Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

Website: http://AIMsciences.org
pp. **X–XX**

LOW DIMENSIONAL INSTABILITY FOR SEMILINEAR AND QUASILINEAR PROBLEMS IN $\mathbb{R}^N$

Daniele Castorina

Dipartimento di Matematica, Universitá di Roma "Tor Vergata"
Via della Ricerca Scientifica, I-00133 Roma, Italy

Pierpaolo Esposito

Dipartimento di Matematica, Universitá di Roma Tre
Largo San Leonardo Murialdo, 1, I-00146 Roma, Italy

Berardino Sciunzi

Dipartimento di Matematica, Universitá della Calabria
V. P. Bucci, I-87036 Arcavacata di Rende (CS), Italy

Abstract. Stability properties for solutions of $-\Delta_m(u) = f(u)$ in $\mathbb{R}^N$ are investigated, where $N \geq 2$ and $m \geq 2$. The aim is to identify a critical dimension $N^\#$ so that every non-constant solution is linearly unstable whenever $2 \leq N < N^\#$. For positive, increasing and convex nonlinearities $f(u)$, global bounds on $\frac{f f''}{(f')^2}$ allows us to find a dimension $N^\#$, which is optimal for exponential and power nonlinearities. In the radial setting we can deal more generally with C^1-nonlinearities and the dimension $N^\#$ we find is still optimal.

1. Introduction and statement of the main results.

Let us consider a solution u of

$$-\Delta_m(u) = f(u) \qquad \text{in } \mathbb{R}^N, \tag{1}$$

where $\Delta_m = \text{div}\,(|\nabla u|^{m-2}\nabla u)$ denotes the m-Laplace operator, $m \geq 2$ and $N \geq 2$.

Due to the singular/degenerate nature of the elliptic operator Δ_m, by [14, 26, 32] the best and natural regularity for a weak-solution u of (1) is $u \in C^{1,\alpha}_{\text{loc}}(\mathbb{R}^N)$, for some $\alpha \in (0,1)$. Therefore equation (1) is to be understood with its weak formulation:

$$\int_{\mathbb{R}^N} |\nabla u|^{m-2}(\nabla u, \nabla \varphi)\, dx = \int_{\mathbb{R}^N} f(u)\varphi\, dx \quad \forall\, \varphi \in C^1_c(\mathbb{R}^N). \tag{2}$$

In the paper, a solution u of (1) is always assumed to be in $C^{1,\alpha}_{\text{loc}}(\mathbb{R}^N)$, $\alpha \in (0,1)$, and to satisfy (2).

We are concerned with stability properties of solutions u of (1). Let us give the following definition:

2000 *Mathematics Subject Classification.* Primary: 35J60,35J70; Secondary: 35B05.

Key words and phrases. $p-$Laplace operator, linear instability, critical dimension.

Authors' research is supported by *MIUR Metodi variazionali ed equazioni differenziali nonlineari.*

Definition 1.1. A solution u of (1) is **stable** if

$$\int_{\mathbb{R}^N} |\nabla u|^{m-2}|\nabla\varphi|^2\,dx + (m-2)\int_{\mathbb{R}^N} |\nabla u|^{m-4}(\nabla u,\nabla\varphi)^2\,dx \\ - \int_{\mathbb{R}^N} f'(u)\varphi^2\,dx \geq 0 \tag{3}$$

for every $\varphi \in C^1_c(\mathbb{R}^N)$. In particular it follows

$$\int_{\mathbb{R}^N} f'(u)\varphi^2\,dx \leq (m-1)\int_{\mathbb{R}^N} |\nabla u|^{m-2}|\nabla\varphi|^2\,dx \tag{4}$$

for every $\varphi \in C^1_c(\mathbb{R}^N)$.

Roughly speaking, the stability condition (3) means that the first eigenvalue of the linearized operator L_u at u is nonnegative. Formally, the linearized operator L_u is defined by duality as

$$L_u(\varphi)[\psi] = \int_{\mathbb{R}^N} |\nabla u|^{m-2}\,(\nabla\varphi,\nabla\psi)\;dx+ \\ + (m-2)\int_{\mathbb{R}^N} |\nabla u|^{m-4}\,(\nabla u,\nabla\varphi)\,(\nabla u,\nabla\psi)\;dx - \int_{\mathbb{R}^N} f'(u)\varphi\psi\,dx \quad \forall\,\psi \in C^1_c(\mathbb{R}^N).$$

For $\varphi \in C^1_c(\mathbb{R}^N)$ the operator L_u is well defined with values in $\left(C^1_c(\mathbb{R}^N)\right)'$. It is possible to define the corresponding first eigenvalue as

$$\lambda_1(L_u) := \inf\{L_u(\varphi)[\varphi] : \; \varphi \in C^1_c(\mathbb{R}^N)\,,\; \int_{\mathbb{R}^N} \varphi^2\,dx = 1\}.$$

Assumption (3) reads exactly as $\lambda_1(L_u) \geq 0$. For our purposes, the functional space $C^1_c(\mathbb{R}^N)$ is sufficiently large to choose good test functions and we won't go deeper into the description of L_u. Let us note that, for $\varphi \in C^1_c(\mathbb{R}^N)$ it is not clear which is the optimal space $L_u(\varphi)$ belongs to and $\lambda_1(L_u)$ is only formally the first eigenvalue of L_u.
However, on a bounded domain Ω with a Dirichlet condition on u this construction has been made rigorous in [6]. In [9, 10] it is shown that $\rho = |\nabla u|^{m-2} \in L^\infty(\Omega)$ with $\rho^{-1} \in L^1(\Omega)$. Then, it is possible to define $H^1_{0,\rho}(\Omega)$ as the completion of $C^1_c(\Omega)$ w.r.t. the weighted norm

$$\|\varphi\|^2_{H^1_{0,\rho}(\Omega)} = \int_\Omega \rho|\nabla\varphi|^2\,dx + \int_\Omega \varphi^2\,dx.$$

In this way L_u is well defined as an operator from the Hilbert space $H^1_{0,\rho}(\Omega)$ into itself. The first eigenvalue $\lambda_1(L_u)$ is attained in $H^1_{0,\rho}(\Omega)$ and has the usual properties.

The aim of the paper is to show that, in low dimensions, stable solutions of (1) are necessarily trivial. A similar phenomenon has been already investigated in other contexts and is strictly related to regularity and compactness issues via a blow up procedure: minimal hypersurfaces in $\mathbb{R}^N$ [31], minimizing harmonic maps from $\mathbb{R}^N$ into spheres/hemispheres [25, 30], De Giorgi's conjecture [1, 12, 13, 21, 23, 28, 29]. Semilinear problems (1) (i.e. $m = 2$) with exponential/polynomial nonlinearities have been considered in [2, 11, 17, 18, 19, 20] and with general nonlinearities in [3]. The quasilinear case $m > 2$ with power nonlinearities has been studied in [8]. Regularity of extremal solutions (or equivalently, compactness of the minimal branch) for nonlinear eigenvalue problems with general nonlinearities have been

considered in [2, 4, 7, 11, 18, 24, 27, 33] in the semilinear case and [5, 6] in the quasilinear case (see also [16] for the compactness of higher branches).
We are interested in obtaining Liouville-type results for (1) both in the semilinear and quasilinear situation, and for general nonlinearities $f(u)$.

First, we focus on the model class composed by polynomial and exponential nonlinearities. Given $\gamma > 0$, define $f_\gamma(u)$ as

$$f_\gamma(u) = \begin{cases} (1+u)^{\frac{1}{1-\gamma}} & \text{if } \gamma < 1 \\ e^u & \text{if } \gamma = 1 \\ (1-u)^{-\frac{1}{\gamma-1}} & \text{if } \gamma > 1. \end{cases} \tag{5}$$

When $f = f_\gamma$ we will consider solutions u of (1) so that $u \geq -1$ if $\gamma < 1$ and $u < 1$ if $\gamma > 1$. Our first main result is the following:

Theorem 1.2. *Let $m \geq 2$ and $\gamma > \frac{m-2}{m-1}$. Assume*

$$2 \leq N < N^\# := \frac{m}{m-1} \frac{2 + 2\sqrt{\gamma(m-1)-(m-2)} + \gamma(m-1)}{\gamma(m-1)-(m-2)}.$$

Then, problem (1) with $f = f_\gamma$ does not possess any non-constant, stable solution u with $f_\gamma(u) \in L^\infty(\mathbb{R}^N)$.

For the nonlinearity $f_\gamma(u)$ in (5), the critical dimension $N^\#$ is optimal. Indeed, for $N \geq N^\#$ the associated nonlinear eigenvalue problem on the unit ball has a non-compact minimal branch u_λ: $\|f_\gamma(u_\lambda)\|_\infty \to +\infty$ as $\lambda \uparrow \lambda^*$, λ^* being the extremal parameter. A suitable rescaling of u_λ converges to a non-constant, stable radial solution u of (1) with $f_\gamma(u) \in L^\infty(\mathbb{R}^N)$. We skip the details of the argument which is by now very well established. We refer to [5] for $m > 1$ and $f(u) = (1+u)^p$, $p > 1$, and to [17] for $m = 2$ and general $f(u)$ in the form (5).
Observe that, when $f_\gamma = (1+u)^p$, we have that $\gamma = \frac{p-1}{p} \to 1^-$ as $p \to +\infty$ and the corresponding critical dimension $N^\# \to m + \frac{4m}{m-1}$ as $p \to +\infty$. This means that for any $N > m + \frac{4m}{m-1}$ we can find p large so that problem (1) with $f = (1+u)^p$ possesses a non-constant, stable bounded solution u which is radial. This will provide the optimality property stated in Theorem 1.4 part c). Moreover, for every $p > 1$ the inequality $N^\# > m + \frac{4m}{m-1}$ holds and explains somehow why in Theorem 1.4 part c) the limiting situation $N = m + \frac{4m}{m-1}$ gives rise to instability.

Theorem 1.2 is a special case of a more general result. Let $f \in C^1[a_0, a_1] \cap C^2(a_0, a_1)$ be a positive, increasing and convex function in (a_0, a_1), where $-\infty \leq a_0 < a_1 < +\infty$ (here and in the sequel, we use the convention $[a_0, a_1] = (a_0, a_1]$ whenever $a_0 = -\infty$).
We will focus on solutions u of (1) so that $a_0 \leq u \leq a_1$ and we assume on $f(u)$ the following condition:

$$\gamma \leq \frac{f(u) f''(u)}{(f')^2(u)} \leq \Gamma \qquad \forall\, u \in (a_0, a_1) \tag{6}$$

for $0 < \gamma \leq \Gamma < +\infty$.

Observe that $f_\gamma(u)$ satisfies assumption (6) with $\gamma = \Gamma$. Viceversa, by a simple integration it is easy to see that the limiting situation $\gamma = \Gamma$ in (6) corresponds exactly to the nonlinearities $f_\gamma(u)$ in (5) (up to a linear change in the variable u and up to a positive factor in front of $f(u)$). Hence, the nonlinearities described by assumption (6) form a class more general than $\{f_\gamma : \gamma > 0\}$.

As far as $f_\gamma(u)$, let us observe that a_0 is -1 when $\gamma < 1$ and $-\infty$ when $\gamma \geq 1$, and $a_1 \in (a_0, +\infty)$ for $\gamma \leq 1$ and $a_1 \in (-\infty, 1)$ for $\gamma > 1$. Theorem 1.2 is then just a consequence of our second main result:

Theorem 1.3. *Let $m \geq 2$ and f be as above. Assume that (6) holds with $\gamma > \frac{m-2}{m-1}$ and let $N^\#$ be defined as*

$$N^\# = \frac{m}{m-1} \frac{2 + 2\sqrt{\gamma(m-1)-(m-2)} + \Gamma(m-1)}{\Gamma(m-1)-(m-2)}. \tag{7}$$

Then, for any $2 \leq N < N^\#$ problem (1) does not possess any non-constant, stable solution u with $a_0 \leq u < a_1$.

In the radial situation, let us consider a general nonlinearity $f \in C^1[a_2, a_3]$, where $-\infty \leq a_2 < a_3 < +\infty$. The approach in [3] for the semilinear case $m = 2$ and bounded solutions u extends to the quasilinear case $m \geq 2$ and to possibly one-side bounded solutions u:

Theorem 1.4. *Let $m \geq 2$. Let f be as above and assume that $f \in L^1(a_2, a_3)$. Either*
a) $N^\#$ is defined as

$$N^\# = \frac{2m + 2\sqrt{m}}{m-1} \tag{8}$$

or
b) $a_2 > -\infty$ and $N^\#$ is defined as

$$N^\# = \frac{3m - 1 + 2\sqrt{2m-1}}{m-1} \tag{9}$$

or
c) $a_2 > -\infty$, $\lim_{u \to a_0} \frac{|f'(u)|}{|u-a_0|^q} \in (0, +\infty)$ for every zero point $a_0 \in \{a \in [a_2, a_3] : f(a) = 0\}$ and for some $q = q(a_0) \geq 0$ and $N^\#$ is defined as

$$N^\# = m + \frac{4m}{m-1}. \tag{10}$$

Then, for any $2 \leq N \leq N^\#$ problem (1) does not possess any radial, non-constant, stable solution u with $a_2 \leq u \leq a_3$. Moreover, the dimension $N^\#$ in case (c) is optimal.

The paper is organized as follows. In Section 2 a class of suitable test functions in the stability assumption (3) yields to strong integrability conditions on $f(u)$ which are impossible in low dimensions for non-constant solutions u. Section 3 is devoted to discuss the radial case contained in Theorem 1.4.

While submitting the paper, we learnt from L. Dupaigne that he and A. Farina have obtained in [15] for the case $m = 2$ stronger results than ours. In particular, a control on $\frac{f f''}{(f')^2}$ near the zeroes of $f(u)$ is sufficient.

2. **Proof of Theorem 1.3.** Our approach is inspired by the techniques developed in [17, 18, 19, 20] for the semilinear case $m = 2$. Let us consider $\alpha > -\min\{\gamma, 1\}$ and set

$$g(u) = \int_{a_0}^{u} f^{\alpha-1}(s)(f')^2(s)\, ds.$$

Observe that $g(u)$ is well defined in $(a_0, a_1]$ and satisfies the crucial estimate

$$g(u) \le \frac{1}{\alpha+\gamma} f^\alpha(u) f'(u) \qquad \forall\, u \in (a_0, a_1] \tag{11}$$

by means of the lower bound in (6). Indeed, fix $a \in (a_0, a_1]$ and compute for $a < u \le a_1$:

$$\begin{aligned}\alpha \int_a^u f^{\alpha-1}(s)(f')^2(s)\, ds &= f^\alpha(u)f'(u) - f^\alpha(a)f'(a) - \int_a^u f^\alpha(s) f''(s)\, ds \\ &\le f^\alpha(u) f'(u) - \gamma \int_a^u f^{\alpha-1}(s)(f')^2(s)\, ds.\end{aligned}$$

Hence, it follows that

$$\int_a^u f^{\alpha-1}(s)(f')^2(s)\, ds \le \frac{1}{\alpha+\gamma} f^\alpha(u) f'(u)$$

for every $u \in (a, a_1]$. Letting $a \to a_0$, $g(u)$ is well defined in $(a_0, a_1]$ and satisfies (11).
The upper bound in (6) implies that the function f'/f^Γ is non-increasing and

$$\frac{f'(u)}{f^\Gamma(u)} \ge \frac{f'(a_1)}{f^\Gamma(a_1)} > 0$$

for every $a_0 < u \le a_1$. Therefore, the estimate

$$f^\Gamma(u) \le k f'(u) \tag{12}$$

does hold for any $u \in (a_0, a_1]$, for some positive constant k.

Let us now explain the strategy. Given $\chi \in C_c^\infty(\mathbb{R}^N)$ so that $0 \le \chi \le 1$, we take $\varphi = \chi^m g(u)$ as a test function in (2) to deduce by (4) some integrability condition on $f(u)$. In low dimensions, this will be a very strong condition and will imply the "triviality" of u.

Since $\nabla\varphi = \chi^m f^{\alpha-1}(u)(f')^2(u)\nabla u + m\chi^{m-1} g(u)\nabla\chi$, by (2) we have

$$\begin{aligned}&\int_{\mathbb{R}^N} \chi^m f^{\alpha-1}(u)(f')^2(u)|\nabla u|^m\, dx \\ &= \int_{\mathbb{R}^N} |\nabla u|^{m-2}(\nabla u, \nabla\varphi)\, dx - m\int_{\mathbb{R}^N} \chi^{m-1} g(u)|\nabla u|^{m-2}(\nabla u, \nabla\chi)\, dx \\ &= \int_{\mathbb{R}^N} \chi^m f(u) g(u)\, dx - m\int_{\mathbb{R}^N} \chi^{m-1} g(u)|\nabla u|^{m-2}(\nabla u, \nabla\chi)\, dx.\end{aligned} \tag{13}$$

As a consequence of the stability condition on u, we have the validity of (4) which, applied to $\varphi = \frac{2}{\alpha+1}\chi^{\frac{m}{2}} f^{\frac{\alpha+1}{2}}(u)$, leads to

$$\begin{aligned}&\frac{4}{(m-1)(\alpha+1)^2}\int_{\mathbb{R}^N} \chi^m f^{\alpha+1}(u) f'(u)\, dx \\ &\le \frac{4}{(\alpha+1)^2}\int_{\mathbb{R}^N} |\nabla u|^{m-2}\big|\nabla(\chi^{\frac{m}{2}} f^{\frac{\alpha+1}{2}}(u))\big|^2\, dx \\ &= \int_{\mathbb{R}^N} \chi^m f^{\alpha-1}(u)(f')^2(u)|\nabla u|^m\, dx \\ &+ \frac{m^2}{(\alpha+1)^2}\int_{\mathbb{R}^N} \chi^{m-2}|\nabla\chi|^2 f^{\alpha+1}(u)|\nabla u|^{m-2}\, dx \\ &+ \frac{2m}{\alpha+1}\int_{\mathbb{R}^N} \chi^{m-1} f^\alpha(u) f'(u)|\nabla u|^{m-2}(\nabla u, \nabla\chi)\, dx.\end{aligned} \tag{14}$$

Formula (13) into (14) yields to

$$\begin{aligned}&\int_{\mathbb{R}^N}\chi^m f(u)\left(\frac{4}{(m-1)(\alpha+1)^2}f^\alpha(u)f'(u)-g(u)\right)dx\\&\le\frac{m^2}{(\alpha+1)^2}\int_{\mathbb{R}^N}\chi^{m-2}|\nabla\chi|^2 f^{\alpha+1}(u)|\nabla u|^{m-2}\,dx\\&+\int_{\mathbb{R}^N}\chi^{m-1}\left(\frac{2m}{\alpha+1}f^\alpha(u)f'(u)-mg(u)\right)|\nabla u|^{m-2}(\nabla u,\nabla\chi)\,dx.\end{aligned}\tag{15}$$

Set $G(u)=\int_{a_0}^u g(u)$ and observe that by (11) we get $0\le G(u)\le Cf^{\alpha+1}(u)$ in view of $\alpha>-1$. We are now in position to show:

Proof. **(of Theorem 1.3)** First, observe that the estimate

$$\begin{aligned}&\int_{\mathbb{R}^N}\chi^m f^{\alpha-1}(u)(f')^2(u)|\nabla u|^m dx\\&\le\frac{1}{1-\varepsilon}\int_{\mathbb{R}^N}\chi^m f(u)g(u)\,dx+D_\varepsilon\int_{\mathbb{R}^N}|\nabla\chi|^m f^{\alpha+2\Gamma-1+m(1-\Gamma)}(u)\,dx\end{aligned}\tag{16}$$

does hold for every $\varepsilon>0$ small and D_ε a suitable large constant. Estimate (16) will be the main tool to control the gradient terms in (15) as we will see.

As far as (16), inserting (11) into (13) we get

$$\begin{aligned}&\int_{\mathbb{R}^N}\chi^m f^{\alpha-1}(u)(f')^2(u)|\nabla u|^m dx\\&\le\int_{\mathbb{R}^N}\chi^m f(u)g(u)\,dx+\frac{m}{\alpha+\gamma}\int_{\mathbb{R}^N}\chi^{m-1}|\nabla\chi|f^\alpha(u)f'(u)|\nabla u|^{m-1}\,dx.\end{aligned}\tag{17}$$

Split now

$$f^\alpha(u)=f^{(\alpha-1)\frac{m-1}{m}}(u)f^{\frac{\alpha-1+m}{m}-\Gamma\frac{m-2}{m}}(u)f^{\Gamma\frac{m-2}{m}}(u)$$

and by (12) obtain that

$$f^\alpha(u)\le k^{\frac{m-2}{m}}f^{(\alpha-1)\frac{m-1}{m}}(u)f^{\frac{\alpha-1+m}{m}-\Gamma\frac{m-2}{m}}(u)(f')^{\frac{m-2}{m}}(u).$$

Exploiting Young inequality with exponents $\frac{m}{m-1}$ and m, we get that

$$\begin{aligned}&\frac{m}{\alpha+\gamma}\chi^{m-1}|\nabla\chi|f^\alpha(u)f'(u)|\nabla u|^{m-1}\\&\le\frac{m}{\alpha+\gamma}k^{\frac{m-2}{m}}\chi^{m-1}f^{(\alpha-1)\frac{m-1}{m}}(u)(f')^{\frac{2(m-1)}{m}}(u)|\nabla u|^{m-1}\times|\nabla\chi|f^{\frac{\alpha-1+m}{m}-\Gamma\frac{m-2}{m}}(u)\\&\le\varepsilon\chi^m f^{\alpha-1}(u)(f')^2(u)|\nabla u|^m+C_\varepsilon|\nabla\chi|^m f^{\alpha+2\Gamma-1+m(1-\Gamma)}(u)\end{aligned}$$

for any $\varepsilon>0$ small, where C_ε is a suitable large constant. This estimate, inserted in (17), yields to the validity of (16).

As far as the first term in the R.H.S. of (15), let us write

$$\begin{aligned}f^{\alpha+1}(u)&=f^{\frac{2}{m}[\alpha+2\Gamma-1+m(1-\Gamma)]+\frac{m-2}{m}(\alpha+2\Gamma-1)}(u)\\&\le k^{2\frac{m-2}{m}}f^{\frac{2}{m}[\alpha+2\Gamma-1+m(1-\Gamma)]+\frac{m-2}{m}(\alpha-1)}(u)(f')^{2\frac{m-2}{m}}(u)\end{aligned}$$

in view of (12). Exploiting now Young inequality with exponents $\frac{m}{m-2}$ and $\frac{m}{2}$, for $m > 2$ we have that

$$
\begin{aligned}
&\frac{m^2}{(\alpha+1)^2}\chi^{m-2}|\nabla\chi|^2 f^{\alpha+1}(u)|\nabla u|^{m-2}\\
&\le \frac{m^2}{(\alpha+1)^2}k^{2\frac{m-2}{m}}\chi^{m-2}f^{\frac{m-2}{m}(\alpha-1)}(u)(f')^{2\frac{m-2}{m}}(u)|\nabla u|^{m-2}\\
&\times|\nabla\chi|^2 f^{\frac{2}{m}[\alpha+2\Gamma-1+m(1-\Gamma)]}(u)\\
&\le \varepsilon\chi^m f^{\alpha-1}(u)(f')^2(u)|\nabla u|^m + C_\varepsilon|\nabla\chi|^m f^{\alpha+2\Gamma-1+m(1-\Gamma)}(u),
\end{aligned}
$$

for every $\varepsilon > 0$ small and C_ε a large constant. Let us note that, when $m = 2$, the above estimate is automatically true with $\varepsilon = 0$ and $C_\varepsilon = \frac{4}{(\alpha+1)^2}$. By (16) the following estimate does hold:

$$
\begin{aligned}
&\frac{m^2}{(\alpha+1)^2}\int_{\mathbb{R}^N}\chi^{m-2}|\nabla\chi|^2 f^{\alpha+1}(u)|\nabla u|^{m-2}\,dx\\
&\le \varepsilon\int_{\mathbb{R}^N}\chi^m f^{\alpha-1}(u)(f')^2(u)|\nabla u|^m\,dx + C_\varepsilon\int_{\mathbb{R}^N}|\nabla\chi|^m f^{\alpha+2\Gamma-1+m(1-\Gamma)}(u)\,dx \qquad (18)\\
&\le \frac{\varepsilon}{1-\varepsilon}\int_{\mathbb{R}^N}\chi^m f(u)g(u)\,dx + D_\varepsilon\int_{\mathbb{R}^N}|\nabla\chi|^m f^{\alpha+2\Gamma-1+m(1-\Gamma)}(u)\,dx,
\end{aligned}
$$

for every $\varepsilon > 0$ small and D_ε a suitable large constant.

As far as the second term in the R.H.S. of (15), let us observe that

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\chi^{m-1}\left(\frac{2m}{\alpha+1}f^\alpha(u)f'(u) - mg(u)\right)|\nabla u|^{m-2}(\nabla u,\nabla\chi)\,dx\\
&\le C_0\int_{\mathbb{R}^N}\chi^{m-1}|\nabla\chi|f^\alpha(u)f'(u)|\nabla u|^{m-1}\,dx
\end{aligned}
$$

in view of (11). By (12) and Young inequality with exponents $\frac{m}{m-1}$ and m, we can write

$$
\begin{aligned}
&C_0\chi^{m-1}|\nabla\chi|f^\alpha(u)f'(u)|\nabla u|^{m-1}\\
&= C_0\chi^{m-1}f^{(\alpha-1)\frac{m-1}{m}+\frac{m-2}{m}\Gamma}f'(u)|\nabla u|^{m-1}\times|\nabla\chi|f^{\frac{1}{m}[\alpha+2\Gamma-1+m(1-\Gamma)]}(u)\\
&\le C_0k^{\frac{m-2}{m}}\chi^{m-1}f^{(\alpha-1)\frac{m-1}{m}}(f')^{2\frac{m-1}{m}}(u)|\nabla u|^{m-1}\times|\nabla\chi|f^{\frac{1}{m}[\alpha+2\Gamma-1+m(1-\Gamma)]}(u)\\
&\le \varepsilon\chi^m f^{\alpha-1}(f')^2(u)|\nabla u|^m + C_\varepsilon|\nabla\chi|^m f^{\alpha+2\Gamma-1+m(1-\Gamma)}(u)
\end{aligned}
$$

for $\varepsilon > 0$ small and C_ε large. By (16) finally we get

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\chi^{m-1}\left(\frac{2m}{\alpha+1}f^\alpha(u)f'(u) - mg(u)\right)|\nabla u|^{m-2}(\nabla u,\nabla\chi)\,dx\\
&\le \varepsilon\int_{\mathbb{R}^N}\chi^m f^{\alpha-1}(f')^2(u)|\nabla u|^m\,dx + C_\varepsilon\int_{\mathbb{R}^N}|\nabla\chi|^m f^{\alpha+2\Gamma-1+m(1-\Gamma)}(u)\,dx \qquad (19)\\
&\le \frac{\varepsilon}{1-\varepsilon}\int_{\mathbb{R}^N}\chi^m f(u)g(u)\,dx + D_\varepsilon\int_{\mathbb{R}^N}|\nabla\chi|^m f^{\alpha+2\Gamma-1+m(1-\Gamma)}(u)\,dx
\end{aligned}
$$

for $\varepsilon > 0$ small and D_ε large.

Set $\delta_\varepsilon = \frac{2\varepsilon}{1-\varepsilon}$. Inserting (18)-(19) into (15) we get

$$\begin{aligned}&\int_{\mathbb{R}^N} \chi^m f(u) \left(\frac{4}{(m-1)(\alpha+1)^2} f^\alpha(u) f'(u) - (1+\delta_\varepsilon) g(u) \right) dx \\ &\leq C_\varepsilon \int_{\mathbb{R}^N} |\nabla \chi|^m f^{\alpha+2\Gamma-1+m(1-\Gamma)}(u)\, dx\end{aligned} \tag{20}$$

for $\varepsilon > 0$ small and C_ε large. By (11) we have that

$$\begin{aligned}&\frac{4}{(m-1)(\alpha+1)^2} f^\alpha(u) f'(u) - (1+\delta_\varepsilon) g(u) \\ &\geq \left(\frac{4}{(m-1)(\alpha+1)^2} - \frac{1+\delta_\varepsilon}{\alpha+\gamma} \right) f^\alpha(u) f'(u).\end{aligned}$$

The constant $\frac{4}{(m-1)(\alpha+1)^2} - \frac{1}{\alpha+\gamma}$ is positive whenever $\alpha \in (\alpha_-, \alpha_+)$, where

$$\alpha_\pm = \frac{1}{m-1}\left[3 - m \pm 2\sqrt{\gamma(m-1) - (m-2)}\right]$$

are well defined by the assumption $\gamma > \frac{m-2}{m-1}$. Moreover, observe that the interval

$$I_0 = (-\min\{\gamma, 1\}, +\infty) \cap (\alpha_-, \alpha_+)$$

is not empty in view of $\alpha_+ > -\min\{\gamma, 1\}$. Fix now $\alpha \in I_0$ and take ε sufficiently small so that $\frac{4}{(m-1)(\alpha+1)^2} - \frac{1+\delta_\varepsilon}{\alpha+\gamma}$ is still a positive number. By (20) we finally get the estimate

$$\int_{\mathbb{R}^N} \chi^m f^{\alpha+1}(u) f'(u)\, dx \leq C \int_{\mathbb{R}^N} |\nabla \chi|^m f^{\alpha+2\Gamma-1+m(1-\Gamma)}(u)\, dx$$

for every $\alpha \in I_0$.

Let us now consider χ in the form $\chi = \phi^k$ ($k \geq 1$, $\phi \in C_0^\infty(\mathbb{R}^N)$ and $0 \leq \phi \leq 1$) to obtain

$$\int_{\mathbb{R}^N} \phi^{mk} f^{\alpha+1+\Gamma}(u)\, dx \leq C \int_{\mathbb{R}^N} \phi^{m(k-1)} |\nabla \phi|^m f^{\alpha+2\Gamma-1+m(1-\Gamma)}(u)\, dx$$

in view of (12). Hölder inequality with $q := \frac{\alpha+1+\Gamma}{\alpha+2\Gamma-1+m(1-\Gamma)}$ and $q' := \frac{\alpha+1+\Gamma}{\Gamma(m-1)-(m-2)}$ ($q > 1$ in view of $\Gamma \geq \gamma > \frac{m-2}{m-1}$) leads now to

$$\int_{\mathbb{R}^N} \phi^{mk} f^{\alpha+1+\Gamma}(u)\, dx \leq C \big(\int_{\mathbb{R}^N} \phi^{m(k-1)q} f^{\alpha+1+\Gamma}(u)\, dx \big)^{\frac{1}{q}} \big(\int_{\mathbb{R}^N} |\nabla \phi|^{mq'}\, dx \big)^{\frac{1}{q'}}.$$

Since $0 \leq \phi \leq 1$, for k large so that $m(k-1)q \geq mk$ it follows that

$$\int_{\mathbb{R}^N} \phi^{mk} f^{\alpha+1+\Gamma}(u)\, dx \leq C \int_{\mathbb{R}^N} |\nabla \phi|^{mq'}\, dx$$

for all $\alpha \in I_0$. If now we further restrict our attention and consider smooth test functions $0 \leq \phi_R \leq 1$ so that $\phi_R = 1$ in $B_R(0)$, $\phi_R = 0$ outside $B_{2R}(0)$ and $|\nabla \phi_R| \leq \frac{2}{R}$, we get that

$$\int_{B_R(0)} f^{\alpha+1+\Gamma}(u)\, dx \leq C R^{N-mq'},$$

and then

$$\int_{\mathbb{R}^N} f^{\alpha+1+\Gamma}(u)\, dx = 0$$

whenever $N - mq' < 0$. Since $N - mq'$ is a decreasing function of α, we evaluate the quantity $N - mq'$ at α_0 which reduces to:

$$N - \frac{m}{m-1} \frac{2 + 2\sqrt{\gamma(m-1) - (m-2)} + \Gamma(m-1)}{\Gamma(m-1) - (m-2)}.$$

By the definition (7) of $N^{\#}$, for $2 \le N < N^{\#}$ the quantity $N - mq'$ is negative at α_0 and, by continuity, is still negative for $\alpha \in (\alpha_0 - \eta, \alpha_0) \cap I_0$, $\eta > 0$ small. For a given $\bar\alpha \in (\alpha_0 - \eta, \alpha_0) \cap I_0$, we get that

$$\int_{\mathbb{R}^N} f^{\bar\alpha + 1 + \Gamma}(u)\, dx = 0.$$

Since $f > 0$ in $(a_0, a_1]$, we get that $a_0 > -\infty$ and $u \equiv a_0$. The solution u is trivial and the proof is complete. □

3. **The radial case.** We will be concerned now with the study of radial solutions to (1). Inspired by the nice argument in [3] for bounded solutions in the semilinear case, our aim is to to cover the case $m \geq 2$ and allow possibly one-side bounded solutions u.

Setting $r = |x|$, let $u = u(r)$ be a stable radial solution of (1). We have $u_r(0) = 0$ and

$$\int_0^\infty r^{N-1}(|u_r|^{m-2}u_r\varphi_r - f(u)\varphi)\, dr = 0 \quad \text{for any } \varphi \in C^1_c[0,\infty). \tag{21}$$

Our first aim is to derive the following fundamental relationship: for every $\psi \in C^1[a,b]$

$$\int_a^b r^{N-1}\left[(m-1)|u_r|^{m-2}u_{rr}\psi_r - f'(u)u_r\psi\right] = -(N-1)\int_a^b r^{N-3}|u_r|^{m-2}u_r\psi\, dr$$
$$- \left[r^{N-1}f(u)\psi + (N-1)r^{N-2}|u_r|^{m-2}u_r\psi\right]\Big|_a^b, \tag{22}$$

where $0 < a < b < +\infty$ are so that u_r has constant sign on $[a,b]$ (positive or negative).

Indeed, by classical elliptic regularity theory [22] u_r is smooth in $[a,b]$ and equation (21) is solved in the classical sense in (a,b):

$$\begin{aligned} r^{1-N}(r^{N-1}|u_r|^{m-2}u_r)_r + f(u) &= \left((m-1)u_{rr} + \frac{N-1}{r}u_r\right)|u_r|^{m-2} + f(u) \\ &= 0. \end{aligned} \tag{23}$$

Multiply now equation (23) by $r^{N-1}\psi_r$, $\psi \in C^1[a,b]$, and integrate by parts on (a,b) to get

$$\begin{aligned}
0 &= \int_a^b \left[(r^{N-1}|u_r|^{m-2}u_r)_r\psi_r + r^{N-1}f(u)\psi_r\right]\, dr \\
&= \int_a^b r^{N-1}\left((m-1)|u_r|^{m-2}u_{rr} + \frac{N-1}{r}|u_r|^{m-2}u_r\right)\psi_r\, dr \\
&\quad - \int_a^b r^{N-1}\left(\frac{N-1}{r}f(u) + f'(u)u_r\right)\psi dr + r^{N-1}f(u)\psi\Big|_a^b \\
&= \int_a^b r^{N-1}\left[(m-1)|u_r|^{m-2}u_{rr}\psi_r - f'(u)u_r\psi + \frac{N-1}{r^2}|u_r|^{m-2}u_r\psi\right]\, dr \\
&\quad - \int_a^b \frac{N-1}{r}[r^{N-1}f(u) + (r^{N-1}|u_r|^{m-2}u_r)_r]\psi\, dr \\
&\quad + \left[r^{N-1}f(u)\psi + (N-1)r^{N-2}|u_r|^{m-2}u_r\psi\right]\Big|_a^b \\
&= \int_a^b r^{N-1}\left[(m-1)|u_r|^{m-2}u_{rr}\psi_r - f'(u)u_r\psi\right] + (N-1)\int_a^b r^{N-3}|u_r|^{m-2}u_r\psi\, dr \\
&\quad + \left[r^{N-1}f(u)\psi + (N-1)r^{N-2}|u_r|^{m-2}u_r\psi\right]\Big|_a^b .
\end{aligned}$$

The validity of (22) easily follows.

Now we want to show that u_r does not change sign on $(0,\infty)$. Given $r_0 > 0$ be such that $u_r(r_0) \neq 0$, by continuity let (a_0, b_0), $0 \leq a_0 < b_0 \leq +\infty$, be the largest interval in $[0,+\infty)$ where $u_r \neq 0$. We claim that necessarily $b_0 = +\infty$ and then, u_r has constant sign on $(0,+\infty)$.
To prove the claim, assume by contradiction $b_0 < +\infty$. For $\varepsilon > 0$ small, use (22) on $(a_0+\varepsilon, b_0-\varepsilon)$ with $\psi = u_r$ to get

$$\begin{aligned}
&\int_{a_0+\varepsilon}^{b_0-\varepsilon} r^{N-1}\left[(m-1)|u_r|^{m-2}u_{rr}^2 - f'(u)u_r^2\right]\, dr \qquad (24)\\
&= -(N-1)\int_{a_0+\varepsilon}^{b_0-\varepsilon} r^{N-3}|u_r|^m\, dr - \left[r^{N-1}f(u)u_r + (N-1)r^{N-2}|u_r|^m\right]\Big|_{a_0+\varepsilon}^{b_0-\varepsilon} .
\end{aligned}$$

By (23) and the Hopital rule we get that for $a_0 = 0$

$$\lim_{r\to 0^+}\frac{|u_r|^{m-2}u_r}{r} = \lim_{r\to 0^+}\frac{(r^{N-1}|u_r|^{m-2}u_r)_r}{(r^N)_r} = -\frac{f(u(0))}{N}$$

while for $a_0 > 0$

$$\lim_{r\to a_0^+}\frac{|u_r|^{m-2}u_r}{r-a_0} = \lim_{r\to a_0^+}(|u_r|^{m-2}u_r)_r = (m-1)\lim_{r\to a_0^+}|u_r|^{m-2}u_{rr} = -f(u(a_0)).$$

In conclusion, there holds

$$|u_r|^{m-1} = O(|r-a_0|) \text{ as } r\to a_0^+; \quad |u_r|^{m-1} = O(|r-b_0|) \text{ as } r\to b_0^-, \qquad (25)$$

and, in particular

$$r^{N-3}|u_r|^m \in L^1(a_0,b_0). \qquad (26)$$

Since $u_r(a_0) = u_r(b_0) = 0$, letting $\varepsilon \to 0$ in (24) by (26) we get that

$$r^{N-1}|u_r|^{m-2}u_{rr}^2 \in L^1(a_0,b_0) \qquad (27)$$

and

$$\int_{a_0}^{b_0} r^{N-1}\left[(m-1)|u_r|^{m-2}u_{rr}^2 - f'(u)u_r^2\right]\,dr = -(N-1)\int_{a_0}^{b_0} r^{N-3}|u_r|^m\,dr. \quad (28)$$

On the other side, the stability of u implies $Q(\varphi) \geq 0$ for any $\varphi \in C_c^1[0,\infty)$, where

$$Q(\varphi) := \int_0^\infty r^{N-1}\left[(m-1)|u_r|^{m-2}\varphi_r^2 - f'(u)\varphi^2\right]\,dr.$$

Given $\varepsilon > 0$, let $0 \leq \Psi \leq 1$ be a smooth cut-off function in $\mathbb{R}$ so that $\Psi \equiv 1$ in $(a_0+2\varepsilon, b_0-2\varepsilon)$, $\Psi \equiv 0$ in $[0,\infty)\setminus(a_0+\varepsilon, b_0-\varepsilon)$ and $\Psi_r^2 \leq \frac{2}{\varepsilon^2}$. Since u_r is smooth in (a_0,b_0), $u_r\Psi \in C_c^1[0,+\infty)$ and then

$$\begin{aligned} Q(u_r\Psi) &= \int_0^\infty r^{N-1}\left[(m-1)|u_r|^{m-2}\left(u_{rr}^2\Psi^2 + 2u_r u_{rr}\Psi\Psi_r + u_r^2\Psi_r^2\right) - f'(u)u_r^2\Psi^2\right] \\ &\geq 0. \end{aligned}$$

We get that by (25)

$$\int_0^\infty r^{N-1}|u_r|^m\Psi_r^2 \leq C\varepsilon^{\frac{1}{m-1}} \to 0$$

and by (27)

$$\begin{aligned} \int_0^\infty r^{N-1}|u_r|^{m-1}|u_{rr}|\Psi|\Psi_r| &\leq \left(\int_{a_0}^{b_0} r^{N-1}|u_r|^{m-2}u_{rr}^2\right)^{\frac{1}{2}}\left(\int_0^\infty r^{N-1}|u_r|^m\Psi_r^2\right)^{\frac{1}{2}} \\ &\leq C\varepsilon^{\frac{1}{2(m-1)}} \to 0 \end{aligned}$$

as $\varepsilon \to 0^+$. By (28) and the Lebesgue Theorem we get that

$$\begin{aligned} 0 &\leq \lim_{\varepsilon\to 0} Q(u_r\Psi) = \int_{a_0}^{b_0} r^{N-1}\left[(m-1)|u_r|^{m-2}u_{rr}^2 - f'(u)u_r^2\right]\,dr \\ &= -(N-1)\int_{a_0}^{b_0} r^{N-3}|u_r|^m\,dr < 0. \end{aligned}$$

Hence, $b_0 = +\infty$ and the claim is proved.

Without loss of generality, we can now suppose $u_r < 0$ in $(0,\infty)$ and, by classical elliptic regularity theory [22], we find that $u \in C^\infty(0,\infty)$ solves (23) in $(0,+\infty)$. Moreover, by (25) we get that $r^{N-3}|u_r|^m \in L^1_{\text{loc}}[0,+\infty)$.
Given $\eta \in C_c^1[0,+\infty)$, apply (22) on (ε, M) with $\psi = \eta^2 u_r$, where M is large so that supp $\eta \subset [0,M]$. Letting $\varepsilon \to 0^+$, we get that $r^{N-1}|u_r|^{m-2}u_{rr}^2 \in L^1_{\text{loc}}[0,+\infty)$ and

$$\begin{aligned} &\int_0^\infty r^{N-1}\left[(m-1)\eta^2|u_r|^{m-2}u_{rr}^2 + 2(m-1)\eta\eta_r|u_r|^{m-2}u_r u_{rr} - f'(u)\eta^2 u_r^2\right] \\ &= -(N-1)\int_0^\infty \eta^2 r^{N-3}|u_r|^m\,dr. \end{aligned} \quad (29)$$

Formula (29) allows to write $Q(\eta u_r)$ as:

$$Q(\eta u_r) = \int_0^\infty r^{N-1}|u_r|^m\left[(m-1)\,\eta_r^2 - \frac{N-1}{r^2}\eta^2\right]\,dr.$$

Arguing as before, by (25) we can get that

$$Q(\eta u_r) = \lim_{\varepsilon\to 0^+} Q(\eta u_r\Psi) \geq 0$$

for a cut-off function $0 \le \Psi \le 1$ so that $\Psi \equiv 1$ in $(2\varepsilon, +\infty)$, $\Psi \equiv 0$ in $[0, \varepsilon)$ and $\Psi_r^2 \le \frac{2}{\varepsilon^2}$. In conclusion, there holds

$$\int_0^\infty r^{N-1}|u_r|^m \left[(m-1)\,\eta_r^2 - \frac{N-1}{r^2}\eta^2\right] dr \ge 0 \tag{30}$$

for every $\eta \in C_c^1[0, +\infty)$. By density (30) is valid also for $\eta \in H_c^1[0, +\infty)$.
Now, for $\alpha > 0$ choose

$$\eta = \begin{cases} 1 & \text{if } r < 1 \\ r^{-\alpha} & \text{if } r \ge 1. \end{cases}$$

Applying (30) to $(\eta - R^{-\alpha})\chi_{(0,R)}$ and letting $R \to \infty$, by monotone convergence we see that

$$\left[(m-1)\,\alpha^2 - (N-1)\right] \int_1^\infty r^{N-2\alpha-3}|u_r|^m \, dr \ge (N-1) \int_0^1 r^{N-3}|u_r|^m > 0, \tag{31}$$

provided

$$\int_1^\infty r^{N-2\alpha-3}|u_r|^m \, dr < \infty. \tag{32}$$

We reach a contradiction whenever we can find some $\alpha \le \sqrt{\frac{N-1}{m-1}}$ so that (32) holds. This will be the case in low dimensions $2 \le N \le N^\#$, for a suitable $N^\#$.

Assume as before $u_r < 0$ in $(0, +\infty)$ and let $u_\infty := \lim_{r\to\infty} u(r)$. Define the energy

$$E(r) = \frac{m-1}{m}|u_r(r)|^m + F(u(r)), \quad F(u) = \int_{u_\infty}^u f(s)\, ds.$$

Since $f \in L^1(a_2, a_3)$, notice that $E(r)$ is well defined for $r > 0$ and $E \in C^1(0, \infty)$. By equation (23) a direct differentiation of $E(r)$ yields to

$$E'(r) = -\frac{N-1}{r}|u_r(r)|^m < 0 \quad \text{ for any } r > 0.$$

In particular, we have

$$(N-1) \int_0^r \frac{|u_r(s)|^m}{s}\, ds = E(0) - E(r) \le \int_{u(r)}^{u(0)} f(s)\, ds$$

which implies

$$\int_0^\infty \frac{|u_r(r)|^m}{r}\, dr \le \int_{a_2}^{a_3} |f|(s)\, ds < \infty.$$

This means that equation (32) is valid whenever $\alpha \ge \frac{N-2}{2}$. We reach a contradiction in the dimensions N so that $\frac{N-2}{2} \le \sqrt{\frac{N-1}{m-1}}$, i.e. $2 \le N \le N^\# = \frac{2m+2\sqrt{m}}{m-1}$ as in (8). This concludes the proof of Theorem 1.4 part a).

To prove part b), notice that $a_2 > -\infty$ leads to $u_\infty > -\infty$. Hence, we necessarily get $\liminf_{r\to\infty} u_r(r) = 0$, and by Lebesgue Theorem we obtain

$$\lim_{r\to\infty} E(r) = \lim_{r\to\infty} \int_{u_\infty}^{u(r)} f(s)\, ds = 0.$$

Thus $0 \le E(r) \le E(0)$ for any $r > 0$, which means that E is bounded. Since $F(u(r))$ is also bounded by assumption, we get $u_r \in L^\infty(0, \infty)$. We then have

$$\int_0^\infty |u_r(r)|^m\, dr \le C \int_0^\infty |u_r(r)|\, dr = -C \int_0^\infty u_r(r)\, dr = C(u(0) - u_\infty) < \infty.$$

Equation (32) is then valid for any $\alpha \geq \frac{N-3}{2}$. We reach a contradiction if $\frac{N-3}{2} \leq \sqrt{\frac{N-1}{m-1}}$, i.e. for any $2 \leq N \leq N^{\#} = \frac{3m-1+2\sqrt{2m-1}}{m-1}$ as in (9). This proves Theorem 1.4 part b).

In order to prove part c), without loss of generality we can suppose $u_\infty = 0$, $u > 0$ and $u_r < 0$ in $(0, \infty)$. By the smoothness of u we can differentiate (23) to get

$$-(m-1)r^{1-N}(r^{N-1}|u_r|^{m-2}u_{rr})' + \frac{N-1}{r^2}|u_r|^{m-2}u_r = f'(u)u_r \text{ in } (0,\infty). \quad (33)$$

By (23) we would easily get that $|u_r|^{m-1} \geq \delta r \to +\infty$ as $r \to +\infty$ when $f(0) > 0$ and $|u_r|^{m-1} \leq -\delta r \to -\infty$ as $r \to +\infty$ when $f(0) < 0$, for some $\delta > 0$. Hence, $f(0) = 0$ and also $f'(0) \leq 0$ does hold. Indeed, if $f'(0) > 0$ for large r, from $Q(\phi) \geq 0$, with ϕ a cut-off function so that $\phi \equiv 0$ in $(0, R) \cup (4R, \infty)$ and $\phi \equiv 1$ in $(2R, 3R)$, we would get $\varepsilon R^N \leq CR^{N-2}$ for R large and some $\varepsilon > 0$ (we are using $u_r \in L^\infty(0, +\infty)$ as previously shown). This is a contradiction for large R.
By our assumptions we have

$$\lim_{s \to 0^+} \frac{f'(s)}{s^q} = b \neq 0. \quad (34)$$

If $b < 0$, we have $f'(s) < 0$ for s small, which means $f'(u(r))u_r(r) \geq 0$ for $r \to \infty$ and then $(r^{N-1}|u_r|^{m-2}u_{rr})' \leq 0$ for large r, in view of (33). Hence $|u_r|^{m-2}u_{rr} \leq Cr^{1-N}$ for sufficiently large r and $C \in \mathbb{R}$. Since $\liminf_{r\to\infty} u_r(r) = 0$, integrating from r to infinity we have

$$|u_r| \leq Cr^{-\frac{N-2}{m-1}} \text{ for any large } r. \quad (35)$$

If $b > 0$, since $f(0) = 0$ and $f'(0) \leq 0$, we necessarily have $q > 0$. This means that $f(s) \geq \delta s^{q+1}$ for some $\delta > 0$ and every small $s > 0$, so that $-(r^{N-1}|u_r|^{m-2}u_r)' \geq \delta r^{N-1}u^{q+1}$ for large r. Integrating on (s, t) we have

$$\begin{aligned} -t^{N-1}|u_r(t)|^{m-2}u_r(t) &\geq \delta \int_s^t r^{N-1}u^{q+1}\,dr - s^{N-1}|u_r(s)|^{m-2}u_r(s) \\ &\geq \delta \int_s^t r^{N-1}u^{q+1}\,dr \end{aligned}$$

for large $s < t$. Now, since $u^{q+1}(r) > u^{q+1}(t)$ for $r < t$, we deduce

$$-|u_r(t)|^{m-2}u_r(t)u^{-(q+1)}(t) \geq \frac{\delta}{N}(t - s^N/t^{N-1})$$

for large $s < t$. This gives $-u_r(t)u^{-\frac{q+1}{m-1}}(t) \geq C(t - s^N/t^{N-1})^{\frac{1}{m-1}} \geq C' t^{\frac{1}{m-1}}$ for some $C, C' > 0$ and large $2s < t$.
Notice that necessarily $q \geq m-2$: if $q < m-2$, integrating the last equation on $(2s, r)$ we would have $-u^{\frac{m-q-2}{m-1}}(r) \geq Cr^{\frac{m}{m-1}}$ for large r which is clearly a contradiction.
Integrating on $(2s, r)$ we get for large r:

$$u^{q+2-m}(r) \leq Cr^{-m}. \quad (36)$$

Now, by (34) we also have $f(s) \leq C's^{q+1}$ for all $s \in [0, u(0)]$. Hence, by (36) we deduce that

$$-(r^{N-1}|u_r|^{m-2}u_r)' \leq C'r^{N-1}u^{q+1} \leq \begin{cases} C & \text{for } 0 \leq r < 1 \\ Cr^{N-1-m}u^{m-1} & \text{for } r \geq 1 \end{cases}$$

for any $r > 0$ and for some $C > 0$. If we integrate on $(0, t)$ with $t > 1$ we obtain

$$t^{N-1}|u_r(t)|^{m-1} \le \begin{cases} C + C\frac{u^{m-1}(0)}{N-m}(t^{N-m} - 1) & \text{when } N \ne m \\ C + Cu^{m-1}(0)\ln t & \text{when } N = m \end{cases}$$

which gives

$$|u_r| \le C \begin{cases} r^{-\frac{N-1}{m-1}} & \text{for } N < m \\ r^{-1}(\ln r)^{\frac{1}{m-1}} & \text{for } N = m \\ r^{-1} & \text{for } N > m \end{cases} \tag{37}$$

for r sufficiently large. Taking into account (35) and (37), we discuss the two cases:
a) when $N < m + 1$, the estimate $|u_r| \le Cr^{-\frac{N-2}{m-1}}$ for $r \ge 1$ yields to the validity of (32) for $\alpha \ge -\frac{1}{2}$ and we reach a contradiction whenever $-\frac{1}{2} \le \sqrt{\frac{N-1}{m-1}}$, i.e. $2 \le N < m + 1$;
b) when $N \ge m + 1$, the estimate $|u_r| \le Cr^{-1}$ for $r \ge 1$ yields to the validity of (32) for $\alpha \ge \frac{N-m-2}{2}$ and we reach a contradiction whenever $\frac{N-m-2}{2} \le \sqrt{\frac{N-1}{m-1}}$, i.e. $m + 1 \le N \le m + \frac{4m}{m-1}$.
In conclusion, a contradiction arises whenever $2 \le N \le N^{\#} = m + \frac{4m}{m-1}$ as in (10), and Theorem 1.4 part c) is established.

Acknowledgements. The third author would like to thank Alberto Farina for the useful discussions on this topic.

REFERENCES

[1] L. Ambrosio and X. Cabré, *Entire solutions of semilinear elliptic equations in* $\mathbb{R}^3$ *and a conjecture of De Giorgi*, J. Amer. Math. Soc., **13** (2000), 725–739.
[2] A. Bahri and P.-L. Lions, *Solutions of superlinear elliptic equations and their Morse indices*, Comm. Pure Appl. Math., **45** (1992), 1205–1215.
[3] X. Cabré and A. Capella, *On the stability of radial solutions of semilinear elliptic equations in all of* $\mathbb{R}^n$, C. R. Math. Acad. Sci. Paris, **338** (2004), 769–774.
[4] X. Cabré and A. Capella, *Regularity of radial minimizers and extremal solutions of semilinear elliptic equations*, J. Funct. Anal., **238** (2006), 709–733.
[5] X. Cabré and M. Sanchón, *Semi-stable and extremal solutions of reactions equations involving the* $p-$*Laplacian*, Commun. Pure Appl. Anal., **6** (2007), 43–67.
[6] D. Castorina, P. Esposito and B. Sciunzi, *Degenerate elliptic equations with singular nonlinearities*, Calc. Var. Partial Differential Equations, **34** (2009), 279–306.
[7] M. G. Crandall and P. H. Rabinowitz, *Some continuation and variational methods for positive solutions of nonlinear elliptic eigenvalue problems*, Arch. Ration. Mech. Anal., **58** (1975), 207–218.
[8] L. Damascelli, A. Farina, B. Sciunzi and E. Valdinoci, *Liouville results for m-Laplace equations of Lane-Emden-Fowler type*, Ann. Inst. H. Poincaré Anal. Non Linéaire, to appear.
[9] L. Damascelli and B. Sciunzi, *Regularity, monotonicity and symmetry of positive solutions of m-Laplace equations*, J. Differential Equations, **206** (2004), 483–515.
[10] L. Damascelli and B. Sciunzi, *Harnack inequalities, maximum and comparison principles, and regularity of positive solutions of m-Laplace equations*, Calc. Var. Partial Differential Equations, **25** (2006), 139–159.
[11] E.N. Dancer, *Finite Morse index solutions of exponential problems*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **25** (2008), 173–179.
[12] M. del Pino, M. Kowalczyk and J. Wei, *On De Giorgi Conjecture in Large Dimensions*, C. R. Math. Acad. Sci. Paris, to appear.
[13] M. del Pino, M. Kowalczyk and J. Wei, *On De Giorgi conjecture in dimensions* $N \ge 9$, preprint, arXiv:0806.3141.
[14] E. Di Benedetto, $C^{1+\alpha}$ *local regularity of weak solutions of degenerate elliptic equations*, Nonlinear Anal., **7** (1983), 827–850.

[15] L. Dupaigne and A. Farina, *Stable solutions of $-\Delta u = f(u)$ in $\mathbb{R}^N$*, preprint, `arXiv:0806.0492`.

[16] P. Esposito, *Compactness of a nonlinear eigenvalue problem with a singular nonlinearity*, Commun. Contemp. Math., **10** (2008), 17–45.

[17] P. Esposito, *Linear instability of entire solutions for a class of non-autonomous elliptic equations*, Proc. Roy. Soc. Edinburgh Sect. A, **138** (2008), 1005–1018.

[18] P. Esposito, N. Ghoussoub and Y. Guo, *Compactness along the branch of semi-stable and unstable solutions for an elliptic problem with a singular nonlinearity*, Comm. Pure Appl. Math., **60** (2007), 1731–1768.

[19] A. Farina, *Stable solutions of $-\Delta u = e^u$ in $\mathbb{R}^N$*, C. R. Math. Acad. Sci. Paris, **345** (2007), 63–66.

[20] A. Farina, *On the classification of solutions of the Lane-Emden equation on unbounded domains of $\mathbb{R}^N$*, J. Math. Pures Appl., **87** (2007), 537–561.

[21] A. Farina, B. Sciunzi and E. Valdinoci, *Bernstein and De Giorgi type problems: new results via a geometric approach*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), to appear.

[22] D. Gilbarg and N. S. Trudinger, "Elliptic partial differential equations of second order," 2^{nd} edition, Springer-Verlag, Berlin, 1983.

[23] N. Ghoussoub and C. Gui, *On a conjecture of De Giorgi and some related problems*, Math. Ann., **311** (1998), 481–491.

[24] N. Ghoussoub and Y. Guo, *On the partial differential equations of electrostatic MEMS devices: stationary case*, SIAM J. Math. Anal., **38** (2006/2007), 1423–1449.

[25] M. Giaquinta and J. Souček, *Harmonic maps into a hemisphere*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), **12** (1985), 81–90.

[26] G. M. Lieberman, *Boundary regularity for solutions of degenerate elliptic equations*, Nonlinear Anal., **12** (1988), 1203–1219.

[27] G. Nedev, *Regularity of the extremal solution of semilinear elliptic equations*, C. R. Math. Acad. Sci. Paris, **330** (2000), 997–1002.

[28] V. O. Savin, *Regularity of flat level sets in phase transitions*, Ann. of Math. (2), **169** (2009), 41–78.

[29] V. O. Savin, B. Sciunzi and E. Valdinoci, *Flat level set regularity of $p-$Laplace phase transitions*, Mem. Amer. Math. Soc., **182** (2006), vi+144 pp.

[30] R. Schoen and K. Uhlenbeck, *Regularity of minimizing harmonic maps into the sphere*, Invent. Math., **78** (1984), 89–100.

[31] J. Simons, *Minimal varieties in riemannian manifolds*, Ann. of Math. (2), **88** (1968), 62–105.

[32] P. Tolksdorf, *Regularity for a more general class of quasilinear elliptic equations*, J. Differential Equations, **51** (1984), 126–150.

[33] D. Ye and F. Zhou, *Boundedness of the extremal solution for semilinear elliptic problems*, Commun. Contemp. Math., **4** (2002), 547–558.

Received xxxx 20xx; revised xxxx 20xx.

E-mail address: `castorin@mat.uniroma2.it`
E-mail address: `esposito@mat.uniroma3.it`
E-mail address: `sciunzi@mat.unical.it`