Partial and full symmetry of solutions of quasilinear elliptic equations, via the Comparison Principle.

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Dedidated to Haïm Brezis on the occasion of his 60th birthday, with great admiration.

ABSTRACT. Using a Comparison Principle for degenerate elliptic equations of the form $\operatorname{div}\{a(x)A(|Du|)Du\} + B(x,u) = 0$, we establish corresponding symmetry results. As a consequence, for balls and annuli, we obtain radial symmetry results for equations of the form

$$\operatorname{div}\{a(r)A(|Du|)Du\} + B(r,u) = 0,$$

when $B(r,z) \in L^\infty_{\text{loc}}(\Omega \times \mathbb{R})$ is non–increasing in z.

1. Introduction

In this paper we consider the equation

$$\operatorname{div}\{a(x)A(|Du|)Du\} + B(x,u) = 0 \qquad \text{in} \quad \Omega, \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ and the solution is understood in the weak distribution sense. We make the following structural assumptions on the operator

(H_1) $a(x) \equiv [a_{ij}(x)]$ $(i,j = 1,\dots,N)$ *is a locally bounded real positive definite symmetric matrix, i.e.*

$$0 < \lambda|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2 \qquad \textit{for all } x \in \Omega \textit{ and } \xi \in \mathbb{R}^N;$$

(H_2) *A is positive and differentiable in $\mathbb{R}^+$. Also $\Phi(t) = o(1)$ as $t \to 0^+$, with $\Phi(t) = tA(t)$ $(\Phi(0) = 0)$ and*

$$\inf_{t>0} \frac{tA'(t)}{A(t)} = c_1 > -1, \qquad \sup_{t>0} \frac{tA'(t)}{A(t)} = c_2 < \infty. \tag{1.2}$$

In some cases we shall not assume (H_2), but only the condition

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(H_3) *A is positive and Φ is strictly increasing in $\mathbb{R}^+$, with $\Phi(t) = o(1)$ as $t \to 0^+$.*

REMARK 1.1. Condition (H_3) is weaker than (H_2). Indeed, if (H_2) holds, then by (1.2), we have

$$\frac{\Phi'(t)}{A(t)} = \frac{tA'(t)}{A(t)} + 1 > 0,$$

so that Φ is strictly increasing in $\mathbb{R}^+$.

Assumptions (H_2), (H_3) include the case $A(t) = t^{p-2}$, $p > 1$, which gives the well known Laplacian when $p = 2$ and $a = \mathbb{I}$. The operator in (1.1) was first studied in [**10**], see also [**2**] and [**11**].

For definiteness in the interpretation of (1.1) we put $\boldsymbol{A}(\xi) = A(|\xi|)\xi$ for $\xi \neq 0$ and $\boldsymbol{A}(0) = 0$. Thus $\boldsymbol{A}$ is continuous on $\mathbb{R}^N$ because of (H_2), and (1.1) can also be written in the form

$$\operatorname{div}\{a(x)\boldsymbol{A}(Du)\} + B(x,u) = 0.$$

Our starting point is to give sufficient conditions to guarantee that the operator is elliptic.

As shown in [**2**], see also [**11**], to this end it is important to know when the product of two positive definite matrices is positive definite. In particular, using the results in [**8, 13, 2**], it was proved in [**11**] that the operator $a(x)A(|Du|)Du$ *is elliptic if*

$$(H_4) \qquad \sqrt{\frac{\Lambda}{\lambda}} < \min\{\phi(c_1), \phi(c_2)\}, \qquad \phi(c) \equiv \frac{2 + c + 2\sqrt{1+c}}{|c|},$$

where c_1 and c_2 are given by (H_2)*;* see [**11**, Lemma 2.3.3] and the following Lemma 2.1.

We use the ellipticity of the operator to get a Comparison Principle, see Proposition 2.1, from which symmetry results for solutions of (1.1) follow. In particular, we consider the case when $B(x,z) \in L^\infty_{\text{loc}}(\Omega \times \mathbb{R})$ is non–increasing in z, and show that if the domain is symmetric in one direction, say e_1, then the solution is symmetric in the same direction, provided that the matrix a and the nonlinearity B are similarly symmetric. No assumption is needed on the sign of the solution nor need the domain be simply connected.

Our main result is Theorem 3.1. Here we point out an interesting corollary.

THEOREM 1.2. *Let $\Omega \subset \mathbb{R}^N$ be a ball or an annulus. Let $u \in W^{1,\infty}_{\text{loc}}(\Omega) \cap C(\overline{\Omega})$ be a solution of the following Dirichlet boundary value problem for* (1.1)*, written in distribution form: for all $\varphi \in C^\infty_0(\Omega)$*

$$\begin{cases} \displaystyle\int_\Omega \langle a(|x|)A(|Du|)Du\,,\, D\varphi\rangle\, dx = \int_\Omega B(|x|,u)\varphi\, dx, \\ u(x) = g(|x|) \qquad \text{on} \qquad \partial\Omega, \end{cases}$$

with[1] *$B(|x|,z) \in L^\infty_{\text{loc}}(\Omega \times \mathbb{R})$ non–increasing in z.*

If the assumptions (H_1), (H_2), (H_4) *hold, then u is unique and radial* (*i.e.* $u = u(|x|)$).

If finally $\Phi(t) \leq \text{const.}\, t^{p-1}$, $p > 1$, the same result holds even for solutions $u \in W^{1,p}_{\text{loc}}(\Omega) \cap C(\overline{\Omega})$.

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[1] Note that the condition at the boundary in Theorem 1.2 means that u is constant on $\partial\Omega$ if Ω is a ball or u assumes two different values on $\partial\Omega$ if Ω is an annulus.

Theorem 1.2 will be proved in Section 3.

In case $a(x) = \rho(|x|)\mathbb{I}$ it is enough in Theorem 1.1 to assume only that condition (H_3) holds and that ρ is positive and locally bounded in $B_R \setminus \{0\}$. See the remark in Section 3 after the proof of Theorem 1.1

Remark 1.3. For the case of the p–Laplace operator, we refer the reader to [**4, 5**] and the references therein. In particular in [**4**] the case of locally Lipschitz continuous nonlinearities with $p \leq 2$ is considered, while [**5**] deals with the case of positive locally Lipschitz continuous nonlinearities for $p \geq 2$.

When $A(t) = t^{p-2}$ and a is not the identity matrix $(0 < \lambda|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2)$, the ellipticity condition (H_4) becomes

$$\sqrt{\frac{\Lambda}{\lambda}} < \frac{p + 2\sqrt{p-1}}{|p-2|}$$

with no condition if $p = 2$, see [**2**] and [**11**, Section 2.3].

Remark 1.4. In spite of the elegance of the results, the reader should observe that nontrivial solutions of the Dirichlet problem in Theorem 1.2 may not exist for arbitrary boundary data.

In particular, consider the zero Dirichlet boundary value problem for (1.1), with B independent of x and non–increasing in z, and $B(0) = 0$. Let u be a solution, and note that it can be used as test function in (1.1), yielding

$$0 \leq \int_\Omega \langle a(x)A(|Du|)Du\,,\, Du\rangle\; dx = \int_\Omega B(u)u\, dx \leq 0.$$

It follows that $Du \equiv 0$, showing that there are no non–trivial solutions!

Nevertheless, there are other cases with different Dirichlet boundary conditions, or with $B(0) \neq 0$, where this difficulty does not arise and nontrivial solutions exist. In these cases we obtain symmetry results which hold for broad classes of operators and domains. In fact we only need to assume that the domains in question are symmetric in some direction to prove that the solution is symmetric (in that direction). We note particularly that the domains need not be convex or even simply connected.

If we consider the semilinear non–degenerate case ($A(t) = 1$ and $a = \mathbb{I}$), there are in the literature many symmetry and monotonicity results obtained exploiting the well known Alexandrov–Serrin [**12**] moving plane method. We mention here the celebrated papers [**1, 7**] where symmetry and monotonicity results are obtained for positive solutions with zero Dirichlet boundary conditions, under general assumptions on the nonlinearity.

There are cases when the moving plane technique can not be exploited. As an example if the domain is not convex (e.g. an annulus) or if we consider operators that depend on the position (as in our case in view of the matrix a). Nevertheless one could expect that if the domain is symmetric then the solution inherits symmetry properties. If the domain is a ball or an annulus and we consider the semilinear non–degenerate case, axial symmetry of the solutions is proved in [**9**] assuming that Morse index information concerning the solution is known and assuming that the nonlinearity is convex.

The idea behind [**9**], which we shall also exploit here in a different (and possibly degenerate) context and with different techniques, is to consider the solution u and its reflection (say $\tilde{u}$) directly in the entire domain.

2. Preliminaries

The following two lemmas can be found in [**11**]. For the reader's convenience we recall the proofs.

Lemma 2.1. (Lemma 2.3.3 of [**11**]). *Assume that conditions* (H_1), (H_2), (H_4) *are fulfilled. Then the operator* $a(x)A(|Du|)Du$ *is elliptic in* $\Omega \times \mathbb{R}^N$, *i.e. the Jacobian matrix* $\partial_\xi\{a(x)A(|\xi|)\xi\}$ *is positive definite in* $\Omega \times \mathbb{R}^N$. *Moreover, for any* $x \in \Omega$ *and* ξ, $\eta \in \mathbb{R}^N$, *with* $\xi \neq \eta$, *we have*

$$\langle a(x)A(|\xi|)\xi - a(x)A(|\eta|)\eta\,,\,\xi - \eta\rangle > 0. \tag{2.1}$$

Proof. By direct calculations we get

$$\partial_\xi\{a(x)A(|\xi|)\xi\} = a(x)A(|\xi|)\left[\mathbb{I} + \frac{|\xi|A'(|\xi|)}{A(|\xi|)}\,\frac{\xi\otimes\xi}{|\xi|^2}\right],$$

that is

$$\partial_\xi\{a(x)A(|\xi|)\xi\} = a(x)A(|\xi|)b(\xi),$$

with

$$b(\xi) \equiv \mathbb{I} + c(|\xi|)\frac{\xi\otimes\xi}{|\xi|^2}, \qquad c(|\xi|) = \frac{|\xi|A'(|\xi|)}{A(|\xi|)}.$$

By linear algebra the eigenvalues of the matrix b are 1 with multiplicity $N-1$ and $1+c$. Note that $1+c>0$ by (H_3). Therefore b is symmetric and positive definite and by [**8, 2, 13**], see Theorem 2.1 of [**2**], the (symmetric) product ab is positive definite, provided

$$\left(\sqrt{\frac{\Lambda}{\lambda}} - 1\right)\left(\sqrt{1+c} - 1\right) < 2 \quad \text{if} \qquad c \geq 0$$

$$\left(\sqrt{\frac{\Lambda}{\lambda}} - 1\right)\left(\sqrt{\frac{1}{1+c}} - 1\right) < 2 \quad \text{if } -1 < c < 0,$$

that is

$$\sqrt{\Lambda/\lambda} < \phi(c).$$

Let us first prove (2.1) assuming that $0 \notin [\xi,\eta]$ (that is, 0 is not on the segment from ξ to η). We have for some $\zeta \in [\xi,\eta]$

$$\begin{aligned}\langle a(x)A(|\xi|)\xi - a(x)A(|\eta|)\eta, \xi-\eta\rangle &= \langle \partial_\xi\{a(x)A(|\zeta|)\zeta\}(\xi-\eta), \xi-\eta\rangle \\ &= A(|\zeta|)\langle a(x)b(\zeta)(\xi-\eta), \xi-\eta\rangle > 0,\end{aligned}$$

since we already proved that ab is positive definite.

When $0 \in [\xi,\eta]$, we can exploit the same arguments in $[\eta,0]$ and $[0,\xi]$, using the fact that $\boldsymbol{A}$ is continuous. □

If $a(x) = \mathbb{I}$ we have

Lemma 2.2. (Lemma 2.3.2 of [**11**]). *Assume* (H_3). *Then for all* ξ *and* η *in* $\mathbb{R}^N$, *with* $\xi \neq \eta$,

$$\langle A(|\xi|)\xi - A(|\eta|)\eta\,,\,\xi-\eta\rangle > 0.$$

PROOF. If either of the vectors is 0 the assertion is trivial since $\boldsymbol{A}(0) = 0$. Otherwise, since $A(t) > 0$ for $t > 0$ and $\langle \xi, \eta \rangle \le |\xi| \cdot |\eta|$, we have

$$\begin{aligned}
&\langle A(|\xi|)\xi - A(|\eta|)\eta \,,\, \xi - \eta \rangle \\
&\qquad = A(|\xi|)|\xi|^2 + A(|\eta|)|\eta|^2 - A(|\xi|)\langle \xi \,,\, \eta \rangle - A(|\eta|)\langle \xi \,,\, \eta \rangle \\
&\qquad \ge \Phi(|\xi|)|\xi| + \Phi(|\eta|)|\eta| - \Phi(|\xi|)|\eta| - \Phi(|\eta|)|\xi| \\
&\qquad = \{\Phi(|\xi|) - \Phi(|\eta|)\}(|\xi| - |\eta|)
\end{aligned}$$

and the conclusion now comes from the strict monotonicity of Φ. □

Exploiting now Lemma 2.1 and Lemma 2.2, we prove the following Comparison Principle, see Theorem 3.3.3 of [**11**].

PROPOSITION 2.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and let u, $v \in W^{1,\infty}_{\rm loc}(\Omega) \cap C(\overline{\Omega})$ be such that*

$$\begin{aligned}
&\int_\Omega \langle a(x)A(|Du|)Du, D\varphi\rangle dx - \int_\Omega B(x,u)\varphi \, dx \\
&\qquad \le \int_\Omega \langle a(x)A(|Dv|)Dv, D\varphi\rangle dx - \int_\Omega B(x,v)\varphi \, dx.
\end{aligned} \tag{2.2}$$

for any $\varphi \in C^\infty_0(\Omega)$. Assume that $B(x,z) \in L^\infty_{\rm loc}(\Omega \times \mathbb{R})$ and is non–increasing in z. Let conditions (H_1), (H_2), (H_4) be fulfilled and suppose that $u \le v$ on $\partial\Omega$. Then

$$u \le v \qquad in \quad \Omega.$$

The same result holds assuming only u, $v \in W^{1,p}_{\rm loc}(\Omega) \cap C(\overline{\Omega})$ when $\Phi(t) \le$ const. t^{p-1}, $p > 1$.

Finally, in the case $a(x) = \rho(|x|)\mathbb{I}$ with ρ positive and locally bounded, the result follows with the assumptions (H_1), (H_2), (H_4) replaced by the weaker condition (H_3).

PROOF. For $\varepsilon > 0$, define $\varphi = \varphi_\varepsilon = (u - v - \varepsilon)^+$. and

$$\Gamma = \Gamma_\varepsilon \equiv \{x \in \Omega \,:\, u(x) - v(x) > \varepsilon\}\,.$$

Then since $u \le v$ on $\partial\Omega$ we have supp $\varphi \subset \Gamma$ and $\Gamma \subset\subset \Omega$ so $\varphi \in W^{1,\infty}_0(\Omega)$.

Therefore, recalling that the matrix a is locally bounded, by density arguments we can use φ as test–function in (2.2) and get

$$\begin{aligned}
&\int_\Gamma \langle a(x)A(|Du|Du - a(x)A(|Dv|)Dv, Du - Dv\rangle dx \\
&\qquad\qquad \le \int_\Gamma [B(x,u) - B(x,v)]\varphi \, dx \le 0.
\end{aligned}$$

where $[B(x,u) - B(x,v)]\varphi$ is non–positive in Γ since $\varphi \ge 0$ and $B(x,z)$ is non–increasing in z. By Lemma 2.1 it now follows easily that $D\varphi = Du - Dv \equiv 0$ in the (open) set Γ.

Also $D\varphi = 0$ a.e. in $\Omega \setminus \Gamma$. Thus $\varphi =$ constant in Ω, and in turn since $\varphi = 0$ in $\Omega \setminus \Gamma$ we get $\varphi = 0$ in Ω. That is $u \le v$ in Ω.

Next assume that u, $v \in W^{1,p}_{\rm loc}(\Omega) \cap C(\overline{\Omega})$ and $\Phi(t) \le$ const. t^{p-1}. Define $\Gamma = \Gamma_\varepsilon$, $\varphi = \varphi_\varepsilon$ as above. In this case $\varphi \in W^{1,p}_0(\Omega)$. Also $\boldsymbol{A}(Du) \le$ const. $|Du|^{p-1}$ so that $\boldsymbol{A}(Du) \in L^{p'}_{\rm loc}(\Omega)$. Consequently, by density arguments, φ can be used as test–function in (2.2) and the result follows as above.

When $a(x) = \rho(|x|)\mathbb{I}$ the result can be proved in exactly the same way using Lemma 2.2 instead of Lemma 2.1. □

3. Symmetry (and monotonicity) results

We exploit the above preliminaries to get symmetry and monotonicity results. Introduce the notation

$$x = (x_1, x'), \qquad x' = (x_2, \ldots, x_N), \qquad \tilde{x} = (-x_1, x').$$

THEOREM 3.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, symmetric with respect to the e_1–direction (that is $x \in \Omega \Leftrightarrow \tilde{x} \in \Omega$). Let $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$ be a solution of the following Dirichlet boundary value problem for* (1.1)*, written in distribution form: for all $\varphi \in C_0^\infty(\Omega)$*

$$\begin{cases} \displaystyle\int_\Omega \langle a(x)A(|Du|)Du, D\varphi\rangle dx = \int_\Omega B(x,u)\varphi\, dx \\ \qquad u = g \qquad \text{on} \qquad \partial\Omega, \end{cases} \tag{3.1}$$

with $B(x,z) \in L^\infty_{\mathrm{loc}}(\Omega \times \mathbb{R})$ non–increasing in z.

Assume that conditions (H_1), (H_2), (H_4) are fulfilled and suppose

$$a(x) = a(\tilde{x}), \qquad B(x,z) = B(\tilde{x}, z), \qquad g(x) = g(\tilde{x}). \tag{3.2}$$

Then u is the only solution of (3.1) *in $W^{1,\infty}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$, and is symmetric with respect to the e_1–direction.*

The same result holds for solutions u only of class $W^{1,p}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$ if $\Phi(t) \le$ const. t^{p-1}, $p > 1$.

PROOF. Let $v \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$ be any other solution of (3.1). Since $u = v$ on $\partial\Omega$, by Proposition 2.1 we get $u \le v$ and $u \ge v$ so that $u = v$. Now let us define

$$\tilde{u}(x) \equiv u(\tilde{x})$$

for any $x \in \Omega$. By the change of variables $x \to \tilde{x}$ it follows that for all $\varphi \in C_0^\infty(\Omega)$

$$\int_\Omega \langle a(\tilde{x})A(|D\tilde{u}|)D\tilde{u}, D\varphi\rangle dx = \int_\Omega B(\tilde{x}, \tilde{u})\varphi\, dx$$
$$\tilde{u}(x) = g(\tilde{x}) \qquad \text{on} \qquad \partial\Omega.$$

By the assumption (3.2) it follows that $\tilde{u}$ is a solution of (3.1) so that $u \equiv \tilde{u}$ and the assertion is proved. □

Proof of Theorem 1.2. By rotation of coordinates, we can use Theorem 3.1 to obtain reflection symmetry with respect to all directions in $\mathbb{R}^N$. Radial symmetry is now apparent; an explicit proof for this is given in [**6**], Lemma 1.8. This completes the proof of Theorem 3.1.

The case when $a(x) = \rho(|x|)\mathbb{I}$ in Theorem 1.1 (and condition (H_3) holds) is proved in the same way, using however the final part of Proposition 2.1.

The ideas of Theorem 3.1 can also be used to prove monotonicity of the solution in the e_1–direction, provided Ω is partially convex and the conditions on the matrix a and the function B are slightly strengthened.

THEOREM 3.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, partially convex in the sense that the intersection of Ω with any line $x' =$ constant is connected, and also symmetric with respect to the e_1–direction. Let $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$ be such that for all $\varphi \in C^\infty_0(\Omega)$*

$$\begin{cases} \displaystyle\int_\Omega \langle a(x)A(|Du|)Du, D\varphi\rangle dx = \int_\Omega B(x,u)\varphi\, dx \\ u > 0 \qquad\qquad \textit{in} \quad \Omega \\ u = 0 \qquad\qquad \textit{on} \quad \partial\Omega, \end{cases}$$

with $B(x,z) \in L^\infty_{\mathrm{loc}}(\Omega \times \mathbb{R})$ non–increasing in z.

Assume that conditions (H_1), (H_2), (H_4) are fulfilled and suppose

$$a(x) = a(x'), \qquad B(x,z) = B(x',z). \tag{3.3}$$

Then u is non–decreasing in the e_1–direction in $\Omega^- = \{x \in \Omega \,|\, x_1 < 0\}$.

The same result holds for solution u only of class $W^{1,p}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$ if $\Phi(t) \le$ const. t^{p-1}, $p > 1$.

PROOF. For $\lambda < 0$ and $x \in \Omega^-$, define

$$\bar{x} = (x_1 + 2(\lambda - x_1), x'),$$

the reflection of the point x across the plane $x_1 = \lambda$. Let $\Omega_\lambda = \{x \in \Omega : x_1 < \lambda\}$. By convexity if $x \in \Omega_\lambda$ then $\bar{x} \in \Omega$, and we can define

$$v(x) = u(\bar{x}).$$

By (3.3) we see that v is a solution of (1.1) in Ω_λ, as also of course is u.

Since $u = 0$ on $\partial\Omega$ and $u > 0$ in Ω, one has $u < v$ on $\partial\Omega_\lambda \cap \partial\Omega$ and $u = v$ on $\partial\Omega_\lambda \setminus \partial\Omega$. Hence by Proposition 2.1 applied in Ω_λ we find that

$$u \le v \qquad \text{in } \ \Omega_\lambda.$$

Now let y, z be two points in Ω^- with $y' = z'$ and $y_1 < z_1 < 0$. Choose specifically

$$\lambda = \frac{y_1 + z_1}{2} < 0.$$

Then $\bar{y} = z$ and so $u(y) \le v(y) = u(\bar{y}) = u(z)$. □

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