabla w| + c(x)w \leqslant 0$ in A for α large enough. Let $\phi \in \mathscr{C}^2$ and $\hat{x} \in A$ such that $w - \phi$ has a local maximum at $\hat{x}$. It is easy to see that $u - \Phi$ has a local maximum at $\hat{x}$, with $\Phi(x) = \phi(x) - \varepsilon v(x)$. Since $v \in \mathscr{C}^2$, so it is Φ, and the definition of u and the structure condition $(F3)$ imply

$$\begin{aligned}
0 &\geqslant \mathscr{P}^-_{\theta,\Theta}\big(D^2\Phi(\hat{x})\big) - \gamma\big|\nabla\Phi(\hat{x})\big| + c(\hat{x})u(\hat{x}) \\
&= \mathscr{P}^-_{\theta,\Theta}\big(D^2\phi(\hat{x}) - \varepsilon D^2v(\hat{x})\big) - \gamma\big|\nabla\phi(\hat{x}) - \varepsilon\nabla v(\hat{x})\big| + c(\hat{x})w(\hat{x}) - \varepsilon c(\hat{x})v(\hat{x}) \\
&\geqslant \mathscr{P}^-_{\theta,\Theta}\big(D^2\phi(\hat{x})\big) - \gamma\big|\nabla\phi(\hat{x})\big| + c(\hat{x})w(\hat{x}) + \varepsilon\mathscr{P}^-_{\theta,\Theta}\big(-D^2v(\hat{x})\big) - \gamma\varepsilon\big|\nabla v(\hat{x})\big| - \varepsilon c(\hat{x})v(\hat{x}).
\end{aligned}$$

Consequently,

$$\mathscr{P}^-_{\theta,\Theta}\big(D^2\phi(\hat{x})\big) - \gamma\big|\nabla\phi(\hat{x})\big| + c(\hat{x})w(\hat{x}) \leqslant \varepsilon\mathscr{P}^+_{\theta,\Theta}\big(D^2v(\hat{x})\big) + \gamma\varepsilon\big|\nabla v(\hat{x})\big| + \varepsilon c(\hat{x})v(\hat{x}).$$

A direct computation yields,

$$\mathscr{P}^{+}_{\theta,\Theta}\big(D^2 v(\hat{x})\big) = e^{\frac{-\alpha|x-y|^2}{2}}\,\mathscr{P}^{+}_{\theta,\Theta}\big(\alpha^2(x-y)\otimes(x-y)-\alpha I\big)$$
$$\leqslant e^{\frac{-\alpha|x-y|^2}{2}}\big(-\alpha^2\rho^2\theta+\alpha n\Theta\big),$$
$$\big|\nabla v(\hat{x})\big| \leqslant \alpha R e^{\frac{-\alpha|x-y|^2}{2}}.$$

Combining the expressions above,

$$\mathscr{P}^{-}_{\theta,\Theta}\big(D^2\phi(\hat{x})\big)-\gamma\big|\nabla\phi(\hat{x})\big|+c(\hat{x})w(\hat{x})$$
$$\leqslant \varepsilon e^{\frac{-\alpha|x-y|^2}{2}}\big(-\alpha^2\rho^2\theta+\alpha(n\Theta-\gamma R)\big)+\varepsilon\|c\|_\infty\cdot\left(1-e^{\frac{-\alpha R^2}{2}}\right)\leqslant 0,$$

for α large enough.

(ii) $w\leqslant 0$ on ∂A for $\varepsilon>0$ small enough. Since $u<0$ on $\partial B_\rho(y)$, we can choose $\varepsilon>0$ small enough such that

$$u+\varepsilon v\leqslant 0 \tag{6}$$

on $\partial B_\rho(y)$. Moreover, as $v=0$ on $\partial B_R(y)$, (6) also holds in the outer boundary.

Hence, from steps (i) and (ii) and the ABP estimate in Proposition 8 we deduce $w\leqslant 0$ in the whole of A. We have arrived at a contradiction, since $0=u(x_0)\leqslant -\varepsilon v(x_0)<0$.

2. Next, we prove the second part of the result, the Hopf Boundary Lemma. Since Ω satisfies an interior sphere condition at z, there exists a ball $B=B_R(y)\subset\Omega$ with $z\in\partial B$. As before, for $0<\rho<R$, we consider $A=\{x\in\Omega\colon\ \rho<|x-y|<R\}$ and define

$$w(x)=u(x)-u(z)+\varepsilon v(x)\quad \text{for } x\in A,$$

with v as in (5), and $\alpha,\varepsilon>0$ some constants to be determined. Proceeding exactly as before, one proves that

$$\mathscr{P}^{-}_{\theta,\Theta}\big(D^2 w\big)-\gamma|\nabla w|+c(x)w\leqslant 0\quad \text{in } A \text{ for } \alpha \text{ large enough},$$

in the viscosity sense.

Moreover, $w\leqslant 0$ on ∂A for $\varepsilon>0$ small enough. Since $u-u(z)<0$ on $\partial B_\rho(y)$, we can choose $\varepsilon>0$ small enough such that

$$\big(u-u(z)+\varepsilon v\big)\leqslant 0 \tag{7}$$

on $\partial B_\rho(y)$. Moreover, since $v=0$ on $\partial B_R(y)$, (7) also holds in the outer boundary.

Again, the ABP estimate in Proposition 8 implies $w\leqslant 0$ in the whole A. Hence, for every non-tangential direction ξ pointing into Ω, one has

$$\lim_{t\to 0^+}\frac{u(z+t\xi)-u(z)}{t}\leqslant -\varepsilon\frac{\partial v}{\partial\xi}(z)=\varepsilon\alpha e^{\frac{-\alpha R^2}{2}}\big\langle(z-y),\xi\big\rangle<0. \qquad\square$$

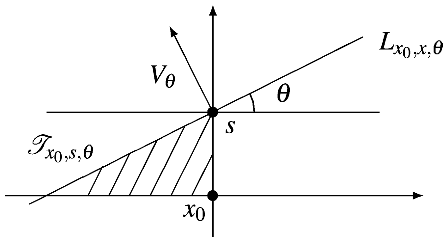

Fig. 1. The domain.

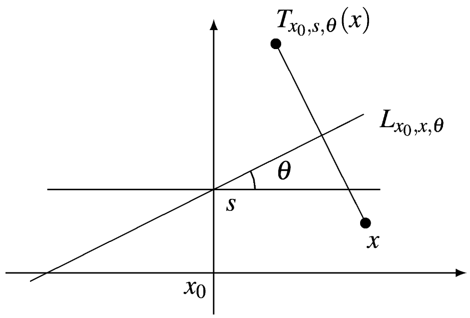

Fig. 2. The reflection.

3. Proof of Theorem 1

Before starting with the proof, let us introduce some necessary notation.

Let $L_{x_0,s,\theta}$ be the line with slope $\tan(\theta)$ passing through the point (x_0, s), and V_θ the vector orthogonal to $L_{x_0,s,\theta}$ such that $\langle V_\theta, e_2\rangle \geqslant 0$. Denote by $\mathcal{T}_{x_0,s,\theta}$ the triangle delimited by $L_{x_0,s,\theta}$, $\{y=0\}$ and $\{x=x_0\}$ (see Fig. 1).

Define $T_{x_0,s,\theta}(x)$ as the point symmetric to x with respect to $L_{x_0,s,\theta}$ (see Fig. 2), and

$$u_{x_0,s,\theta}(x) = u\big(T_{x_0,s,\theta}(x)\big),$$

and,

$$w_{x_0,s,\theta} = u - u_{x_0,s,\theta}.$$

For simplicity we shall denote $u_{x_0,s,0} = u_s$.

An important point in the proof of Theorem 1 is the fact that $u_{x_0,s,\theta}$ is still a viscosity solution of (1), which we prove next.

Lemma 12. *The function $u_{x_0,s,\theta}$ is a viscosity solution of*

$$F\big(\nabla u_{x_0,s,\theta}, D^2 u_{x_0,s,\theta}\big) = f(u_{x_0,s,\theta}) \quad \textit{in } \mathcal{T}_{x_0,s,\theta}.$$

Proof. Let us consider for example the subsolution case, since the supersolution case is analogous.

Take $\phi \in \mathcal{C}^2$ and $\hat{x} \in \mathcal{T}_{x_0,s,\theta}$ such that $u_{x_0,s,\theta} - \phi$ has a local maximum at $\hat{x}$. Define $\phi_{x_0,s,\theta}(x) = \phi(T_{x_0,s,\theta}(x))$. It is easy to see that $u - \phi_{x_0,s,\theta}$ has a local maximum at $\hat{y} = T_{x_0,s,\theta}(\hat{x})$. Then,

$$\nabla_y \phi_{x_0,s,\theta}(y) = \nabla_x \phi\big(T_{x_0,s,\theta}(y)\big)^t \cdot A_\theta^{-1} B A_\theta$$

and

$$D_y^2\phi_{x_0,s,\theta}(y) = \left(A_\theta^{-1}BA_\theta\right)^t \cdot D_x^2\phi\left(T_{x_0,s,\theta}(y)\right) \cdot A_\theta^{-1}BA_\theta$$

where

$$A_\theta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Finally, by definition of u as a viscosity solution of (1), and the invariance hypothesis $(F4)$ we get

$$F\left(\nabla_x\phi(\hat{x}), D_x^2\phi(\hat{x})\right) \leqslant f\left(u_{x_0,s,\theta}(\hat{x})\right)$$

which is what we aimed for. $\square$

Given any $x \in \mathbb{R}$, since f is locally Lipschitz continuous and $f(0) \geqslant 0$, it is standard to see that u satisfies an equation like (4). Then Proposition 11 implies that the solution u is actually positive in the interior of the domain and by Hopf Lemma (Proposition 11) for every $x \in \mathbb{R}$,

$$u_y(x,0) = \frac{\partial u}{\partial y}(x,0) > 0.$$

However, $u_y(x,0)$ possibly goes to 0 if $x \to \pm\infty$. So, we fix x_0 and h such that

$$\frac{\partial u}{\partial y}(x,y) \geqslant \gamma > 0, \quad \forall (x,y) \in Q_h(x_0),$$

where

$$Q_h(x_0) = \left\{(x,y)\colon |x - x_0| \leqslant h,\ 0 \leqslant y \leqslant 2h\right\}, \tag{8}$$

as shown in Fig. 3. Note that such $\gamma > 0$ exists as a consequence of the $C^{1,\alpha}$ regularity of u, see Remark 3.

Also, since $u \in C^{1,\alpha}$, we may assume that there exists

$$\delta_1 = \delta_1(h, \gamma, x_0) > 0 \tag{9}$$

such that, if $|\theta| \leqslant \delta_1$ (and consequently $V_\theta \approx e_2$), we have

$$\frac{\partial u}{\partial V_\theta} \geqslant \frac{\gamma}{2} > 0 \quad \text{in } Q_h(x_0). \tag{10}$$

Claim 1. *Let $Q_h(x_0)$ as in (8) and δ_1 defined in (9) and fix $\theta \neq 0$ with $|\theta| \leqslant \delta_1$. Then it is possible to find $\bar{s} = \bar{s}(\theta)$ such that for any $s \leqslant \bar{s}$ the triangle $\mathscr{T}_{x_0,s,\theta}$ is contained in $Q_h(x_0)$ and $u < u_{x_0,s,\theta}$ in $\mathscr{T}_{x_0,s,\theta}$ (with $u \leqslant u_{x_0,s,\theta}$ on $\partial\mathscr{T}_{x_0,s,\theta}$), see Fig. 4.*

To prove Claim 1, fix θ such that $|\theta| \leqslant \delta_1$ and set $\bar{s} \leqslant h$ such that, for $s \leqslant \bar{s}$:

- The triangle $\mathscr{T}_{x_0,s,\theta}$ is contained in $Q_h(x_0)$ as well as the triangle obtained from $\mathscr{T}_{x_0,s,\theta}$ by reflection with respect to the line $L_{x_0,s,\theta}$ (see Fig. 4). Note that this is possible by simple geometric considerations.

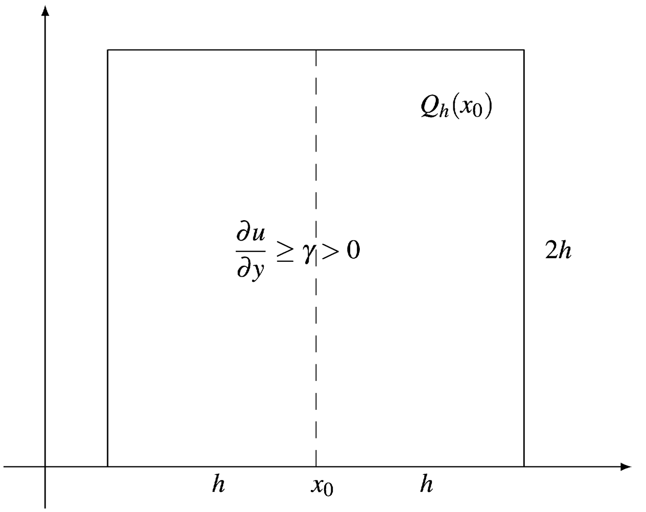

Fig. 3. Monotonicity region.

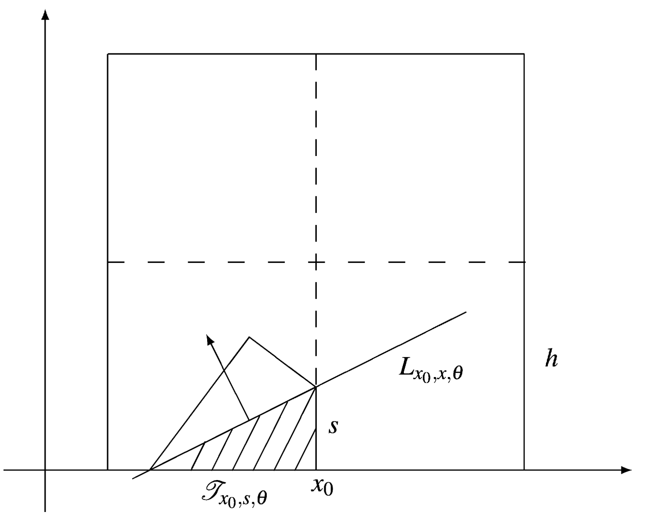

Fig. 4. The starting domain.

- $u \leqslant u_{x_0,s,\theta}$ on $\partial\mathcal{T}_{x_0,s,\theta}$. In fact, since $|\theta| \leqslant \delta_1$ then $u \leqslant u_{x_0,s,\theta}$ on the line (x_0, y) for $0 \leqslant y \leqslant s$, as a consequence of the monotonicity in the V_θ-direction, by construction, see (10). Also $u \leqslant u_{x_0,s,\theta}$ if $y = 0$ by the Dirichlet assumption, and the fact that u is positive in the interior of the domain. Finally $u \equiv u_{x_0,s,\theta}$ on $L_{x_0,s,\theta}$.

With this construction, for $0 < s \leqslant \bar{s}$, we have that

$$w_{x_0,s,\theta} = u - u_{x_0,s,\theta} \leqslant 0 \quad \text{on } \partial\mathcal{T}_{x_0,s,\theta}. \tag{11}$$

Indeed, $w_{x_0,s,\theta}$ satisfies a differential inequality (in the viscosity sense) in the triangle $\mathcal{T}_{x_0,s,\theta}$ to which the Maximum Principle in small domains (Proposition 10 via Lemma 13) applies; this is the content of the following lemma.

Lemma 13. *The difference $w_{x_0,s,\theta} = u - u_{x_0,s,\theta}$ satisfies*

$$\mathcal{P}^-_{\theta,\Theta}\big(D^2 w_{x_0,s,\theta}(x)\big) - \gamma\big|\nabla w_{x_0,s,\theta}(x)\big| \leqslant c_{x_0,s,\theta}(x) w_{x_0,s,\theta}(x) \quad \text{in } \mathcal{T}_{x_0,s,\theta}, \tag{12}$$

in the viscosity sense, with

$$c_{x_0,s,\theta}(x) = \begin{cases} \frac{f(u(x)) - f(u_{x_0,s,\theta}(x))}{u - u_{x_0,s,\theta}}, & \text{if } u_{x_0,s,\theta}(x) \neq u(x), \\ 0, & \text{otherwise}. \end{cases}$$

Remark 14. Notice that since f is Lipschitz, we have $c_{x_0,s,\theta}(x) \in L^\infty$. It is also worth emphasizing that the difficulty of the above result is the lack of regularity of u, as the result follows obviously from the structure condition $(F3)$ and Lemma 12 when u is of class $\mathcal{C}^2$.

Proof of Lemma 13. The proof follows the ideas in [12]. To this end, let $\phi \in \mathcal{C}^2$ such that $u - u_{x_0,s,\theta} - \phi$ has a local maximum at some point $\hat{x} \in \mathcal{T}_{x_0,s,\theta}$. As usual in the theory of viscosity solutions, let us introduce for every $\varepsilon > 0$

$$\Phi_\varepsilon(x, y) = u(x) - u_{x_0,s,\theta}(y) - \phi(x) - \frac{|x-y|^2}{\varepsilon^2} - |x - \hat{x}|^4.$$

For ε small enough, Φ_ε attains a maximum in $\mathcal{T}_{x_0,s,\theta} \times \mathcal{T}_{x_0,s,\theta}$ at some point $(x_\varepsilon, y_\varepsilon) \in B_r(\hat{x}) \times B_r(\hat{x})$ for some $r > 0$. Since $\hat{x}$ is a local strict maximum of

$$x \mapsto u(x) - u_{x_0,s,\theta}(x) - \phi(x) - |x - \hat{x}|^4,$$

standard results of the theory of viscosity solutions (see [11, Proposition 3.7]) yield $x_\varepsilon, y_\varepsilon \to \hat{x}$ and $\frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon^2} \to 0$ as $\varepsilon \to 0$.

In addition, defining $\psi(x, y) = \phi(x) + \frac{|x-y|^2}{\varepsilon^2} + |x - \hat{x}|^4$, Theorem 3.2 in [11] implies that for any given $\alpha > 0$, there exist matrices $X, Y \in S^2$ such that

$$\begin{aligned} \big(\nabla_x \psi(x_\varepsilon, y_\varepsilon), X\big) &\in \bar{J}^{2,+} u(x_\varepsilon), \\ \big(-\nabla_y \psi(x_\varepsilon, y_\varepsilon), Y\big) &\in \bar{J}^{2,-} u_{x_0,s,\theta}(y_\varepsilon), \end{aligned} \tag{13}$$

and

$$-\left(\frac{1}{\alpha} + \|A\|\right) I \leqslant \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leqslant A + \alpha A^2,$$

where $A = D^2\psi(x_\varepsilon, y_\varepsilon)$. From this, setting $\alpha = \varepsilon^2$, an elementary computation yields

$$X - Y \leqslant D^2\phi(x_\varepsilon) + O\big(\varepsilon^2 + |x_\varepsilon - \hat{x}|^2\big) I.$$

By Definition 5 of viscosity solutions and (13), we get

$$F\big(\nabla_x \psi(x_\varepsilon, y_\varepsilon), X\big) \leqslant f\big(u(x_\varepsilon)\big) \quad \text{and} \quad F\big(-\nabla_y \psi(x_\varepsilon, y_\varepsilon), Y\big) \geqslant f\big(u_{x_0,s,\theta}(y_\varepsilon)\big),$$

and subtracting the previous inequalities we obtain

$$\begin{aligned} f\big(u(x_\varepsilon)\big) - f\big(u_{x_0,s,\theta}(y_\varepsilon)\big) &\geqslant F\big(\nabla_x \psi(x_\varepsilon, y_\varepsilon), X\big) - F\big(-\nabla_y \psi(x_\varepsilon, y_\varepsilon), Y\big) \\ &\geqslant \mathscr{P}^-_{\theta,\Theta}(X - Y) - \gamma \big|\nabla_x \psi(x_\varepsilon, y_\varepsilon) + \nabla_y \psi(x_\varepsilon, y_\varepsilon)\big| \\ &\geqslant \mathscr{P}^-_{\theta,\Theta}\big(D^2\phi(x_\varepsilon)\big) - \gamma \big|\nabla \phi(x_\varepsilon)\big| + O\big(\varepsilon^2 + |x_\varepsilon - \hat{x}|^2\big). \end{aligned}$$

Letting $\varepsilon \to 0$, we get

$$f\big(u(\hat{x})\big) - f\big(u_{x_0,s,\theta}(\hat{x})\big) \geqslant \mathscr{P}^-_{\theta,\Theta}\big(D^2\phi(\hat{x})\big) - \gamma \big|\nabla \phi(\hat{x})\big|,$$

and we deduce that (12) holds in the viscosity sense. $\square$

Possibly reducing $\bar{s}$, we can assume that the triangle $\mathscr{T}_{x_0,s,\theta}$ has sufficiently small measure in order to exploit the Maximum Principle in small domains (Proposition 10). Then, from (11) and (12), we get

$$w_{x_0,s,\theta} \leqslant 0 \quad \text{in } \mathscr{T}_{x_0,s,\theta}.$$

Also, since the case $w_{x_0,s,\theta} \equiv 0$ is clearly impossible, by the Strong Maximum Principle (Proposition 11), we have

$$w_{x_0,s,\theta} < 0 \quad \text{in } \mathscr{T}_{x_0,s,\theta}$$

and Claim 1 follows.

In the sequel we shall make repeated use of a technique which is the product of the “moving plane technique”, the “rotating plane technique” and the “sliding plane technique”. Let us explain next these techniques in an axiomatic way for future use.

Given (x_0, s, θ) and $\mathscr{T}_{x_0,s,\theta}$ as above, assume that,

$$w_{x_0,s,\theta} \leqslant 0 \quad \text{on } \partial\mathscr{T}_{x_0,s,\theta}, \quad \text{and} \quad w_{x_0,s,\theta} < 0 \quad \text{in } \mathscr{T}_{x_0,s,\theta}, \tag{$\star$}$$

and suppose that for some (s', θ') sufficiently close to (s, θ) so that, $\mathscr{T}_{x_0,s',\theta'} \approx \mathscr{T}_{x_0,s,\theta}$, we have,

$$w_{x_0,s',\theta'} \leqslant 0 \quad \text{on } \partial\mathscr{T}_{x_0,s',\theta'}. \tag{I}$$

Since $w_{x_0,s,\theta} < 0$ in $\mathscr{T}_{x_0,s,\theta}$, we can carve a compact set $K \subset \mathscr{T}_{x_0,s,\theta}$ where $w_{x_0,s,\theta} \leqslant \rho < 0$. If (s', θ') are chosen appropriately close to (s, θ), we can assume without loss of generality that $K \subset \mathscr{T}_{x_0,s',\theta'}$,

$$w_{x_0,s',\theta'} \leqslant \frac{\rho}{2} < 0 \quad \text{in } K, \tag{14}$$

and the Lebesgue measure of $\mathscr{T}_{x_0,s',\theta'} \setminus K$ is small enough for the Maximum Principle in small domains to apply.

Therefore, since $w_{x_0,s',\theta'} \leqslant 0$ on $\partial(\mathscr{T}_{x_0,s',\theta'} \setminus K)$ by (I) and (14), the Maximum Principle in small domains (Proposition 10) yields,

$$w_{x_0,s',\theta'} \leqslant 0 \quad \text{in } \mathscr{T}_{x_0,s',\theta'} \setminus K$$

and consequently in the whole $\mathscr{T}_{x_0,s',\theta'}$. Then, by the Strong Maximum Principle (see Theorem 11), we get

$$w_{x_0,s',\theta'} < 0 \quad \text{in } \mathscr{T}_{x_0,s',\theta'}.$$

Summarizing, the outcome of the above argument is that after small translations and rotations, we can recover for $\mathcal{T}_{x_0,s',\theta'}$ the same situation we initially had in $\mathcal{T}_{x_0,s,\theta}$, that is $(\star)$. More explicitly, we get that for (s',θ') sufficiently close to (s,θ),

$$w_{x_0,s',\theta'} \leqslant 0 \quad \text{on } \partial\mathcal{T}_{x_0,s',\theta'} \quad \text{and} \quad w_{x_0,s',\theta'} < 0 \quad \text{in } \mathcal{T}_{x_0,s',\theta'}.$$

Let us now show that, the fact that we can make small translations and rotations of $\mathcal{T}_{x_0,s,\theta}$ towards $\mathcal{T}_{x_0,s',\theta'}$ when $(s',\theta') \approx (s,\theta)$, implies that *we can also make larger translations and rotations*.

More precisely, let us fix (s,θ) for which $(\star)$ holds and let $(\bar{s},\bar{\theta})$ be such that there exists a continuous function

$$g : [0,1] \to \mathbb{R}^2,$$
$$t \mapsto \big(s(t),\theta(t)\big)$$

with $g(0) = (s,\theta)$, $g(1) = (\bar{s},\bar{\theta})$ and $\theta(t) \neq 0$ for every $t \in [0,1)$. Finally, suppose that (I) holds for every $t \in [0,1)$, that is, suppose that,

$$w_{x_0,s(t),\theta(t)} \leqslant 0 \text{ and not identically zero on } \partial\mathcal{T}_{x_0,s(t),\theta(t)}, \ \forall t \in [0,1).$$

The above arguments imply that we can find some small $\tilde{t} > 0$ such that, for $0 < t \leqslant \tilde{t}$,

$$w_{x_0,s(t),\theta(t)} \leqslant 0 \quad \text{on } \partial\mathcal{T}_{x_0,s(t),\theta(t)} \quad \text{and} \quad w_{x_0,s(t),\theta(t)} < 0 \quad \text{in } \mathcal{T}_{x_0,s(t),\theta(t)}. \tag{15}$$

We now let,

$$\overline{T} \equiv \big\{\tilde{t} \in [0,1] \text{ s.t. (15) holds for any } 0 \leqslant t \leqslant \tilde{t}\big\} \text{ and set } \bar{t} = \sup_{t\in\overline{T}} t.$$

Notice that we have proved that $\bar{t} > 0$. The argument concludes by showing that, actually, $\bar{t} = 1$. To prove this, we proceed by contradiction and assume $\bar{t} < 1$. Then, by continuity

$$w_{x_0,s(\bar{t}),\theta(\bar{t})} \leqslant 0 \quad \text{in } \mathcal{T}_{x_0,s(\bar{t}),\theta(\bar{t})}$$

and, by the Strong Maximum Principle

$$w_{x_0,s(\bar{t}),\theta(\bar{t})} < 0 \quad \text{in } \mathcal{T}_{x_0,s(\bar{t}),\theta(\bar{t})}.$$

As we are now in the situation described in $(\star)$ and (I), we can argue as above and show that it is still possible to push the plane slightly further, that is, to find a sufficiently small $\varepsilon > 0$ so that (15) holds for any $0 \leqslant t \leqslant \bar{t} + \varepsilon$, a contradiction with the definition of $\bar{t}$, therefore implying $\bar{t} = 1$. Summarizing, by means of this argument we get, $w_{x_0,\bar{s},\bar{\theta}} \leqslant 0$ on $\partial\mathcal{T}_{x_0,\bar{s},\bar{\theta}}$ and $w_{x_0,\bar{s},\bar{\theta}} < 0$ in $\mathcal{T}_{x_0,\bar{s},\bar{\theta}}$.

Now, we are going to apply the techniques just described axiomatically to the proof of Theorem 1. Let x_0, $Q_h(x_0)$ and δ_1 as in (8), (9). Define,

$$\Sigma_t = \big\{(x,y) \bigm| 0 < y < t\big\}.$$

We aim to prove the following,

Claim 2. *Given any $\tilde{s}$ with $0 < \tilde{s} \leqslant h$, we have $u < u_{\tilde{s}}$ in $\Sigma_{\tilde{s}}$, and also clearly $u \leqslant u_{\tilde{s}}$ on $\partial\Sigma_{\tilde{s}}$ (recall that $u_{\tilde{s}}$ stands for $u_{x_0,\tilde{s},0}$).*

To prove this, let us first fix θ such that $|\theta| \leqslant \delta_1$. Consequently, by Claim 1, we can find some $s = s(\theta) \leqslant \tilde{s}$ such that the triangle $\mathscr{T}_{x_0,s,\theta}$ is contained in $Q_h(x_0)$ (see Fig. 4), $u < u_{x_0,s,\theta}$ in $\mathscr{T}_{x_0,s,\theta}$ (and $u \leqslant u_{x_0,s,\theta}$ on $\partial\mathscr{T}_{x_0,s,\theta}$).

Our purpose now is to enlarge the triangle $\mathscr{T}_{x_0,s,\theta}$ by applying the axiomatic arguments above to particular cases of the transformation $g(t)$, mainly, translations and rotations. The idea is to show that we can actually reach $\Sigma_{\tilde{s}}$ with these small perturbations of the initial triangle. In order to be able to do so, we shall have to check the hypothesis corresponding to $(\star)$, (I) and (15).

Sliding technique: We start moving the line $L_{x_0,s,\theta}$ in the e_2-direction towards the line $L_{x_0,\tilde{s},\theta}$, keeping θ fixed and moving $s \to \tilde{s}$. In the notation above, we have $g(t) = (s(t), \theta)$ with $s(0) = s$ and $s(1) = \tilde{s}$, where in particular we can assume that $s(t) \leqslant \tilde{s}$.

We note that for every $s(t) \leqslant \tilde{s}$ we have $u \leqslant u_{x_0,s(t),\theta}$ on $\partial\mathscr{T}_{x_0,s(t),\theta}$. To see this, notice that since $|\theta| \leqslant \delta_1$, then $u \leqslant u_{x_0,s(t),\theta}$ on the line (x_0, y) for $0 \leqslant y \leqslant s(t)$, because of the monotonicity in the V_θ-direction, that we have by construction. Also $u < u_{x_0,s(t),\theta}$ if $y = 0$ by the Dirichlet assumption, and the fact that u is positive in the interior of the domain. And finally, $u \equiv u_{x_0,s(t),\theta}$ on $L_{x_0,s(t),\theta}$ by definition.

This shows that we have the right conditions (I) on the boundary for every $s(t) \leqslant \tilde{s}$; therefore by the technique described above, we get,

$$u < u_{x_0,\tilde{s},\theta} \quad \text{in } \mathscr{T}_{x_0,\tilde{s},\theta} \quad \text{and} \quad u \leqslant u_{x_0,\tilde{s},\theta} \quad \text{on } \partial\mathscr{T}_{x_0,\tilde{s},\theta}.$$

Rotating technique: After having reached $\tilde{s}$, we start rotating the line $L_{x_0,\tilde{s},\theta}$ towards the line $\{y = \tilde{s}\}$, keeping $\tilde{s}$ fixed and letting $\theta \to 0$. That is, in the notation above, we consider $g(t) = (\tilde{s}, \theta(t))$ with $\theta(0) = \theta$ and $\theta(1) = 0$, where in particular we can assume that $\theta(t) \neq 0$ if $t \neq 1$. It can be easily checked, exactly in the same way as in the sliding technique, that we still have the right conditions on the boundary, namely, that (I) holds.

Note that to start from θ we need $|\theta| \leqslant \delta_1$. Therefore we can assume either that $0 < \theta(t) \leqslant \delta_1$, or that $0 < -\theta(t) \leqslant \delta_1$. Consider first the case when θ is positive. By the rotating plane technique, at the limit it follows that,

$$u(x, y) < u_{\tilde{s}}(x, y) = u(x, 2\tilde{s} - y) \quad \text{in } \Sigma_{\tilde{s}} \cap \{x \leqslant x_0\}.$$

In the second case, we start from a negative θ, whence it follows that

$$u(x, y) < u_{\tilde{s}}(x, y) = u(x, 2\tilde{s} - y) \quad \text{in } \Sigma_{\tilde{s}} \cap \{x \geqslant x_0\}.$$

Finally,

$$u(x, y) < u_{\tilde{s}}(x, y) = u(x, 2\tilde{s} - y) \quad \text{in } \Sigma_{\tilde{s}}$$

for every $0 < \tilde{s} \leqslant h$, proving Claim 2.

We now point out some consequences. First, note that u is strictly monotone increasing in the e_2-direction in Σ_h. In fact, given (x, y_1) and (x, y_2) in Σ_h (say $0 \leqslant y_1 < y_2 \leqslant h$), we have proved in Claim 2 that $u(x, y_1) < u_{\frac{y_1+y_2}{2}}(x, y_1)$ which yields,

$$u(x, y_1) < u(x, y_2).$$

This immediately gives $\frac{\partial u}{\partial y} \geqslant 0$ in Σ_h, but, actually, it is possible to show that

$$\frac{\partial u}{\partial y} > 0 \quad \text{in } \Sigma_h.$$

To do this, we use that for every $\tilde{s} \leqslant h$, the difference $w_{\tilde{s}} = u - u_{\tilde{s}}$ fulfills an equation of the type of (12) in $\Sigma_{\tilde{s}}$ (see Lemma 13). Then, Hopf's Lemma (Proposition 11) implies that,

$$\frac{\partial u}{\partial y}(x, \tilde{s}) = \frac{1}{2}\frac{\partial w_{\tilde{s}}}{\partial y}(x, \tilde{s}) > 0, \quad \forall \tilde{s} \leqslant h. \tag{16}$$

Let us define,

$$\Lambda = \left\{\lambda \in \mathbb{R}^+ \colon u < u_{\lambda'} \text{ in } \Sigma_{\lambda'} \text{ for every } \lambda' < \lambda\right\},$$

and,

$$\bar{\lambda} = \sup_{\lambda \in \Lambda} \lambda. \tag{17}$$

From Claim 2, we know that $\bar{\lambda} \geqslant h > 0$. As before, $u \leqslant u_{\bar{\lambda}}$ by continuity, which implies $u < u_{\bar{\lambda}}$ by the Strong Maximum Principle. Moreover, as in (16), we have,

$$\frac{\partial u}{\partial y} > 0 \quad \text{in } \Sigma_{\bar{\lambda}}. \tag{18}$$

To finish the proof of Theorem 1 we have to show

Claim 3. *Actually, $\bar{\lambda} = \infty$.*

The proof is by contradiction, so we assume that $\bar{\lambda} < \infty$. We shall show that we can find $\varepsilon > 0$ small enough so that,

$$u < u_{\lambda'} \quad \text{for } 0 < \lambda' \leqslant \bar{\lambda} + \varepsilon, \tag{19}$$

a contradiction with the definition of $\bar{\lambda}$. This would finish the proof of Theorem 1.

In the proof of Claim 3, we shall need the following result, whose proof follows from [4,13]. We provide the details for the sake of completeness.

Lemma 15. *Consider $x_0 \in \mathbb{R}$ and $\bar{\lambda} > 0$ such that,*

1. $\frac{\partial u}{\partial y}(x_0, y) > 0$ *for every* $y \in [0, \bar{\lambda}]$.
2. *For every $\lambda \in (0, \bar{\lambda}]$ we have $u(x_0, y) < u_\lambda(x_0, y) = u(x_0, 2\lambda - y)$ for $y \in [0, \lambda)$.*

Then there exists $\delta_2 > 0$ such that, for any θ such that $|\theta| \leqslant \delta_2$ and for any $\lambda \in (0, \bar{\lambda} + \delta_2]$, we have

$$u(x_0, y) < u_{x_0,\lambda,\theta}(x_0, y) \quad \textit{for } y \in [0, \lambda).$$

Proof. We argue by contradiction. Were the claim false, we could find a sequence $\delta_n \to 0$ and corresponding sequences y_n, λ_n and θ_n, such that,

(i) $-\delta_n \leqslant \theta_n \leqslant \delta_n$.
(ii) $0 < \lambda_n \leqslant \bar{\lambda} + \delta_n$.
(iii) $0 \leqslant y_n < \lambda_n$ with $u(x_0, y_n) \geqslant u_{x_0,\lambda_n,\theta_n}(x_0, y_n)$.

Note that, up to subsequences, $\lambda_n \to \tilde{\lambda} \leqslant \bar{\lambda}$ and $y_n \to \tilde{y}$ for some $\tilde{y} \leqslant \tilde{\lambda}$. Let us prove that actually $\tilde{y} = \tilde{\lambda}$. If $\tilde{\lambda} = 0$, this follows by the fact that $\tilde{y} = \tilde{\lambda} = 0$ since $0 \leqslant y_n < \lambda_n$. Otherwise if $\tilde{\lambda} > 0$, it follows by continuity that $u(x_0, \tilde{y}) \geqslant u_{\tilde{\lambda}}(x_0, \tilde{y})$. Consequently $\tilde{y} = \tilde{\lambda}$ since we know that $u < u_\lambda$ for every $\lambda \leqslant \bar{\lambda}$ and $y \in [0, \lambda)$.

Since $u(x_0, y_n) \geqslant u_{x_0,\lambda_n,\theta_n}(x_0, y_n)$, it follows from the mean value theorem that

$$\frac{\partial u}{\partial V_{\theta_n}}(\tilde{x}_n, \tilde{y}_n) \leqslant 0$$

at some point $(\tilde{x}_n, \tilde{y}_n)$ belonging to the segment connecting (x_0, y_n) to $T_{x_0,\lambda_n,\theta_n}(x_0, y_n)$. We recall that the vector V_{θ_n} is orthogonal to the line $L_{x_0,\lambda_n,\theta_n}$ and $V_{\theta_n} \to e_2$ since $\theta_n \to 0$. Therefore, passing to the limit, it follows that,

$$\frac{\partial u}{\partial y}(x_0, \tilde{\lambda}) \leqslant 0,$$

which is impossible by assumption. $\square$

Remark 16. If $\bar{\lambda}$ is given by (17) we are in the hypothesis of Lemma 15 for any $x_0 \in \mathbb{R}$, since the difference $w_{\bar{\lambda}} = u - u_{\bar{\lambda}}$ fulfills an equation of the type (12) and we can argue as in (16) to prove that,

$$\frac{\partial u}{\partial y}(x_0, \bar{\lambda}) > 0.$$

We are going to prove (19) with $\varepsilon = \delta_2$ given by Lemma 15. Let us fix θ with $|\theta| \leqslant \delta_2$ and λ small enough so that Claim 1 applies. From Claim 1, we get that the triangle $\mathscr{T}_{x_0,\lambda,\theta}$ is contained in $Q_h(x_0)$ (see Fig. 4), and

$$u < u_{x_0,\lambda,\theta} \quad \text{in } \mathscr{T}_{x_0,\lambda,\theta} \quad \text{with} \quad u \leqslant u_{x_0,\lambda,\theta} \quad \text{on } \partial\mathscr{T}_{x_0,\lambda,\theta}.$$

Following the proof in Claim 2 we now start sliding the line $L_{x_0,\lambda,\theta}$ in the e_2-direction towards the line $L_{x_0,\bar{\lambda}+\delta_2,\theta}$, keeping θ fixed and letting $\lambda \to \bar{\lambda} + \delta_2$. First we have to check that the appropriate boundary conditions hold, that is to show that for every $\lambda \leqslant \bar{\lambda} + \delta_2$ we have $u \leqslant u_{x_0,\lambda,\theta}$ on $\partial\mathscr{T}_{x_0,\lambda,\theta}$. In fact, since $|\theta| \leqslant \delta_2$ then by Lemma 15 $u < u_{x_0,\lambda,\theta}$ on the line (x_0, y) for $0 \leqslant y < \lambda$. As before, $u \leqslant u_{x_0,\lambda,\theta}$ if $y = 0$ by the Dirichlet assumption, and finally $u = u_{x_0,\lambda,\theta}$ on $L_{x_0,\lambda,\theta}$. Therefore the sliding technique described above, yields,

$$u < u_{x_0,\bar{\lambda}+\delta_2,\theta} \quad \text{in } \mathscr{T}_{x_0,\bar{\lambda}+\delta_2,\theta}, \quad \text{and} \quad u \leqslant u_{x_0,\bar{\lambda}+\delta_2,\theta} \quad \text{on } \partial\mathscr{T}_{x_0,\bar{\lambda}+\delta_2,\theta}.$$

We would like to stress that in the application of the sliding and rotating techniques during the proof of Claim 2, it was crucial to ensure that the vertical side of the triangle $\mathscr{T}_{x_0,s,\theta}$ and the segment resulting from its reflection with respect to $L_{x_0,s,\theta}$ were always inside $Q_h(x_0)$, as this fact was necessary in order to check the right boundary conditions ($\star$) and (I). The role of Lemma 15 is to show that when the perturbations are small enough, the right conditions still hold even if the vertical side of the triangle is outside $Q_h(x_0)$ and we cannot rely on monotonicity anymore.

We now start rotating the line $L_{x_0,\bar{\lambda}+\delta_2,\theta}$ towards the line $y = \bar{\lambda} + \delta_2$, freezing $\bar{\lambda} + \delta_2$ and letting $\theta \to 0$ as in the rotating technique. Again, we use Lemma 15 to check that we have the right boundary conditions. Exactly as in Claim 2, if we keep θ positive then at the limit $\theta \to 0$ we get $u < u_{\bar{\lambda}+\delta_2}$ in $\Sigma_{\bar{\lambda}+\delta_2} \cap \{x \leqslant x_0\}$. Otherwise if θ is negative, it follows that $u < u_{\bar{\lambda}+\delta_2}$ in $\Sigma_{\bar{\lambda}+\delta_2} \cap \{x \geqslant x_0\}$. Finally,

$$u < u_{\bar{\lambda}+\delta_2} \quad \text{in } \Sigma_{\bar{\lambda}+\delta_2},$$

a contradiction with the definition of $\bar{\lambda}$. This proves Claim 3 and hence, according to (18), concludes the proof of Theorem 1.

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