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Spectral theory for linearized p-Laplace equations

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ABSTRACT

We continue and completely set up the spectral theory initiated in Castorina et al. (2009) [5] for the linearized operator arising from $\Delta_p u + f(u) = 0$. We establish existence and variational characterization of all the eigenvalues, and by a weak Harnack inequality we deduce Hölder continuity for the corresponding eigenfunctions, this regularity being sharp. The Morse index of a positive solution can be now defined in the classical way, and we will illustrate some qualitative consequences one should expect to deduce from such information. In particular, we show that zero Morse index (or more generally, non-degenerate) solutions on the annulus are radial.

1. Introduction

Let $u \in C^{1,\alpha}(\Omega)$ be a weak solution of the problem

$$\begin{cases} -\Delta_p u = f(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geqslant 2$, $\Delta_p u = \operatorname{div}(|Du|^{p-2}Du)$ is the p-Laplace operator, and f is a positive ($f(s) > 0$ for $s > 0$) locally Lipschitz continuous nonlinearity. The Hölder continuity of ∇u is in general optimal [1–3] and Eq. (1.1) is always meant in a weak sense.
The linearized operator L_u associated to (1.1) at a given solution u is defined by duality as $L_u : v \in H_0 \to L_u(v) \in H_0'$, where $L_u(v) : \varphi \in H_0 \to L_u(v, \varphi)$ and

$$L_u(v, \varphi) := \int_\Omega |\nabla u|^{p-2}(\nabla v, \nabla \varphi) + (p-2)\int_\Omega |\nabla u|^{p-4}(\nabla u, \nabla v)(\nabla u, \nabla \varphi) - \int_\Omega f'(u) v \varphi. \tag{1.2}$$

The Hilbert space H_0 will be rigorously introduced in Section 2 according to [4] and is roughly composed by functions v vanishing on the boundary so that $\int_\Omega |\nabla u|^{p-2}|\nabla v|^2 < \infty$. In this way, the operator L_u is well defined, and in [5] it is shown that the first eigenvalue of L_u

$$\mu_1 = \inf_{v \in H_0,\ v \neq 0} \frac{L_u(v, v)}{\int_\Omega v^2}$$

is simple and attained at a nonnegative first eigenfunction v_1. The study in [5] can be pushed further to set up a complete spectral theory for L_u as summarized in the following.

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Theorem 1.1. *The eigenvalues of L_u have finite multiplicity and form a sequence*

$$\mu_1 < \mu_2 \le \mu_3 \le \cdots$$

(with repetitions according to the multiplicity) so that $\mu_j \to +\infty$ as $j \to \infty$. Moreover, the μ_j's can be characterized variationally as

$$\mu_j = \min_{\substack{V \subset H_0 \\ \dim V = j}} \max_{v \in V,\ v \neq 0} \frac{L_u(v,v)}{\int_\Omega v^2} = \min_{\substack{V \subset H_0 \\ \dim V = j-1}} \min_{v \in V^\perp,\ v \neq 0} \frac{L_u(v,v)}{\int_\Omega v^2}, \tag{1.3}$$

where the orthogonal space $V^\perp$ is meant in the $L^2(\Omega)$-sense. The corresponding eigenfunctions $v_j \in H_0$ solve the equation

$$L_u(v_j, \varphi) = \mu_j \int_\Omega v_j \varphi \quad \forall \varphi \in H_0$$

and form an orthonormal basis in $L^2(\Omega)$. Moreover, v_j belongs to $C^{0,\alpha}(\Omega)$ for some $\alpha \in (0,1)$ provided $p > \frac{2N+2}{N+2}$.

The existence of μ_j's is based on the Fredholm alternative and makes a crucial use of the compact embedding $H_0 \hookrightarrow L^2(\Omega)$ as established in [5]. The Hölder regularity of v_j follows by a weak Harnack inequality and is essentially optimal. Indeed, the derivatives of u are in the kernel of the linearized operator (but they do not fulfill the zero boundary condition) and are in general just Hölder continuous.
Once the spectral theory for L_u is available, one can classically define the notion of Morse index $m(u)$ and non-degeneracy for a solution u of (1.1). We believe that an information on $m(u)$ should carry relevant properties on u. From a qualitative viewpoint, this is well explained by the following.

Theorem 1.2. *Let Ω be a bounded radially symmetric domain and u be a solution of (1.1). Assume that either $m(u) = 0$ or u is a non-degenerate solution. Then, u is radially symmetric.*

The assumption $m(u) = 0$ is simply equivalent to the semi-stability of u: $L_u(v,v) \geqslant 0$ for all $v \in H_0$. The latter condition has been used (for v in a suitable space of test functions) as a definition of semi-stability in cases where a spectral theory was not available or not attainable (see [6,7] and references therein). When Ω is a ball, the radial symmetry of u follows by the moving plane method in [4] and the difficult case concerns the annulus.

Notice that any local minimum point u of the corresponding energy functional actually satisfies $m(u) = 0$, since the bilinear form L_u represents the second derivative of the energy functional. Noticing that in general $H_0 \neq W_0^{1,p}(\Omega)$ and that the energy functional is not C^2 even in $W_0^{1,p}(\Omega)$ when $p < 2$, the computation of such a second derivative is a very delicate issue which has been established to hold exactly in H_0 (see [5]).

When $f(u)$ is replaced by $\lambda f(u)$, the corresponding nonlinear eigenvalue problem admits, in several situations, a branch u_λ of minimal solutions – for λ in a natural range – with $m(u_\lambda) = 0$ (see [6,5] and references therein). For $p = 2$ and non-decreasing convex nonlinearities $f(u)$, it is well known that u_λ is the unique zero Morse index solution, which is also radial when Ω is an annulus. In this respect, Theorem 1.2 is still already known on the annulus when $p = 2$. However, for $p \neq 2$ it is not known that zero Morse index solutions need to be unique and Theorem 1.2 is no longer obvious. Such an uniqueness has been shown [8] to hold for radial solutions on the ball and $1 < p \le 2$.

Let us provide a last example. For a subcritical exponent q ($p - 1 < q < \frac{N(p-1)+p}{N-p}$ when $p < N$ and $p - 1 < q < \infty$ when $p \ge N$), the compact embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{q+1}(\Omega)$ yields to a minimizer $u > 0$ of

$$m_q = \inf_{u \in W_0^{1,p}(\Omega),\ u \neq 0} \frac{\int_\Omega |\nabla u|^p}{\left(\int_\Omega |u|^{q+1}\right)^{\frac{p}{q+1}}}.$$

The function u can be normalized to give a solution of (1.1) with $f(u) = m_q u^q$ corresponding to a Mountain Pass solution and $m(u) = 1$. Indeed, since $u \in H_0$ and $\int_\Omega |u|^{q+1} = 1$ we can easily compute $L_u(u,u) = m_q(p - 1 - q) < 0$ so as to have $\mu_1 < 0$. On the other hand, we can fix $v \in H_0$ and compute

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \frac{\int_\Omega |\nabla(u+tv)|^p}{\left(\int_\Omega |u+tv|^{q+1}\right)^{\frac{p}{q+1}}} = pL_u(v,v) + pm_q(q-p+1)\left(\int_\Omega u^q v\right)^2.$$

Since u is a minimizer, we get that $L_u(v,v) \ge 0$ for every $v \in V = \left\{v \in H_0 : \int_\Omega u^q v = 0\right\}$ and by Theorem 1.1 we deduce $\mu_2 \ge 0$ so as to establish $m(u) = 1$.
Starting from [9–11], there has been an intensive study of the nonlinear eigenvalues λ_j's of $-\Delta_p$ but still very little is known. They have been used to obtain, by non-standard variational methods, solutions of $-\Delta_p u - \lambda|u|^{p-2}u = f(u)$. There is a large literature on this topic. We refer the reader in particular to [12] and to [13,14](see also the references therein).

The linear eigenvalues μ_j's play the same role here as the eigenvalues of $-\Delta - f'(u)$ but the picture is more complicate due to the degenerate and nonlinear nature of $-\Delta_p$. First, one might wonder if the non-degeneracy of a solution u allows for a local analysis in the spirit of the Implicit Function Theorem. Recall that here the energy functional is not C^2 and the space H_0 depends on the solution itself. Secondly, regularity and compactness results for finite Morse index solutions should be in order in low dimensions (depending on the nonlinearity), as it has been already established for $p = 2$ [15–18]. Finally, it would be really of interest, the study of symmetry properties for finite Morse index solutions as in the case $p = 2$ [19], a flavor of it having been given in Theorem 1.2.

2. Spectral theory for L_u

Given a solution u of (1.1), for $p \geq 2$ we define the Hilbert space $H = H^{1,2}_\rho(\Omega)$, $\rho = |\nabla u|^{p-2}$, where (as in [4]) $H^{1,2}_\rho(\Omega)$ is the completion of $C^\infty(\Omega)$ w.r.t. the norm

$$\|v\|_H^2 = \int_\Omega v^2 + \int_\Omega |\nabla u|^{p-2}|\nabla v|^2. \tag{2.4}$$

Since Ω is smooth, H is equivalently composed by the functions v which have distributional derivative and satisfy $\|v\|_H < \infty$. The space H_0 is defined as the completion of $C_0^\infty(\Omega)$ w.r.t. the norm $\|\cdot\|_H$. Letting $\|v\|_{H_0}^2 = \int_\Omega |\nabla u|^{p-2}|\nabla v|^2$ for every $v \in H_0$, the following Sobolev inequality does hold [4]:

$$\|v\|_{L^q(\Omega)} \leq S_q \|v\|_{H_0} \quad \forall\, v \in H_0, \tag{2.5}$$

where $1 \leqslant q < \frac{2N(p-1)}{N(p-1)-2}$ and $S_q > 0$ is a positive constant. In particular, for $q = 2$ (2.5) provides a Poincaré inequality which implies the equivalence in H_0 of the two norms $\|\cdot\|_H$ and $\|\cdot\|_{H_0}$. Moreover, the embedding $H_0 \hookrightarrow L^q(\Omega)$ is compact for any $1 \leq q < \frac{2N(p-1)}{N(p-1)-2}$, (see [5]). Since $C_0^\infty(\Omega) \subset H_0$, we also have that H_0 is dense in $L^2(\Omega)$.

For $1 < p < 2$, we define H_0 simply as

$$H_0 = \{v \in H_0^1(\Omega) : \ \|v\|_{H_0} < \infty\}.$$

Since $\|v\|_{H_0^1} \leqslant \|\nabla u\|_\infty^{\frac{2-p}{2}} \|v\|_{H_0}$, it follows that H_0 is compactly embedded in $L^q(\Omega)$, for any $1 \leq q < \frac{2N}{N-2}$. Since $Z_u = \{x \in \bar{\Omega} : \nabla u(x) = 0\} \subset \Omega$ has zero Lebesgue measure [4], we can always approximate a function $v \in L^2(\Omega)$ by a sequence $v_n \in C_0(\Omega \setminus Z_u) \cap H^1(\Omega) \subset H_0$ so as to provide the density of H_0 in $L^2(\Omega)$. In conclusion, H_0 is dense and embeds compactly in $L^2(\Omega)$ for every $p > 1$.

To develop the linear theory for L_u as contained in Theorem 1.1, we exploit a standard procedure which may be found for example in [20].

First, for $\Lambda \in \mathbb{R}$ we let

$$a_\Lambda(v, w) := L_u(v, w) + \Lambda \int_\Omega vw \quad \forall v, w \in H_0.$$

Through the Hölder and the (weighted) Poincaré inequalities, it is easy to see that a_Λ is continuous: $|a_\Lambda(v, w)| \leq C\|v\|_{H_0}\|w\|_{H_0}$. Furthermore, if we set $C_1 = \min\{p-1, 1\}$ and $C_2 = \max\{p-1, 1\}$, since

$$C_1|\nabla u|^{p-2}|\nabla v|^2 \leq |\nabla u|^{p-2}|\nabla v|^2 + (p-2)|\nabla u|^{p-4}(\nabla u, \nabla v)^2 \leq C_2|\nabla u|^{p-2}|\nabla v|^2,$$

we can achieve the coercivity of a_Λ: $a_\Lambda(v, v) \geq C_0\|v\|_{H_0}^2$ for $C_0 > 0$, whenever $\Lambda \geq \|f'(u)\|_\infty$. By the Lax–Milgram theorem, we can then define the resolvent operator $G : f \in L^2(\Omega) \to G_f \in H_0$, where G_f is the unique solution of

$$a_\Lambda(G_f, w) = \int_\Omega f w \quad \forall\, w \in H_0.$$

Now $G : L^2(\Omega) \to L^2(\Omega)$ is clearly a self-adjoint operator by the symmetry of a_Λ, whereas its compactness follows by the estimate

$$C_0\|G_f\|_{H_0}^2 \leq a_\Lambda(G_f, G_f) = \int_\Omega fG_f \leq S_2\|f\|_{L^2}\|G_f\|_{H_0}$$

and the compact embedding $H_0 \hookrightarrow L^2(\Omega)$.

By the Riesz–Fredholm theory, the eigenvalues β_j's of G have finite multiplicity and form a sequence of positive numbers which converges to zero. It is clear that (β, v) is an eigenpair of G if and only if $a_\Lambda(v, \varphi) = \beta^{-1}\int_\Omega v\varphi$ does hold for every $\varphi \in H_0$. Hence, the linearized operator L_u has a sequence of eigenvalues $\mu_j = \beta_j^{-1} - \Lambda \to +\infty$ of finite multiplicity:

$$\mu_1 < \mu_2 \leqslant \mu_3 \leqslant \cdots$$

(with repetitions according to the multiplicity). The corresponding eigenfunction $v_j \in H_0$ satisfies $L_u(v_j, \varphi) = \mu_j \int_\Omega v_j\varphi$ for all $\varphi \in H_0$. By the self-adjointness of G the v_j's can be normalized so as to form an orthonormal basis in $L^2(\Omega)$. The operator G can be also seen as acting from H_0 into itself, and is still a compact self-adjoint operator whenever H_0 is endowed with the equivalent norm $|v|^2 = a_\Lambda(v, v)$. Since $v_j \in H_0$, also in this case G has the β_j's as eigenvalues and the renormalization $\tilde{v}_j = \beta_j^{\frac{1}{2}} v_j$ form an orthonormal basis in H_0 in view of

$$a_\Lambda(\tilde{v}_j, \tilde{v}_k) = \beta_k^{\frac{1}{2}} \beta_j^{-\frac{1}{2}} \int_\Omega v_j v_k = \delta_{jk}.$$

Recall that μ_1 is simple and satisfies (1.3) in view of

$$\mu_1 = \min_{v \in H_0,\ v \neq 0} \frac{L_u(v, v)}{\int_\Omega v^2},$$

and the associated eigenspace is one-dimensional, generated by a first nonnegative eigenfunction v_1. Setting

$$\mathcal{R}(v) = \frac{L_u(v, v)}{\int_\Omega v^2},$$

we can compute

$$\mathcal{R}(v) = \frac{\sum\limits_{k=1}^{j} \alpha_k^2 \mu_k}{\sum\limits_{k=1}^{j} \alpha_k^2} \le \mu_j$$

for every $v = \sum_{k=1}^{j} \alpha_k v_k \in \mathrm{Span}\{v_1, \ldots, v_j\}$, $v \neq 0$. Since the equality holds when $\alpha_1 = \cdots = \alpha_{j-1} = 0$ and $\alpha_j = 1$, we get that

$$\mu_j = \max_{v \in \mathrm{Span}\{v_1,\ldots,v_j\} \setminus \{0\}} \mathcal{R}(v). \tag{2.6}$$

Given $v \perp v_1, \ldots, v_{j-1}$ in $L^2(\Omega)$, we have that

$$v = \sum_{k=j}^{\infty} \alpha_k v_k, \quad \alpha_k = \int_\Omega v v_k$$

in $L^2(\Omega)$, so as to get

$$\int_\Omega v^2 = \sum_{k=j}^{\infty} \alpha_k^2.$$

Since

$$a_\Lambda(\tilde{v}_k, v) = \beta_k^{-\frac{1}{2}} \int_\Omega v v_k \quad \forall\, k \in \mathbb{N},$$

similarly we have that

$$v = \sum_{k=j}^{\infty} \tilde{\alpha}_k \tilde{v}_k, \quad \tilde{\alpha}_k = \beta_k^{-\frac{1}{2}} \alpha_k$$

in $(H_0, |\cdot|)$, and then

$$a_\Lambda(v, v) = \sum_{k=j} \beta_k^{-1} \alpha_k^2 = \sum_{k=j} (\mu_k + \Lambda) \alpha_k^2.$$

Hence, we deduce that

$$\mathcal{R}(v) = \frac{\sum\limits_{k=j}^{\infty} \mu_k \alpha_k^2}{\sum\limits_{k=j}^{\infty} \alpha_k^2} \ge \mu_j,$$

and in turn

$$\mu_j = \min_{v \perp v_1,\ldots,v_{j-1}} \mathcal{R}(v). \tag{2.7}$$

Given V with $\dim V = j$, we can always find $\bar{v} \in V$, $\bar{v} \neq 0$, such that $\bar{v} \perp v_1 \ldots, v_{j-1}$, and by (2.7) we get that

$$\max_{v \in V,\ v \neq 0} \mathcal{R}(v) \geqslant \mu_j,$$

and in turn

$$\mu_j = \min_{\substack{V\subset H_0\\ \dim V=j}} \max_{v\in V,\ v\neq 0} \mathcal{R}(v)$$

does hold since by (2.6) the minimum is achieved exactly at $V = \mathrm{Span}\{v_1, \ldots, v_j\}$. The first relation in (1.3) has been established. As far as the second one, similarly, we can deduce by (2.6) that

$$\min_{v\in V^{\perp},\ v\neq 0} \mathcal{R}(v) \leqslant \mu_j,$$

for every V such that $\dim V = j - 1$. Hence, there holds

$$\mu_j = \max_{\substack{V\subset H_0\\ \dim V=j-1}} \min_{v\in V^{\perp},\ v\neq 0} \mathcal{R}(v)$$

since by (2.7) the maximum is achieved exactly at $V = \mathrm{Span}\{v_1, \ldots, v_{j-1}\}$. The first part of Theorem 1.1 has been completely established.

3. $C^{0,\alpha}$-regularity of the eigenfunctions

We prove here that any eigenfunction of the linearized operator L_u is Hölder continuous. To this aim, we prove a Harnack inequality for an operator slightly more general than L_u, i.e.

$$\mathcal{L}(v,\varphi) = \int_\Omega |\nabla u|^{p-2}(\nabla v, \nabla\varphi) + (p-2)\int_\Omega |\nabla u|^{p-4}(\nabla u, \nabla v)(\nabla u, \nabla\varphi) - \int_\Omega cv\varphi - \int_\Omega g\varphi \tag{3.8}$$

for $v, \varphi \in H_0$, where $c, g \in L^\infty(\Omega)$.

We can prove the following weak Harnack inequality for $\mathcal{L}$:

Theorem 3.1. *Let $v \in H \cap L^\infty(\Omega)$ be a nonnegative weak supersolution of (3.8). For $p > 2$, consider s so that $0 < s < \frac{N(p-1)}{(N-2)(p-1)+2(p-2)}$ and $x_0 \in \Omega$ so that $B(x_0, 5R) \subset \Omega$. Then we find a constant $C > 0$ such that*

$$R^{-\frac{n}{s}}\|v\|_{L^s(B(x_0,2R))} \le C\left(\inf_{B(x_0,R)} v + R^p\|g\|_{L^\infty}\right). \tag{3.9}$$

If $\frac{2N+2}{N+2} < p < 2$ the same result holds for $0 < s < \frac{2^*}{s^\sharp}$, with $\frac{2}{s^\sharp} = 1 - \frac{1}{s}$ and $s < \frac{p-1}{2-p}$.

Proof. The function v solves $\mathcal{L}(v,\varphi) \ge 0$ for any $0 \le \varphi \in H_0$. Remark that we may always assume $v \ge \tau > 0$. Otherwise, we can consider $v + \tau$, replace g by $g + \tau c$ and let $\tau \to 0$. Rescaling (3.8) with $y = \frac{x-x_0}{R}$ we get

$$\int_{\Omega'} [|\nabla u'|^{p-2}(\nabla v', \nabla\varphi) + (p-2)|\nabla u'|^{p-4}(\nabla u', \nabla v')(\nabla u', \nabla\varphi) - \tilde{c}v'\varphi - \tilde{g}\varphi] \ge 0, \tag{3.10}$$

where $\Omega' = \frac{\Omega - x_0}{R}$, $w'(y) = w(x_0 + Ry)$ for every function w in Ω, $\tilde{c} = R^p c'$ and $\tilde{g} = R^p g'$. Inequality (3.10) does hold for every $\phi \in H_0'$, where H_0' is defined as H_0 with Ω, u replaced by Ω', u'. Consider the function $\tilde{v}$ defined by $\tilde{v} = v' + \|\tilde{g}\|_\infty$. Taking into account (3.10) it follows that $\tilde{v}$ fulfills

$$\int_{\Omega'} [|\nabla u'|^{p-2}(\nabla\tilde{v}, \nabla\varphi) + (p-2)|\nabla u'|^{p-4}(\nabla u', \nabla\tilde{v})(\nabla u', \nabla\varphi) - \bar{c}\tilde{v}\varphi] \ge 0 \tag{3.11}$$

for all $\varphi \in H_0'$, with $\bar{c} = (\tilde{c}v' + \tilde{g})\tilde{v}^{-1}$. Since the zero order coefficient is bounded:

$$|\bar{c}(y)| \le \left|\frac{\tilde{c}(y)v'(y)}{v'(y) + \|\tilde{g}\|_\infty}\right| + \left|\frac{\tilde{g}}{v'(y) + \|\tilde{g}\|_\infty}\right| \le \|\tilde{c}\|_\infty + 1 < \infty,$$

we can develop an iterative Moser-type scheme [21] to prove a weak Harnack inequality for $\tilde{v}$. For all the details of the proof, we refer the readers to the Appendix of [22], where the iterative Moser-type technique was developed in a similar setting in the spirit of [23]. Here we only start the procedure taking care of the fact that the operator we are considering is more general than the one in [22].

We define

$$\phi \equiv \eta^2\tilde{v}^\beta, \quad \beta < 0,$$

with $0 \le \eta \in C_0^1(B(0,5))$. Since $\nabla\phi = 2\eta\tilde{v}^{\beta}\nabla\eta + \beta\eta^2\tilde{v}^{\beta-1}\nabla\tilde{v}$, we have that $\phi \in H_0'$ and then can be used as a test function in (3.11) so as to get

$$\begin{aligned}&\int_{\Omega'}[\beta\rho'|\nabla\tilde{v}|^2\eta^2\tilde{v}^{\beta-1} + \beta(p-2)|\nabla u'|^{p-4}(\nabla u', \nabla\tilde{v})^2\eta^2\tilde{v}^{\beta-1}]\\ &\quad + \int_{\Omega'}[2\eta\tilde{v}^{\beta}\rho'(\nabla\tilde{v}, \nabla\eta) + 2\eta(p-2)\tilde{v}^{\beta}|\nabla u'|^{p-4}(\nabla u', \nabla\eta)(\nabla u', \nabla\tilde{v})]\\ &\geqslant \int_{\Omega'}\bar{c}\eta^2\tilde{v}^{\beta+1},\end{aligned}$$

where $\rho' = |\nabla u'|^{p-2}$. Note that the weight that appears in [4] is $\rho = |\nabla u|^{p-2}$, and the properties of this weight are crucial. Our rescaled weight has the same summability properties and everything works.

Since $\beta < 0$, for $p > 2$ the term $\beta(p-2)|\nabla u'|^{p-4}(\nabla u', \nabla\tilde{v})^2\eta^2\tilde{v}^{\beta-1}$ is negative and we have

$$\beta\rho'|\nabla\tilde{v}|^2\eta^2\tilde{v}^{\beta-1} + \beta(p-2)|\nabla u'|^{p-4}(\nabla u', \nabla\tilde{v})^2\eta^2\tilde{v}^{\beta-1} \le \beta\rho'|\nabla\tilde{v}|^2\eta^2\tilde{v}^{\beta-1}.$$

For $1 < p < 2$ we can use the fact that $\beta(p-2) > 0$ to show

$$\beta\rho'|\nabla\tilde{v}|^2\eta^2\tilde{v}^{\beta-1} + \beta(p-2)|\nabla u'|^{p-4}(\nabla u', \nabla\tilde{v})^2\eta^2\tilde{v}^{\beta-1} \le (p-1)\beta\rho'|\nabla\tilde{v}|^2\eta^2\tilde{v}^{\beta-1}.$$

In conclusion, for every $p > 1$ we get that

$$\min\{1, p-1\}|\beta|\int_{\Omega'}\rho'|\nabla\tilde{v}|^2\eta^2\tilde{v}^{\beta-1} \leqslant 2(p-1)\int_{\Omega'}\rho'\eta\tilde{v}^{\beta}|\nabla\tilde{v}|\,|\nabla\eta| + \int_{\Omega'}|\bar{c}|\eta^2\tilde{v}^{\beta+1}.$$

By the Young's inequality $2(p-1)ab \leqslant \min\{1, p-1\}|\beta|a^2 \setminus 2 + 2(p-1)^2b^2 \setminus |\beta|\min\{1, p-1\}$ we obtain

$$\min\{1, p-1\}\frac{|\beta|}{2}\int_{\Omega'}\rho'|\nabla\tilde{v}|^2\eta^2\tilde{v}^{\beta-1} \leqslant \frac{C_2}{|\beta|}\int_{\Omega'}\rho'\tilde{v}^{\beta+1}|\nabla\eta|^2 + \int_{\Omega'}|\bar{c}|\eta^2\tilde{v}^{\beta+1},$$

and by $\|\bar{c}\|_\infty < \infty$

$$\int_{\Omega'}\rho'|\nabla\tilde{v}|^2\eta^2\tilde{v}^{\beta-1} \leqslant \frac{C}{|\beta|}\left(1 + \frac{1}{|\beta|}\right)\int_{\Omega'}\tilde{v}^{\beta+1}[\eta^2 + \rho'|\nabla\eta|^2]. \tag{3.12}$$

Let us now define

$$w \equiv \begin{cases}\tilde{v}^{\frac{\beta+1}{2}} & \text{if } \beta \neq -1\\ \log(\tilde{v}) & \text{if } \beta = -1\end{cases}$$

and set $r \equiv \beta + 1$. With these definitions we can write (3.12) as follows

$$\int_{\Omega'}\rho'\eta^2|\nabla w|^2 \leqslant \begin{cases}C\left(1 + \dfrac{1}{|\beta|}\right)^2\displaystyle\int_{\Omega'}w^2[\eta^2 + \rho'|\nabla\eta|^2] & \beta \neq -1\\ C_0\displaystyle\int_{\Omega'}[\eta^2 + \rho'|\nabla\eta|^2] & \beta = -1.\end{cases} \tag{3.13}$$

We have now that (3.13) is exactly (A.9) in [22]. We can therefore use the proof in [22] (from (A.9) to the end of the proof of Theorem 3.1) and get that there exists $C > 0$ such that

$$\|\tilde{v}\|_{L^s(B(0,2))} \le C\inf_{B(0,1)}\tilde{v}.$$

Finally, scaling back to the coordinates $x = x_0 + Ry$ and recalling that $\tilde{v} = v' + \|\tilde{g}\|_\infty = v' + R^p\|g\|_\infty$ we get exactly Eq. (3.9). □

Theorem 3.2. *Let u be a solution of (1.1) and $v_i \in H_0$ be any eigenfunction of the linearized operator L_u. Assume that $\frac{2N+2}{N+2} < p < \infty$. There exists $\alpha \in (0,1)$ such that $v_i \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$.*

Proof. The eigenfunction $v = v_i$ solves

$$\int_{\Omega}[|\nabla u|^{p-2}(\nabla v_i, \nabla\varphi) + (p-2)|\nabla u|^{p-4}(\nabla u, \nabla v_i)(\nabla u, \nabla\varphi) - (f'(u) + \mu_i)v_i\varphi] = 0$$

for all $\varphi \in H_0$. Define $M_k \equiv \sup_{B(x_0,kR)} v$ and $m_k \equiv \inf_{B(x_0,kR)} v$. Considering the functions $M_4 - v$ and $v - m_4$, we see that they are nonnegative supersolutions of (3.8) with $c(x) = -f'(u) - \mu_i$ (resp. $c(x) = f'(u) + \mu_i$) and $g(x) = M_4(f'(u) + \mu_i)$

(resp. $g(x) = m_4(f'(u) + \mu_i)$), which are both bounded. Hence by Theorem 3.1 they satisfy (3.9) for any $0 < s < \chi$, with $\chi > 1$. Now, applying Eq. (3.9) with $s = 1$ and adding up we can estimate

$$M_4 - m_4 = c_n R^{-n} \int_{B(x_0,2R)} (M_4 - v) + (v - m_4)\,\mathrm{d}x \\ \leq C\left\{[(M_4 - M_1) + (m_1 - m_4)] + (M_4 + m_4)(\|f'(u)\|_\infty + |\mu_i|)R^p\right\}.$$

Thus, if we set $\omega(kR) = M_k - m_k$ and $k(R) = 2M_4(\|f'(u)\|_\infty + |\mu_i|)R^p$, we finally get the existence of $\gamma > 0$ such that

$$\omega(R) \leq \gamma\omega(4R) + k(R). \tag{3.14}$$

Since $k(R)$ is non-decreasing for any $R > 0$, by (3.14) we can apply Lemma 8.23 page 201 of [24] to obtain that there exists $\alpha \in (0, 1)$ such that

$$\omega(R) \leq CR^\alpha.$$

This yields to the desired Hölder regularity and concludes the proof. □

4. An application

We are now in position to follow the literature for the non-degenerate case and give the following

Definition 4.1. Let u be any solution of (1.1). We define the *Morse index* $m(u)$ of u as the number of negative eigenvalues of L_u in H_0, i.e. $m(u) = j$ iff $\mu_j(L_u) < 0$ and $\mu_{j+1}(L_u) \geq 0$. We say that u is non-degenerate if 0 is not an eigenvalue of L_u, i.e. $\mu_j(L_u) \neq 0$ for any j.

According to Theorem 1.1 the *Morse index* $m(u)$ of u defined as above is exactly the maximal dimension of a subspace of H_0 where L_u is defined to be negative, the latter being a definition also used in a setting where a complete spectral theory is not available (see for example [18]).

We can then prove the following

Proposition 4.2. *Let u be a solution of (1.1) and $\Omega \subset \mathbb{R}^N$ be an annulus, $N \geq 2$. Assume that either $m(u) = 0$ or u is non-degenerate. Then, u is radially symmetric.*

Proof. For simplicity, let us start considering the case $N = 2$. Taking into account the radial symmetry of Ω, it is convenient to write $u = u(r, \theta)$ in polar coordinates, where $r = |x|$ and θ is the angular variable. Notice that, if u is not radial, u_θ necessarily changes sign in Ω. An explicit computation shows that there exists $C = C(\Omega, N) > 0$ such that

$$\int_\Omega |\nabla u|^{p-2}|\nabla u_\theta|^2 \leq C \int_\Omega |\nabla u|^{p-2}|D^2 u|^2, \tag{4.15}$$

where u_θ is considered as a function of the variables $(x_1, x_2) \in \mathbb{R}^2$. By [4] we know that $u_{x_i} \in H$ for any $i = 1, \ldots, N$, so as to provide the finiteness of the quantities in (4.15). Moreover, by the boundary conditions we see that $u_\theta = 0$ on $\partial\Omega$.

This means that $u_\theta \in H_0$, so that we can plug it into the linearized equation (1.2). Moreover, integrating by parts in polar coordinates as in [4] (see Lemma 2.1), we have that

$$L_u(u_\theta, \varphi) = 0 \quad \forall \varphi \in H_0. \tag{4.16}$$

That is, u_θ is an eigenfunction of the linearized operator with eigenvalue 0. In this way, either $u_\theta = 0$ in Ω or u_θ is the first eigenfunction of L_u and has constant sign in Ω. This necessarily implies that u is radially symmetric.

For $N > 2$, we can still write the solution $u = u(r, \theta)$ in polar coordinates, where $r = |x|$ and $\theta = (\theta_1, \ldots, \theta_{n-1})$ are the $n - 1$ angular variables. Notice that, also in this case, if u is not radial, then $u_{\theta_i} \neq 0$ and u_{θ_i} changes sign, for some $i \in \{1, \ldots, n - 1\}$. We can repeat the argument above for u_{θ_i} so as to show that necessarily u is radial. This concludes the proof. □

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