Monotonicity of the solutions of quasilinear elliptic equations in the half-plane with a changing sign nonlinearity.

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ABSTRACT. We consider weak positive solutions of the equation $-\Delta_m u = f(u)$ in the half-plane with zero Dirichlet boundary conditions and we prove a monotonicity result. We assume that the nonlinearity f is Locally Lipschitz continuous and changing sign: in particular we refer to the model $f(s) = s^q - \lambda s^{m-1}$, $q > m-1$. Our results extend to the case of sign changing nonlinearities the recent results in [**DS3**].

1. Introduction and statement of the main results

In this paper we consider the problem

$$(1) \qquad \begin{cases} -\Delta_m u = f(u), & \text{in } D \equiv \{(x,y) \in \mathbb{R}^2 \mid y > 0\} \\ u(x,y) > 0, & \text{in } D \\ u(x,0) = 0, & \forall x \in \mathbb{R} \end{cases}$$

where $\frac{3}{2} < m \leq 2$ and $\Delta_m u \equiv div(|\nabla u|^{m-2}\nabla u)$. It is well known that solutions of m-Laplace equations are generally of class $C^{1,\alpha}$ (see [**Di, Li, To**]), and the equation has to be understood in the weak distributional sense.

We extend here to the case of some sign changing nonlinearities the monotonicity results recently obtained in [**DS3**]. We restrict our attention to the case $m \leqslant 2$, since weak comparison principles hold true in this case, as proved in [**DP**]. Also some strong maximum and comparison principles obtained in [**Sc**] are exploited.

Our proof combines the geometric technique in [**DS3**] (which goes back to [**BCN**]), with the method of moving planes as developed in [**DP**].

We assume for f (see Figure 1) the following hypotheses:

(f1) f is locally Lipschitz continuous;

(f2) $$f(s) := \begin{cases} 0 & \text{if } s = 0 \text{ or } s = k; \\ < 0 & \text{if } s \in (0,k); \\ > 0 & \text{if } s \in (k, +\infty); \end{cases} \qquad \text{for some } k > 0;$$

(f3) there exists some $\varepsilon > 0$ such that f is non-decreasing in $(k-\varepsilon, k+\varepsilon)$.

As a typical example we refer to the case $f(s) = s^q - \lambda s^{m-1}$ with $q > m-1$ and $\lambda > 0$. We have the following

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Con il nostro ordine discreto dentro il cuore......

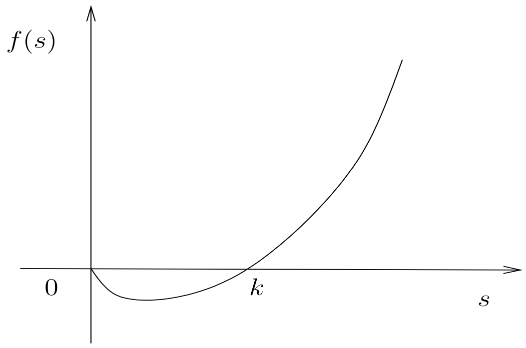

FIGURE 1. The nonlinearity $f(s)$.

THEOREM 1. *Let u be a weak $C^{1,\alpha}_{loc}$ solution of* (1). *Assume $\frac{3}{2} < m \leq 2$ and that hypotheses* $(f1)$, $(f2)$ *and* $(f3)$ *hold true for the nonlinearity f. Then, u is monotone increasing in the e_2-direction and*

$$\frac{\partial u}{\partial y}(x,y) > 0, \quad \forall (x,y) \in \overline{D}.$$

If moreover we assume that u is bounded, it follows that u is one dimensional, that is

$$u(x,y) = \bar{u}(y).$$

2. Notations

We let $L_{x_0,s,\theta}$ the line, with slope $\tan(\theta)$ passing through (x_0,s) and V_θ is the vector orthogonal to $L_{x_0,s,\theta}$

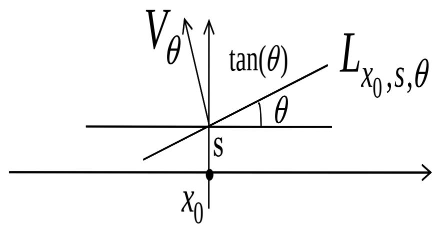

FIGURE 2

such that $(V_\theta, e_2) \geqslant 0$. We define

$$\mathcal{T}_{x_0,s,\theta}$$

as the triangle delimited by $L_{x_0,s,\theta}$, $\{y=0\}$ and $\{x=x_0\}$, see Figure 4. We set

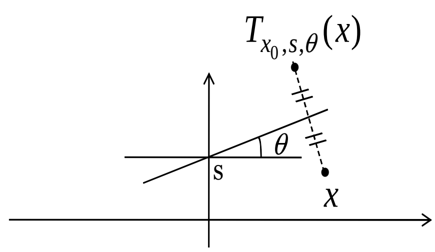

FIGURE 3

$$u_{x_0,s,\theta}(x) = u(T_{x_0,s,\theta}(x)),$$

where $T_{x_0,s,\theta}(x)$ is the point symmetric to x, w.r.t. $L_{x_0,s,\theta}$ (see Figure 3) and

$$w_{x_0,s,\theta} = u - u_{x_0,s,\theta}.$$

It is well known that $u_{x_0,s,\theta}$ still fulfills

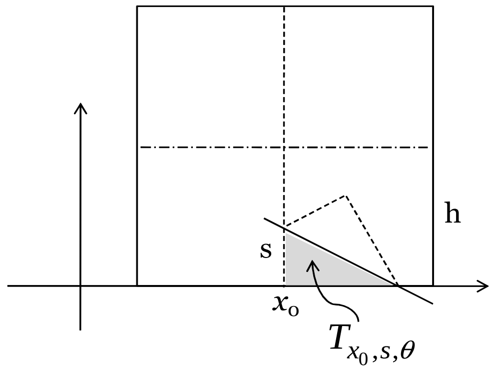

FIGURE 4

$$-\Delta_m u_{x_0,s,\theta} = f(u_{x_0,s,\theta}).$$

For simplicity

$$u_{x_0,s,0} = u_s.$$

3. Proof of Theorem 1

Given any $x \in \mathbb{R}$, by Hopf boundary Lemma, (see [**PS3, Va**]), it follows that

$$u_y(x,0) = \frac{\partial u}{\partial y}(x,0) > 0,$$

obviously, $u_y(x,0)$ possibly goes to 0 if $x \to \pm\infty$. Let x_0 be fixed and h such that

$$\frac{\partial u}{\partial y}(x,y) \geqslant \gamma > 0, \forall x,y \in Q_h(x_0),$$

where

(2) $$Q_h(x_0) = \{\ |x - x_0| \leqslant h, 0 \leqslant y \leqslant 2h\}$$

as shown in Figure 5. Note that such $\gamma > 0$ exists since $u \in C^{1,\alpha}$. Also, since $u \in C^{1,\alpha}$, we may assume

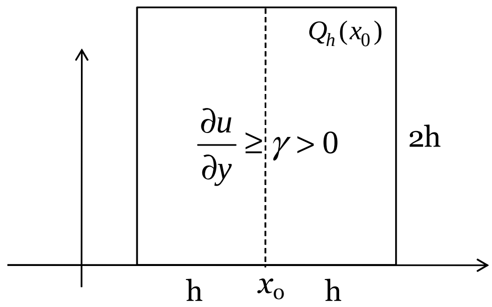

FIGURE 5

that there exists

(3) $$\delta_1 = \delta_1(h, \gamma, x_0) > 0$$

such that, if $|\theta| \leqslant \delta_1$ (and consequently $V_\theta \approx e_2$), we have

$$\frac{\partial u}{\partial V_\theta} \geqslant \frac{\gamma}{2} > 0, \quad \text{in } Q_h(x_0). \tag{4}$$

Then, let $Q_h(x_0)$ as in (2) and δ_1 defined in (3). Consider $\theta \neq 0$ fixed such that $|\theta| \leqslant \delta_1$. Then we find

$$\bar{s} = \bar{s}(\theta)$$

such that, for any $s \leqslant \bar{s}$ we have that
the triangle $\mathcal{T}_{x_0,s,\theta}$ is contained in $Q_h(x_0)$ (see Figure 4)
and $u < u_{x_0,s,\theta}$ in $\mathcal{T}_{x_0,s,\theta}$ (and $u \leqslant u_{x_0,s,\theta}$ on $\partial(\mathcal{T}_{x_0,s,\theta})$).

This follows by the monotonicity in $Q_h(x_0)$.
Let now θ be fixed with $|\theta| \leqslant \delta_1$, we set $\bar{s} \leqslant h$ in such a way that

- the triangle $\mathcal{T}_{x_0,s,\theta}$ is contained in $Q_h(x_0)$ as well as the triangle obtained from $\mathcal{T}_{x_0,s,\theta}$ by reflection with respect to the line $L_{x_0,s,\theta}$ (see Figure 4). Note that this is possible by simple geometric considerations;
- $u \leqslant u_{x_0,s,\theta}$ on $\partial(\mathcal{T}_{x_0,s,\theta})$. In fact, since $|\theta| \leqslant \delta_1$ then $u \leqslant u_{x_0,s,\theta}$ on the line (x_0, y) for $0 \leqslant y \leqslant s$, since of the monotonicity in the V_θ-direction, by construction (see (4)). Also $u \leqslant u_{x_0,s,\theta}$ if $y = 0$ by the Dirichlet assumption, and the fact that u is positive in the interior of the domain. And finally $u \equiv u_{x_0,s,\theta}$ on $L_{x_0,s,\theta}$;
- possibly reducing $\bar{s}$, we assume that the Lebesgue measure $\mathcal{L}(\mathcal{T}_{x_0,s,\theta})$ is sufficiently small in order to exploit the weak comparison principle in small domains.

Therefore, for $0 \leqslant s \leqslant \bar{s}$, if we define

$$w_{x_0,s,\theta} = u - u_{x_0,s,\theta}$$

we have that

$$w_{x_0,s,\theta} \leqslant 0 \text{ on } \partial\mathcal{T}_{x_0,s,\theta},$$

therefore, by the weak comparison principle, which works in our case thanks to [**DP**] since $m \leqslant 2$, we get

$$w_{x_0,s,\theta} \leqslant 0 \text{ in } \mathcal{T}_{x_0,s,\theta}.$$

Using repeatedly now the *moving plane technique*, the *rotating plane technique* and the *sliding plane technique*, as made in [**DS3**], together with the weak comparison principle proved in [**DP**], one has that

$$u \leqslant u_{\tilde{s}} \quad \text{in } \overline{\Sigma}_{\tilde{s}}, \quad \forall \tilde{s} \in (0, h], \tag{5}$$

where

$$\Sigma_t \equiv \{(x, y) \,|\, 0 < y < t\}.$$

We now point out some consequences:
First of all, we have also proved that u is monotone increasing in the e_2-direction in Σ_h. In fact, given (x, y_1) and (x, y_2) in Σ_h (say $0 \leqslant y_1 < y_2 \leqslant h$), by equation (5), one has that

$$u(x, y_1) \leqslant u_{\frac{y_1+y_2}{2}}(x, y_1),$$

which gives exactly

$$u(x, y_1) \leqslant u(x, y_2).$$

Also we note that, this immediately gives

$$\frac{\partial u}{\partial y}(u) \geqslant 0 \quad \text{in } \Sigma_h. \tag{6}$$

Claim: Let us show now that actually $\dfrac{\partial u}{\partial y}(u) > 0$ in Σ_h. We point out that the nonlinearity f change sign, see Figure 1. We remark that since the local weighted Sobolev type inequality is local in nature (see [**DS1, Sc**] for example), then a Sobolev type inequality follows in regions where $f(s)$ is negative or positive.

Therefore a strong maximum principle for the linearized operator follows in such regions (see [**Sc**]) and, when applied to the derivative of u, we readily get

$$\frac{\partial u}{\partial y} > 0,$$

in regions where the nonlinearity is positive or negative. To get the same result in regions where f change sign, we use a *sliding balls technique* as made in [**Sc**, Theorem 6.1]. Here, for the reader's convenience, we give just some details. Let us consider a ball $B^1_\rho(x,y)$, with ρ sufficiently small such that is contained in the region where f does not change sign. Then we move it in the positive y-direction up to reach the level set $f(s) \equiv k$ at some point P_1. Next we repeat the same technique sliding a second ball $B^2_\rho(x,y)$ with radius ρ sufficiently small up to reach the level set $f(s) \equiv k$ in a second point P_2, see Figure 6. By Hopf boundary

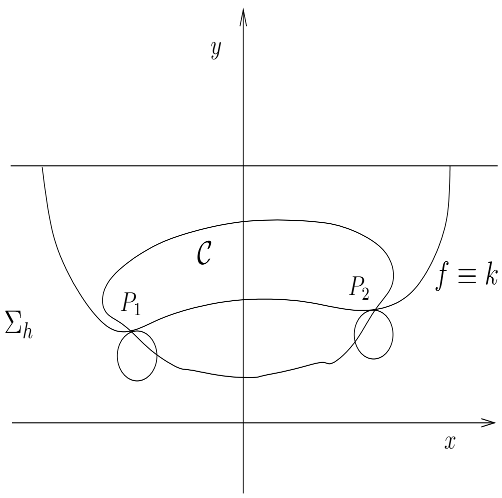

FIGURE 6. The sliding method.

Lemma ([**Va**]) we get that

$$\begin{cases} \dfrac{\partial u}{\partial n_2}(P_1) \neq 0; \\ \\ \dfrac{\partial u}{\partial n_2}(P_2) \neq 0. \end{cases} \tag{7}$$

Here n_1 (resp. n_2) denote the outer normal vector at P_1 (at P_2) to the boundary of $B^1_\rho(x,y)$ (of $B^2_\rho(x,y)$). From (7) and by continuity it follows that ∇u is different from zero in a sufficiently small neighborhood of P_1 and P_2. Therefore a Strong Maximum Principle holds near P_1 and P_2, and in connection with equation (6), it gives

$$\frac{\partial u}{\partial y}(P_1) > 0 \quad \text{and} \quad \frac{\partial u}{\partial y}(P_2) > 0.$$

Then let us fix an open set $\mathcal{C}$ in such a way its closure cross the two points P_1 and P_2, see Figure 6. Also, let us assume that $\mathcal{C} \subset \{k - \varepsilon < u < k + \varepsilon\}$, where ε is the one in condition f(3) and with no loss of generality $\dfrac{\partial u}{\partial y} > 0$ on $\partial\mathcal{C}$. Then by a Strong Maximum Principle for the linearized equation (see in particular [**Sc**, Theorem 6.1]) one has that

$$\frac{\partial u}{\partial y} > 0 \quad \forall y \in \mathcal{C},$$

and consequently the strictly monotonicity of u in the strip Σ_h with

$$\frac{\partial u}{\partial y} > 0 \quad \forall y \in \Sigma_h,$$

since $\mathcal{C}$ was arbitrary. It follows now that by equation (5), we have that $w = u - u_{\tilde{s}} \leqslant 0$ in $\overline{\Sigma}_{\tilde{s}}$. By the Strong Comparison Principle which holds now since $\nabla u \neq 0$ in $\overline{\Sigma}_{\tilde{s}}$ we reduce to the case

$$w < 0,$$

being the case $w \equiv 0$ easily excluded.

Remark 2. *We point out that to apply the Strong Maximum Principle, here we need to require that $f(\cdot)$ is non-decreasing in a neighborhood of its nodal point, see $f(3)$.*

Some Notations

Let us set

$$\Lambda = \{\lambda \in \mathbb{R}^+ \, : \, u < u_{\lambda'} \;\; \forall \lambda' < \lambda\}$$

and define

$$\bar{\lambda} = \sup_{\lambda \in \Lambda} \lambda$$

so that $u \leqslant u_{\bar{\lambda}}$ by continuity and also $u < u_{\bar{\lambda}}$ arguing as above. Consequently, exactly as above, this implies that u is strictly monotone increasing in the e_2-direction with

$$\frac{\partial u}{\partial y} > 0 \tag{8}$$

in $\Sigma_{\bar{\lambda}}$. To prove the theorem, we have to show that actually $\bar{\lambda} = \infty$. To do this we now assume $\bar{\lambda} < \infty$ and show that we can take $\delta > 0$ such that

$$u < u_\lambda \text{ for } 0 < \lambda \leqslant \bar{\lambda} + \delta$$

which would implies $\lambda > \bar{\lambda}$ and then the thesis. To prove this let us consider θ fixed with $|\theta| \leqslant \delta$, and consequently set λ small such that

$$\text{the triangle } \mathcal{T}_{x_0,\lambda,\theta} \text{ is contained in } Q_h(x_0) \text{ (see Figure 4)},$$
$$u < u_{x_0,\lambda,\theta} \text{ in } \mathcal{T}_{x_0,\lambda,\theta} \text{ (and } u \leqslant u_{x_0,\lambda,\theta} \text{ on } \partial(\mathcal{T}_{x_0,\lambda,\theta})).$$

In the following we need to follow the proof of **Claim-1** and **Claim-2** in [**DS3**]. In particular we need to show that we may and do assume

$$\nabla u(x_0, \overline{\lambda}) \neq 0.$$

To do this we have to generalize the arguments in **Claim-1** and **Claim-2** in [**DS3**] since, in our case, the nonlinearity change sign with respect to the case considered in [**DS3**]. Anyway we get the same conclusion distinguishing two different cases:

(1) it may occur that $u(x, \overline{\lambda}) \equiv k, \; \forall x \in \mathbb{R}$. In this situation $f(u) = 0$ by (f2) on $\{y = \overline{\lambda}\}$. Then, since $u(x, \cdot)$ is strictly increasing in the strip Σ_λ, we easily prove the existence of some point $(x_0, \overline{\lambda})$ (actually for any $(x, \overline{\lambda})$) where the gradient is different from zero, by using standard Hopf Lemma;
(2) otherwise the nonlinearity f could change sign on the line $y = \overline{\lambda}$. Without loss of generality, since f is continuous, we find some neighborhood $I_\sigma = \; (x_0 - \sigma, x_0 + \sigma)$, with σ sufficiently small, where $f(u(t, \overline{\lambda}))$ is strictly positive (or negative) when $t \in (x_0 - \sigma, x_0 + \sigma)$. Then the conclusion in the previous case follows as in Theorem 1.1 in [**DS3**].

Therefore, in all two cases we get the existence of some point x_0 where

$$\nabla u(x_0, \overline{\lambda}) \neq 0.$$

Now we use the above arguments:

(i) *the sliding technique:*

we move the line $L_{x_0,\lambda,\theta}$ in the e_2-direction towards the line $L_{x_0,\bar{\lambda}+\delta,\theta}$, letting θ fixed and moving $\lambda \to \bar{\lambda} + \delta$. We note that for every $\lambda \leqslant \bar{\lambda} + \delta$ we have $u \leqslant u_{x_0,\lambda,\theta}$ on $\partial(\mathcal{T}_{x_0,\lambda,\theta})$. In fact, since $|\theta| \leqslant \delta$ can be taken as small as we like, then following closely **Claim-1** and **Claim-2** [1] in [**DS3**] one has that $u < u_{x_0,\lambda,\theta}$ on the line (x_0, y) for $0 \leqslant y < \lambda$. Also $u \leqslant u_{x_0,\lambda,\theta}$ if $y = 0$ by the Dirichlet assumption. And finally $u \equiv u_{x_0,\lambda,\theta}$ on

[1]The fact that $\nabla u(x_0, \overline{\lambda}) \neq 0$ is needed to exploit the Hopf Comparison Lemma in $(x_0, \overline{\lambda})$ as in [**DS3**].

$L_{x_0,\lambda,\theta}$.
Therefore by the sliding technique described above, we get

$$u < u_{x_0,\bar{\lambda}+\delta,\theta} \text{ in } \mathcal{T}_{x_0,\bar{\lambda}+\delta,\theta}.$$

Now start with

(*ii*) *the rotating plane technique:*

rotating the line $L_{x_0,\bar{\lambda}+\delta,\theta}$ towards the line $\{y = \bar{\lambda} + \delta\}$, letting $\bar{\lambda} + \delta$ fixed and moving $\theta \to 0$. We still have the right conditions on the boundary of $\mathcal{T}_{x_0,\bar{\lambda}+\delta,\theta}$ and at same way starting from positive θ at the limit $(\theta \to 0)$ we get $u < u_{\bar{\lambda}+\delta}$ in $\Sigma_{\bar{\lambda}+\delta} \cap \{x \leqslant x_0\}$. If else we start from a negative θ, it follows $u < u_{\bar{\lambda}+\delta}$ in $\Sigma_{\bar{\lambda}+\delta} \cap \{x \geqslant x_0\}$. Finally

$$u < u_{\bar{\lambda}+\delta} \text{ in } \Sigma_{\bar{\lambda}+\delta},$$

proving that $\lambda > \bar{\lambda}$, that is $\lambda = +\infty$.

If now we assume that u is bounded, we have that the gradient is bounded too. Consequently, exploiting Theorem 1.1 in [**FSV**], we may follow exactly the proof of Theorem 1.4 in [**DS3**], and get that u is one dimensional [2], that is there exists $\overline{u} : \mathbb{R} \to \mathbb{R}$ such that

$$u(x, y) = \overline{u}(y),$$

concluding the proof.

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[2]Observe that there are no problems in defining the notion of stable solution, since $\{\nabla u = 0\} = \emptyset$ by our monotonicity result.