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Qualitative Properties of Solutions of m-Laplace Systems

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Abstract

We prove regularity results for the solutions of the equation $-\Delta_m u = h(x)$, such as summability properties of the second derivatives and summability properties of $\frac{1}{|Du|}$. Analogous results were recently proved by the authors for the equation $-\Delta_m u = f(u)$. These results allow us to extend to the case of systems of m-Laplace equations, some results recently proved by the authors for the case of a single equation. More precisely we consider the problem

$$\begin{cases} -\Delta_{m_1}(u) & = f(v) \quad u > 0 \quad \text{in} \quad \Omega \quad , \quad u = 0 \quad \text{on} \quad \partial\Omega \\ -\Delta_{m_2}(v) & = g(u) \quad v > 0 \quad \text{in} \quad \Omega \quad , \quad v = 0 \quad \text{on} \quad \partial\Omega \end{cases}$$

and we prove regularity properties of the solutions as well as qualitative properties of the solutions. Moreover we get a geometric characterization of the critical sets $Z_u \equiv \{x \in \Omega \,|\, Du(x) = 0\}$ and $Z_v \equiv \{x \in \Omega \,|\, Dv(x) = 0\}$. In particular we prove that in convex and symmetric domains we have $Z_u = \{0\} = Z_v$, assuming that 0 is the center of symmetry.

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1 Introduction and statement of the results

Let $(u,v) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$ be a weak solution of the problem

$$\begin{cases} -\Delta_{m_1}(u) & = f(v) \quad u > 0 \quad \text{in} \quad \Omega \quad , u = 0 \quad \text{on} \quad \partial\Omega \\ -\Delta_{m_2}(v) & = g(u) \quad v > 0 \quad \text{in} \quad \Omega \quad , v = 0 \quad \text{on} \quad \partial\Omega \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geqslant 2$, $\Delta_m(u) = \operatorname{div}(|Du|^{m-2}Du)$ is the m-Laplace operator, $1 < m_1, m_2 < \infty$ and the nonlinearities f, g are positive ($f(s), g(s) > 0$ for $s > 0$), nondecreasing and locally Lipschitz continuous.

We study regularity and qualitative properties of the solutions of (1.1) such as symmetry and monotonicity properties. Moreover we study geometric properties of the critical sets Z_u and Z_v, where

$$Z_u \equiv \{x \in \Omega \,|\, Du(x) = 0\}, \qquad \text{and} \qquad Z_v \equiv \{x \in \Omega \,|\, Dv(x) = 0\}. \tag{1.2}$$

We exploit the techniques recently introduced by the authors in [13] and in [14] where the case of a single equation is considered. In particular, to extend the techniques introduced by the authors in [13] to the case of systems of m-Laplace equations, it is necessary first to extend some regularity results proved in [13], where the case $-\Delta_m(u) = f(u)$ is considered. In details (see Theorem 2.1) we prove summability properties of the second derivatives of the equation :

$$-\Delta_m(u) = h(x)^1 \tag{1.3}$$

Then (see Theorem 2.2) we prove summability properties of $\frac{1}{|Du|^{m-2}}$ for any solution u of the problem:

$$\begin{cases} -\Delta_m(u) & = h(x) \qquad \text{in}\, \Omega \\ \quad u & > 0 \qquad \text{in}\, \Omega \\ \quad u & = 0 \qquad \text{on}\, \partial\Omega \end{cases} \tag{1.4}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geqslant 2$, $\Delta_m(u) = \operatorname{div}(|Du|^{m-2}Du)$ is the m-Laplace operator, $1 < m < \infty$.

In both cases we have the following hypothesis for h:

(*) $h \in C^{0,\alpha} \cap W^{1,q}_{loc}(\Omega)$ with $q \geq \max\{\frac{N}{2}, 2\}$.

The results we get may be summarized as follows:

Theorem 1.1 *Let $u \in C^1(\Omega)$ be a weak solution of* (1.3)*, with h satisfying (*) $1 < m < \infty$. Then for any $E \subset\subset \Omega$ and for every $i, j = 1, \ldots, N$, we have*

$$\sup_{x\in\Omega} \int_{E \setminus \{u_{x_i}=0\}} \frac{|Du|^{m-2}}{|u_{x_i}|^{\beta}|x-y|^{\gamma}} |Du_i|^2 \, dy < C$$

[1] we devote an entire section to this equation, which is interesting in its own.

where $\beta < 1$, $\gamma < N - 2$ *if* $N \geqslant 3$, $\gamma = 0$ *if* $N = 2$ *and* $C = C(\beta, \gamma, E)$. *Moreover*

$$\sup_{x \in \Omega} \int_{E \setminus Z_u} \frac{|Du|^{m-2-\beta}}{|x-y|^{\gamma}} \|D^2 u\|^2 \, dy < C,$$

where $Z_u = \{x \in \Omega : Du(x) = 0\}$ *is the critical set of the solution. Furthermore, if* u *is a weak solution of* (1.4) *with* $h(s) > 0$ *for* $s > 0$, *then, for any* $x \in \Omega$ *and for every* $r < 1$, *we have that* ($|Z_u| = 0$ *and)*

$$\int_{\Omega} \frac{1}{|Du|^{(m-1)r}} \frac{1}{|x-y|^{\gamma}} \, dy \leqslant C$$

where C *does not depend on* x, $\gamma < N - 2$ *if* $N \geqslant 3$ *and* $\gamma = 0$ *if* $N = 2$.

In particular these regularity results apply to problem (1.1) with $h = f(v)$ or $h = g(u)$ (see Theorem 3.1). Therefore we get summability properties of $\frac{1}{\rho_u}$ and $\frac{1}{\rho_v}$, where

$$\rho_u \equiv |Du|^{m-2} \qquad \text{and} \qquad \rho_v \equiv |Dv|^{m-2}. \tag{1.5}$$

The summability properties we get are exactly those needed in [13] to prove weighted Sobolev and Poincaré inequalities. We refer to [25] and [32] for the theory of weighted Sobolev spaces $H^{1,p}_{\rho}(\Omega)$. Moreover in Section 3 we briefly recall the relevant definitions and the properties. Therefore also in our case we get the following:

Theorem 1.2 *Let* $(u, v) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$ *be a weak solution of* (1.1) *where* Ω *is a bounded smooth domain in* $\mathbb{R}^N$, $N \geqslant 2$, $1 < m_1, m_2 < \infty$ *and the nonlinearities* f, g *are positive(*$f(s), g(s) > 0$ *for* $s > 0$*) and locally Lipschitz continuous. Then, if we consider* $\rho_u = |Du|^{m_1-2}$ *and* $\rho_v = |Dv|^{m_2-2}$, *we get, for every* $p \geqslant 2$

$$\|\xi\|_{L^p(\Omega)} \leqslant C_1(|\Omega|) \|D\xi\|_{L^p(\Omega, \rho_u)} \qquad \textit{for every } \xi \in H^{1,p}_{0,\rho_u}(\Omega) \tag{1.6}$$

and

$$\|\eta\|_{L^p(\Omega)} \leqslant C_2(|\Omega|) \|D\eta\|_{L^p(\Omega, \rho_v)} \qquad \textit{for every } \eta \in H^{1,p}_{0,\rho_v}(\Omega) \tag{1.7}$$

where $C_1(|\Omega|), C_2(|\Omega|) \to 0$ *if* $|\Omega| \to 0$. *In particular,* (1.6) *and* (1.7) *hold for every* $\xi \in H^{1,2}_{0,\rho_u}(\Omega)$ *or* $\eta \in H^{1,2}_{0,\rho_v}(\Omega)$.

Moreover, by Theorem 1.1, if $(u, v) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$ is a weak solution of (1.1), we get that $|Du|^{m_1-2}Du \in W^{1,2}(\Omega, \mathbb{R}^N)$ and $|Dv|^{m_2-2}Du \in W^{1,2}(\Omega, \mathbb{R}^N)$, and we can define the linearized operator at a fixed solution (u, v):

$$L_{(u,v)}((u_{x_i}, v_{x_j}), (\varphi, \psi)) \equiv (L^1_{(u,v)}((u_{x_i}, v_{x_j}), (\varphi, \psi)), L^2_{(u,v)}((u_{x_i}, v_{x_j}), (\varphi, \psi))$$

where

$$L^1_{(u,v)}((u_{x_i}, v_{x_j}), (\varphi, \psi)) \equiv$$

$$\int_{\Omega} [|Du|^{m_1-2}(Du_{x_i}, D\varphi) + (m_1 - 2)|Du|^{m_1-4}(Du, Du_{x_i})(Du, D\varphi) - f'(v) v_{x_i} \varphi] dx$$

and

$$L^2_{(u,v)}((u_{x_i}, v_{x_j}), (\varphi, \psi)) \equiv$$

$$\int_\Omega [|Dv|^{m_2-2}(Dv_{x_i}, D\psi) + (m_2 - 2)|Dv|^{m_2-4}(Dv, Dv_{x_i})(Dv, D\psi) - g'(u)u_{x_i}\psi]dx$$

for any φ , $\psi \in C^1_0(\Omega)$ and $1 < m_1, m_2 < \infty$. Moreover the following equation holds

$$L_{(u,v)}((u_{x_i}, v_{x_j}), (\varphi, \psi)) = 0 \qquad \forall(\varphi, \psi) \in C^1_0(\Omega) \times C^1_0(\Omega), \quad i, j = 1, \dots, N. \quad (1.8)$$

More generally, if $(w, h) \in H^{1,2}_{\rho_u}(\Omega) \times H^{1,2}_{\rho_v}(\Omega)$, we can also define $L_{(u,v)}((w, h), (\varphi, \psi))$ as above. In this case we say that $(w, h) \in H^{1,2}_{\rho_u}(\Omega) \times H^{1,2}_{\rho_v}(\Omega)$ is a weak solution of the linearized equation if for any φ , $\psi \in C^1_0(\Omega)$

$$L_{(u,v)}((w, h), (\varphi, \psi)) \equiv (L^1_{(u,v)}((w, h), (\varphi, \psi)), L^2_{(u,v)}((w, h), (\varphi, \psi)) \equiv (0, 0). \quad (1.9)$$

In particular, by density arguments we can suppose , $(\varphi, \psi) \in H^{1,2}_{0,\rho_u}(\Omega) \times H^{1,2}_{0,\rho_v}(\Omega)$. Here, given a general weight $\rho \in L^1(\Omega)$, $H^{1,p}_{0,\rho}(\Omega)$ is defined as the closure of $C^1_c(\overline{\Omega})$ (or $C^\infty_c(\overline{\Omega})$) in $H^{1,p}_\rho(\Omega)$ (see Section 3).

Exploiting the linearized equation and the weighted Poincaré inequality proved before, we can use the results of [14] and prove the following:

Theorem 1.3 *Let* $(w, h) \in H^{1,2}_{\rho_u}(\Omega) \times H^{1,2}_{\rho_v}(\Omega)$ *be nonnegative weak solutions of* (3.6) *in a bounded smooth domain* Ω *of* $\mathbb{R}^N$, $N \geqslant 2$, $2 < m_1, m_2 < \infty$ *and suppose that the nonlinearities* f, g *are positive* ($f(s), g(s) > 0$ *for* $s > 0$)*, nondecreasing and locally Lipschitz continuous. Suppose that* $\overline{B(x, 5\delta)} \subset \Omega$*. Let us set*

$$\frac{1}{\overline{2}^*} = \frac{1}{2} - \frac{1}{N} + \frac{1}{N}\left(\frac{m-2}{m-1}\right)$$

(consequently $\overline{2}^* > 2$ *for* $m > 2$*) and let* 2^* *be any real number such that* $2 < 2^* < \overline{2}^*$*. Then for every* $0 < s < \chi$, $\chi \equiv \frac{2^*}{2}$*, there exists* $C_1, C_2 > 0$ *such that*

$$\|w\|_{L^s(B(x,2\delta))} \leqslant C_1 \inf_{B(x,\delta)} w \quad (1.10)$$

and

$$\|h\|_{L^s(B(x,2\delta))} \leqslant C_2 \inf_{B(x,\delta)} h \quad (1.11)$$

where C_1, C_2 *are constants depending on* x, s, N, u, m, f.

If $\frac{2N+2}{N+2} < m_1 < 2$ *or* $\frac{2N+2}{N+2} < m_2 < 2$*, the same result holds with* χ *replaced by* $\chi' \equiv \frac{2^\sharp}{s^\sharp}$ *where* $2^\sharp$ *is the classical Sobolev exponent,* $\frac{2}{s^\sharp} \equiv 1 - \frac{1}{s}$ *and* $s < \frac{m_1-1}{2-m_1}$ *or* $s < \frac{m_2-1}{2-m_2}$ *respectively.*

As a consequence we get a strong maximum principle for the linearized operator:

Theorem 1.4 (Strong Maximum Principle) *Let* $(w,h) \in H^{1,2}_{\rho_u}(\Omega) \times H^{1,2}_{\rho_v}(\Omega) \cap C^0(\Omega) \times C^0(\Omega)$ *be nonnegative weak solutions of* (1.9) *in a bounded smooth domain* Ω *of* $\mathbb{R}^N$*,* $N \geqslant 2$*,* $2 < m_1, m_2 < \infty$ *and suppose that the nonlinearities* f, g *are positive(*$f(s), g(s) > 0$ *for* $s > 0$*), nondecreasing and locally Lipschitz continuous. Then, for any domain* $\Omega' \subset \Omega$ *with* $w \geqslant 0$ *in* Ω' *and* $h \geqslant 0$ *in* Ω'*, we have* $w \equiv 0$ *in* Ω' *or* $w > 0$ *in* Ω' *and* $h \equiv 0$ *in* Ω' *or* $h > 0$ *in* Ω'*.*

These preliminary results allow us to exploit the Alexandrov-Serrin moving plane method and get symmetry and monotonicity properties of the solutions of (1.1). To this aim the key tool is a weak comparison principle in small domains that we will prove exploiting the weighted Poincaré inequality obtained. More precisely we have the following

Theorem 1.5 (Weak Comparison Principle) *Let* $(u,v) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$ *and* $(\bar{u}, \bar{v}) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$ *be weak solutions of* (1.1) *where* Ω *is a bounded smooth domain in* $\mathbb{R}^N$*,* $N \geqslant 2$*,* $1 < m_1, m_2 < \infty$ *and the nonlinearities* f, g *are both positive (*$f(s), g(s) > 0$ *for* $s > 0$*), nondecreasing and locally Lipschitz continuous.*

Let $\Omega' \subseteq \Omega$ *be open and suppose* $u \leqslant \bar{u}$ *on* $\partial\Omega'$ *and* $v \leqslant \bar{v}$ *on* $\partial\Omega'$*. Then there exists* $\delta > 0$ *such that, if* $|\Omega'| \leqslant \delta$*, then* $u \leqslant \bar{u}$ *in* Ω' *and* $v \leqslant \bar{v}$ *in* Ω'*.*

We then exploit Theorem 1.5 and prove the following result. We refer to Section 5 for the definitions of $u(x^\nu_\lambda)$, $v(x^\nu_\lambda)$, Ω^ν_λ, and other definitions which are customary in the Alexandrov-Serrin moving planes method.

Theorem 1.6 *Let* $(u,v) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$ *be a weak solution of* (1.1) *where* Ω *is a bounded smooth domain in* $\mathbb{R}^N$*,* $N \geqslant 2$*,* $1 < m_1, m_2 < \infty$ *and the nonlinearities* f, g *are positive(*$f(s), g(s) > 0$ *for* $s > 0$*), nondecreasing and locally Lipschitz continuous.*
For any direction ν *and for* λ *in the interval* $(a(\nu), \lambda_2(\nu)]$*, we have*

$$u(x) \leqslant u(x^\nu_\lambda) \quad \textit{and} \quad v(x) \leqslant v(x^\nu_\lambda) \quad \forall x \in \Omega^\nu_\lambda. \tag{1.12}$$

Moreover, for any λ*, with* $a(\nu) < \lambda < \lambda_2(\nu)$*, we have*

$$u(x) < u(x^\nu_\lambda) \quad \forall x \in \Omega^\nu_\lambda \setminus Z^\nu_{u\lambda} \tag{1.13}$$

where $Z^\nu_{u\lambda} \equiv \{x \in \Omega^\nu_\lambda \,:\, Du(x) = Du^\nu_\lambda(x) = 0\}$*, and*

$$v(x) < v(x^\nu_\lambda) \quad \forall x \in \Omega^\nu_\lambda \setminus Z^\nu_{v\lambda}. \tag{1.14}$$

where $Z^\nu_{v\lambda} \equiv \{x \in \Omega^\nu_\lambda \,:\, Du(x) = Du^\nu_\lambda(x) = 0\}$*. Finally*

$$\frac{\partial u}{\partial \nu}(x) > 0 \quad \forall x \in \Omega^\nu_{\lambda_2(\nu)} \setminus Z_u \tag{1.15}$$

where $Z_u = \{x \in \Omega : Du(x) = 0\}$*, and*

$$\frac{\partial v}{\partial \nu}(x) > 0 \quad \forall x \in \Omega^\nu_{\lambda_2(\nu)} \setminus Z_v \tag{1.16}$$

where $Z_v = \{x \in \Omega : Dv(x) = 0\}$*.*

Theorem 1.6 was proved by C. Azizieh in [2] for the case $1 < m_1 \leq 2$ and $1 < m_2 \leq 2$. The proof in [2] relies on the techniques introduced by L. Damascelli and F. Pacella in [11] and [12]. We will instead follow the proof proposed by the authors in [13] where a general result on the geometric properties of the critical set (see Theorem 5.1) allows to avoid local symmetry phenomena in a very simple way. At the same time, this proof applies to a larger class of domains (see e.g. the smoothed rectangle).

Remark 1.1 The results in [2] have been used in [4] (see also [3]), where, following [7, 8], existence results and a priori estimate for the solutions of some elliptic systems involving m-laplace equations are proved.

The literature about semilinear (nondegenerate) elliptic systems, is wide. We refer to [6, 7, 8, 15] and the references therein for some results about existence and a priori estimates for the solutions.

In the case $\frac{2N+2}{N+2} < m_1, m_2 < \infty$, using Theorem 1.4, we improve considerably Theorem 1.6. In particular we can prove a result (see Theorem 5.3) that, in the case of convex and symmetric domains, the following holds:

Theorem 1.7 *Let $(u, v) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$ be a weak solution of (1.1) where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geqslant 2$, $\frac{2N+2}{N+2} < m_1, m_2 < \infty$ and the nonlinearities f, g are positive($f(s), g(s) > 0$ for $s > 0$), nondecreasing and locally Lipschitz continuous.*

If the domain Ω is convex with respect to a direction ν and symmetric with respect to the hyperplane $T_0^\nu = \left\{x \in \mathbb{R}^N : x \cdot \nu = 0\right\}$ then u and v are symmetric, i. e. $u(x) = u(x_0^\nu)$ and $v(x) = v(x_0^\nu)$, and nondecreasing in the ν–direction in Ω_0^ν with[2] $\frac{\partial u}{\partial \nu}(x) > 0$ in Ω_0^ν and $\frac{\partial v}{\partial \nu}(x) > 0$ in Ω_0^ν.

In particular $Z_u \subset T_0^\nu$ and $Z_v \subset T_0^\nu$. Therefore if for N orthogonal directions e_i the domain Ω is symmetric with respect to any hyperplane $T_0^{e_i}$ and $\lambda_2(e_i) = \lambda_2(-e_i) = 0$, then

$$Z_u \equiv \{x \in \Omega \,|\, D(u)(x) = 0\} = \{0\} = Z_v \equiv \{x \in \Omega \,|\, D(v)(x) = 0\} \tag{1.17}$$

assuming that 0 is the center of symmetry.

Finally, since the m-Laplace operator in not degenerate in $\Omega \setminus \{0\}$, we get

$$u \in C^2(\Omega \setminus \{0\}) \quad \text{and} \quad v \in C^2(\Omega \setminus \{0\}).$$

The paper is organized as follows:
In Section 2 we prove some general regularity results for the solutions of (1.4). In Section 3 we exploit these results in the case of problem (2.2) proving in particular a weak maximum principle in small domains for the solutions of (2.2) and Theorem 1.4. In Section 5 we prove Theorem 1.3 Theorem 1.6 and Theorem 1.7.

[2] The crucial novelty here is that we get $\frac{\partial u}{\partial \nu}(x) > 0$ and $\frac{\partial v}{\partial \nu}(x) > 0$ in Ω_0^ν. Previously, by Theorem 1.6 this was known only in $\Omega_0^\nu \setminus Z_u$ or in $\Omega_0^\nu \setminus Z_v$.

2 Regularity results

In this section we prove some general regularity results for weak $C^1(\overline{\Omega})$ solutions of the equation

$$-\Delta_m(u) = h(x) \tag{2.1}$$

and in particular for solutions of the problem

$$\begin{cases} -\Delta_m(u) & = h(x) \quad \text{in}\,\Omega \\ \quad u & > 0 \quad\quad\ \text{in}\,\Omega \\ \quad u & = 0 \quad\quad\ \text{on}\,\partial\Omega \end{cases} \tag{2.2}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geqslant 2$, $\Delta_m(u) = \operatorname{div}(|Du|^{m-2}Du)$ is the m-Laplace operator, $1 < m < \infty$, and we have the following hypotheses on h:

(*) $h \in C^{0\,,\,\alpha} \cap W^{1,q}_{loc}(\Omega)$ with $q \geq \max\{\frac{N}{2}, 2\}$ $\alpha \in (0,1)$.

We recall that the set where problem (2.2) is degenerate is exactly the critical set Z_u of u, where

$$Z_u \equiv \{x \in \Omega \,|\, Du(x) = 0\}. \tag{2.3}$$

Therefore in $\Omega \setminus Z_u$ we can use standard elliptic regularity (see e.g. Theorem 6.4 in [23]) and deduce that

$$u \in C^{2,\alpha}(\Omega \setminus Z_u).$$

Also if $h \in C^{0,\alpha}(\overline{\Omega})$ it follows that $u \in C^{2,\alpha}(\overline{\Omega} \setminus Z_u)$.

We extend here to (2.1) and (2.2) some regularity results recently obtained by the authors [13] for the problem

$$-\Delta_m(u) = f(u).$$

In particular, we prove summability properties of the second derivatives of the solutions and summability properties of $\frac{1}{|Du|^{m-2}}$. We recall that summability properties of $\frac{1}{|Du|^{m-2}}$ are the key tool which allows to get a weighted Sobolev inequality.

Summability properties of the second derivatives of the solutions will be deduced using the linearized equation. Since at the moment the linearized equation is well defined only in $\Omega \setminus Z_u$ where u is smooth, for the time being we use the definition of the linearized operator at the fixed solution u only with test function $\varphi \in W^{1,2}(\Omega)$ *with compact support in* $\Omega \setminus Z_u$. Later, our regularity results will allow us to define the linearized operator in the entire region Ω.

Let us first observe that, arguing exactly as in Lemma 2.1 of [13], we get the following:

Lemma 2.1 *Let $u \in C^1(\Omega)$ be a weak solution of* (2.1), *with h satisfying (*). Then, for every $\varphi \in W^{1,2}(\Omega)$ with compact support in $\Omega \setminus Z_u$, $L_u(u_{x_i}, \varphi)$ is well defined by*

$$L_u(u_{x_i}, \varphi) \equiv$$

$$\int_\Omega [|Du|^{m-2}(\tilde{D}u_i, D\varphi) + (m-2)|Du|^{m-4}(Du, \tilde{D}u_i)(Du, D\varphi) - \frac{\partial h}{\partial x_i}\varphi]dx.$$

Moreover, we have

$$L_u(u_{x_i}, \varphi) = 0 \tag{2.4}$$

for every $\varphi \in W^{1,2}(\Omega)$ *with compact support in* $\Omega \setminus Z_u$.

Theorem 2.1 *Let* $u \in C^1(\Omega)$ *be a weak solution of* (2.1), *with* h *satisfying (*)* $1 < m < \infty$. *Then for any* $E \subset\subset \Omega$ *and for every* $i, j = 1, \ldots, N$, *we have*

$$\sup_{x\in\Omega} \int_{E\setminus\{u_{x_i}=0\}} \frac{|Du|^{m-2}}{|u_{x_i}|^\beta |x-y|^\gamma} |Du_i|^2\, dy < C$$

where $\beta < 1$, $\gamma < N-2$ *if* $N \geqslant 3$, $\gamma = 0$ *if* $N = 2$ *and* $C = C(\beta, \gamma, E)$. *Moreover*

$$\sup_{x\in\Omega} \int_{E\setminus Z_u} \frac{|Du|^{m-2-\beta}}{|x-y|^\gamma} \|D^2u\|^2\, dy < C,$$

where $Z_u = \{x \in \Omega : Du(x) = 0\}$ *is the critical set of the solution.*

Proof. Let us observe that we can suppose that $x \in E$ without loss of generality. In fact, suppose that we prove that for every measurable set $E \subset\subset \Omega$ we have

$$\sup_{x\in E} \int_{E\setminus\{u_{x_i}=0\}} \frac{|Du|^{m-2}}{|u_{x_i}|^\beta |x-y|^\gamma} |\tilde{D}u_i|^2\, dy \leqslant K(\beta, \gamma, E).$$

Then if $0 < \delta \leqslant \frac{1}{2}$ dist $(E, \partial\Omega)$ and $E_\delta = \{x \in \Omega :$ dist $(x, E) \leqslant \delta\}$, considering the two cases $x \in E_\delta$ and $x \in \Omega \setminus E_\delta$, it follows that

$$\sup_{x\in\Omega} \int_{E\setminus\{u_{x_i}=0\}} \frac{|Du|^{m-2}}{|u_{x_i}|^\beta |x-y|^\gamma} |\tilde{D}u_i|^2\, dy \leqslant K(\beta, \gamma, E_\delta) + \frac{1}{\delta^\gamma} K(\beta, 0, E).$$

Let $E \subset\subset \Omega$, $x \in E$, and consider a cut-off function $\varphi \in C_c^\infty(\Omega)$ such that $\varphi \geqslant 0$ in Ω, and $\varphi \equiv 1$ in $E_\delta = \{x \in \Omega \,|$ dist $(x, E) \leqslant \delta\}$ where $0 < \delta \leqslant \frac{1}{2}$ dist $(E, \partial\Omega)$.

Let G_ϵ be defined by

$$\begin{cases} G_\epsilon(s) = 0 & \text{if } |s| \leqslant \epsilon \\ G_\epsilon(s) = 2s - 2\epsilon & \text{if } \epsilon \leqslant s \leqslant 2\epsilon \\ G_\epsilon(s) = 2s + 2\epsilon & \text{if } -2\epsilon \leqslant s \leqslant -\epsilon \\ G_\epsilon(s) = s & \text{if } |s| \geqslant 2\epsilon \end{cases}$$

so that G_ϵ is a Lipschitz continuous function and $0 \leqslant G'_\epsilon \leqslant 2$. To obtain our result, we will consider the case $x \in E \cap Z_u$ and $x \in E \setminus Z_u$ separately.

Case 1. Suppose first that $x \in E \cap Z_u$. In this case define $\psi_{\epsilon,x}(y) = \frac{G_\epsilon(u_{x_i})(y)}{|u_{x_i}(y)|^\beta} \frac{\varphi(y)}{|x-y|^\gamma}$ with $\beta < 1$, $\gamma < N-2$ and $N \geqslant 3$. If $N = 2$, we use $\psi_{\epsilon,x} = \frac{G_\epsilon(u_{x_i})}{|u_{x_i}|^\beta}\varphi$. Since $G_\epsilon(u_{x_i})$

vanishes in a neighborhood of each critical point, in particular in a neighborhood of $y = x$, we can use $\psi_{\epsilon,x}$ as a test function in (2.4) and get

$$\begin{aligned}
&\int_\Omega \frac{|Du|^{m-2}}{|u_{x_i}|^\beta}\frac{|\tilde{D}u_i|^2}{|x-y|^\gamma}(G'_\epsilon(u_{x_i}) - \beta\frac{G_\epsilon(u_{x_i})}{u_{x_i}})\varphi\, dy+\\
&+(m-2)\int_\Omega \frac{|Du|^{m-4}(Du,\tilde{D}u_i)^2}{|u_{x_i}|^\beta|x-y|^\gamma}(G'_\epsilon(u_{x_i}) - \beta\frac{G_\epsilon(u_{x_i})}{u_{x_i}})\varphi\, dy+\\
&+\int_{\Omega\setminus E_\delta}|Du|^{m-2}(\tilde{D}u_i, D\varphi)\frac{G_\epsilon(u_{x_i})}{|u_{x_i}|^\beta}\frac{1}{|x-y|^\gamma}\,dy+\\
&+(m-2)\int_{\Omega\setminus E_\delta}|Du|^{m-4}(Du,\tilde{D}u_i)(Du,D\varphi)\frac{G_\epsilon(u_{x_i})}{|u_{x_i}|^\beta}\frac{1}{|x-y|^\gamma}\,dy+\\
&+\int_\Omega |Du|^{m-2}(\tilde{D}u_i, D_y(\frac{1}{|x-y|^\gamma}))\frac{G_\epsilon(u_{x_i})}{|u_{x_i}|^\beta}\varphi\, dy+\\
&+(m-2)\int_\Omega |Du|^{m-4}(Du,\tilde{D}u_i)(Du,D_y(\frac{1}{|x-y|^\gamma}))\frac{G_\epsilon(u_{x_i})}{|u_{x_i}|^\beta}\varphi\, dy=\\
&\int_\Omega \frac{\partial\, h}{\partial x_i}\frac{G_\epsilon(u_{x_i})}{|u_{x_i}|^\beta}\frac{1}{|x-y|^\gamma}\varphi\, dy.
\end{aligned}$$

By the definition of G_ϵ it follows that $(G'_\epsilon(u_{x_i}) - \beta\frac{G_\epsilon(u_{x_i})}{u_{x_i}}) \geqslant 0$ in Ω. Therefore we get

$$\begin{aligned}
&\int_\Omega \frac{|Du|^{m-2}|\tilde{D}u_i|^2}{|u_{x_i}|^\beta|x-y|^\gamma}(G'_\epsilon(u_{x_i}) - \beta\frac{G_\epsilon(u_{x_i})}{u_{x_i}})\varphi\, dy \leqslant\\
&\leqslant (m-1)\int_{\Omega\setminus E_\delta}\frac{|Du|^{m-2}|\tilde{D}u_i||D\varphi|}{|x-y|^\gamma}\frac{G_\epsilon(u_{x_i})}{|u_{x_i}|^\beta}\,dy+\\
&\gamma(m-1)\int_\Omega \frac{|Du|^{m-2}|\tilde{D}u_i|}{|x-y|^{\gamma+1}}\frac{G_\epsilon(u_{x_i})}{|u_{x_i}|^\beta}\varphi\, dy+\\
&+\int_\Omega \frac{|\frac{\partial\, h}{\partial x_i}||u_{x_i}|^{1-\beta}}{|x-y|^\gamma}\varphi\, dy.
\end{aligned}$$

By the definition of E_δ, since $x \in E$, we know that $\sup\limits_{y\in\Omega\setminus E_\delta}\dfrac{1}{|x-y|^\gamma} \leqslant \dfrac{1}{\delta^\gamma}$ and, using the fact that $|Du|^{m-2}|\tilde{D}u_i| \in L^2_{loc}(\Omega)$, since φ has compact support in Ω, we get

$$\int_{\Omega\setminus E_\delta}\frac{|Du|^{m-2}|\tilde{D}u_i||D\varphi|}{|x-y|^\gamma}\frac{G_\epsilon(u_{x_i})}{|u_{x_i}|^\beta}\,dy \leqslant C_1,$$

where C_1 does not depend on x.

Let us now note that, since Ω is bounded, then $\int_\Omega \frac{1}{|x-y|^s\, dx}$ is uniformly bounded for any fixed $s < N$. Therefore, since u is C^1 we get

$$\int_\Omega \frac{|\frac{\partial h}{\partial x_i}||u_{x_i}|^{1-\beta}}{|x-y|^\gamma}\varphi\, dy \leq \leq const \int_\Omega \frac{|\frac{\partial h}{\partial x_i}|}{|x-y|^\gamma}\varphi\, dy \tag{2.5}$$

where $\gamma < N-2$.

By Young's inequality with exponents $\frac{N}{N-2}$ and $(\frac{N}{N-2})' = \frac{N}{2}$ (note that $\gamma\frac{N}{N-2} < N$), we get

$$\begin{aligned}&\int_\Omega \frac{|\frac{\partial h}{\partial x_i}||u_{x_i}|^{1-\beta}}{|x-y|^\gamma}\varphi\, dy \leq \\ &\leq const\,\Big(\int_{supp\,\varphi}|\frac{\partial h}{\partial x_i}|^{\frac{N}{2}}\, dy + \int_{supp\,\varphi}\frac{1}{|x-y|^{\gamma\frac{N}{N-2}}}\Big) \leq \\ &\leq const\,\Big(\|h\|^{\frac{N}{2}}_{W^{1,\frac{N}{2}}(supp\,\varphi)} + const\Big) \leq C_2\end{aligned} \tag{2.6}$$

where C_2 does not depend on x. From now on, the proof is exactly the one of Theorem 2.2 in [13].

Case 2. Suppose now that $x \in E \setminus Z_u$. In this case, consider E and E_δ as above, and for $\epsilon > 0$ small consider a cut-off function $\varphi_{\epsilon,x} \in C_c^\infty(\Omega)$ such that $\varphi_{\epsilon,x} \geqslant 0$ in Ω, $\varphi_{\epsilon,x} \equiv 0$ in $B_\epsilon(x)$, $\varphi_{\epsilon,x} \equiv 1$ in $E_\delta \setminus B_{2\epsilon}(x)$, $|D\varphi_{\epsilon,x}| \leqslant \frac{C}{\epsilon}$ in $B_{2\epsilon}(x) \setminus B_\epsilon(x)$ and $|D\varphi_{\epsilon,x}| \leqslant c_1$ outside $B_{2\epsilon}(x)$. Moreover suppose that there exists a set $A \subset\subset \Omega$ such that supp $(\varphi_{\epsilon,x}) \subset A$ for every ϵ and $x \in E$.
Using $\psi_{\epsilon,x} = \frac{G_\epsilon(u_{x_i})}{|u_{x_i}|^\beta}\frac{1}{|x-y|^\gamma}\varphi_{\epsilon,x}$ as a test function in (2.4) and arguing as in Theorem 2.2 in [13], the thesis follows.

As a consequence of the previous estimates we can prove

Corollary 2.1 *Let $u \in C^1(\Omega)$ be a weak solution of* (2.2) *with h satisfying (*), $1 < m < \infty$. Then $u \in C^{2,\alpha}(\Omega \setminus Z_u)$, where $Z_u = \{x \in \Omega : Du(x) = 0\}$ is the critical set of the solution, $|Du|^{m-2}Du \in W^{1,2}_{loc}(\Omega, \mathbb{R}^N)$, and therefore $|Du|^{m-1} \in W^{1,2}_{loc}(\Omega)$. If moreover, Ω is smooth, $u \in C^1(\overline{\Omega})$ and $h \in C^{o,\alpha}(\overline{\Omega})$ is nonnegative, then $Z_u \cap \partial\Omega = \emptyset$, $u \in C^{2,\alpha}(\overline{\Omega} \setminus Z_u)$, $|Du|^{m-2}Du \in W^{1,2}(\Omega, \mathbb{R}^N)$ and $|Du|^{m-1} \in W^{1,2}(\Omega)$.*

Proof. By elliptic regularity, $u \in C^{2,\alpha}(\Omega \setminus Z_u)$ (see e.g. Theorem 6.4 in [23]), since it satisfies an uniformly elliptic equation in a neighborhood of each regular point $x \in \Omega \setminus Z_u$. Recall that by Theorem 2.1 (where we have used test function with compact support in $\Omega \setminus Z_u$ only) we obtain that

$$\int_{E\setminus Z_u} |Du|^{m-2-\beta}\|D^2u\|^2\, dx < C, \tag{2.7}$$

where $\beta < 1$, $Z_u = \{x \in \Omega : Du(x) = 0\}$ is the critical set of the solution, and E is any compact set contained in Ω.

Let us now set

$$\phi_n \equiv G_{\frac{1}{n}}(|Du|^{m-2}u_{x_i})$$

where $G_{\frac{1}{n}}$ is defined as in Theorem 2.1, $n \in \mathbb{N}$ and $i \in \{1, \ldots, N\}$. By the definition of $G_{\frac{1}{n}}$ we get that $\phi_n \in W^{1,2}(E)$ and

$$\frac{\partial}{\partial x_j}\phi_n = G'_{\frac{1}{n}}(|Du|^{m-2}u_{x_i})\frac{\partial}{\partial x_j}(|Du|^{m-2}u_{x_i}). \tag{2.8}$$

Therefore, exploiting (2.7), we get

$$\|\phi_n\|_{W^{1,2}(E)} \leqslant K \quad \forall\, n \in \mathbb{N} \tag{2.9}$$

where we also use the fact that $(|Du|^{m-2})2 \le c|Du|^{m-2-\beta}$ for $1 < m < \infty$ and $\beta > (2-m)$ if $1 < m < 2$.

Since $W^{1,2}(E)$ has a compact embedding in $L^2(E)$, up to subsequences there exists $\varpi \in W^{1,2}(E)$ such that

$$\phi_n \longrightarrow \varpi \text{ strongly in } L^2(E),$$

as n tends to infinity and

$$\phi_n \longrightarrow \varpi \text{ almost everywhere in } E.$$

Since $\phi_n \longrightarrow |Du|^{m-2}u_{x_i}$ almost everywhere in E, we get

$$|Du|^{m-2}u_{x_i} \equiv \varpi \in W^{1,2}(E). \tag{2.10}$$

Since $i \in \{1, \ldots, N\}$ is arbitrary, the thesis follows and $|Du|^{m-2}Du \in W^{1,2}_{loc}(\Omega, \mathbb{R}^N)$. If moreover Ω is smooth, $u \in C^1(\overline{\Omega})$ and $h \in C^{o,\alpha}(\overline{\Omega})$ is nonnegative, then $Z_u \cap \partial\Omega = \emptyset$ by the Hopf's lemma. By standard elliptic regularity it follows that u belongs to the class $C^{2,\alpha}$ in a neighborhood of the boundary, so that $u \in C^{2,\alpha}(\overline{\Omega} \setminus Z_u)$ and $|Du|^{m-2}Du \in W^{1,2}(\Omega, \mathbb{R}^N)$.

Remark 2.1 Since a $C^1(\Omega)$ solution u of (2.1) with h satisfying (*) is regular in $\Omega \setminus Z_u$, the generalized derivatives of $|Du|^{m-2}u_{x_i}$, coincide there with the classical ones. Moreover in $\{u_{x_i} = 0\}$, by Stampacchia's Theorem (see e.g. [31] Theorem 1.56, p. 79), the generalized derivatives of $|Du|^{m-2}u_{x_i}$ are zero almost everywhere. From now on we will do all computations taking into account this fact. In particular we get

$$\frac{\partial}{\partial x_j}(|Du|^{m-2}u_{x_i}) \equiv (|Du|^{m-2}\tilde{u}_{ij} + (m-2)|Du|^{m-4}(Du, \tilde{D}u_j)u_{x_i})$$

where $\frac{\partial}{\partial x_j}$ stands for the distributional derivative and $\tilde{u}_{ij}$ are defined by

$$\tilde{u}_{ij} = \begin{cases} u_{x_i x_j} & \text{in} \quad \Omega \setminus Z_u \\ 0 & \text{in} \quad Z_u \end{cases} \tag{2.11}$$

and $\tilde{D}u_i$ stands for the "gradient" $(\tilde{u}_{i1}, \ldots, \tilde{u}_{iN})$.

Lemma 2.2 *Let $u \in C^1(\Omega)$ be a weak solution of (2.2), with h satisfying (*). Then the linearized operator $L_u(u_{x_i}, \varphi)$ is well defined by*

$$L_u(u_{x_i}, \varphi) \equiv$$
$$\int_\Omega [|Du|^{m-2}(\tilde{D}u_i, D\varphi) + (m-2)|Du|^{m-4}(Du, \tilde{D}u_i)(Du, D\varphi) - \frac{\partial h}{\partial x_i}\varphi] dx$$

for every $\varphi \in W^{1,2}(\Omega)$ with compact support in Ω. Moreover we have

$$L_u(u_{x_i}, \varphi) = 0 \tag{2.12}$$

for every $\varphi \in W^{1,2}(\Omega)$ with compact support in Ω.

Furthermore, if Ω is smooth, and h is a nonnegative function in $C^{0,\alpha} \cap W^{1,2}(\Omega)$, then $L_u(u_{x_i}, \varphi) = 0$ for every $\varphi \in W_0^{1,2}(\Omega)$.

Proof. By Corollary 2.1, $|Du|^{m-2}u_{x_i} \in W^{1,2}_{loc}(\Omega)$, so that we can argue as in Lemma 2.1 integrating by parts and, if $\varphi \in C_c^\infty(\Omega)$, we get

$$\begin{aligned} \int_\Omega [|Du|^{m-2}(\tilde{D}u_i, D\varphi) + (m-2)|Du|^{m-4}(Du, \tilde{D}u_i)(Du, D\varphi)]\, dx + \\ - \int_\Omega [\frac{\partial h}{\partial x_i}\varphi]\, dx = 0 \end{aligned} \tag{2.13}$$

i.e.

$$L_u(u_{x_i}, \varphi) = 0.$$

By density we get the general case of $\varphi \in W^{1,2}(\Omega)$ with compact support. If, moreover, Ω is smooth, and h is a nonnegative function in $C^{0,\alpha} \cap W^{1,2}(\Omega)$ then again by Corollary 2.1, $|Du|^{m-2}u_{x_i} \in W^{1,2}(\Omega)$, and, since $h \in W^{1,2}(\Omega)$, by density, we can consider $\varphi \in W_0^{1,2}(\Omega)$.

The results proved in this section allow us finally to get the summability properties of the inverse of the weight $\rho = |Du|^{m-2}$ stated above.

Theorem 2.2 *Let Ω be a smooth domain in $\mathbb{R}^N$, $u \in C^1(\overline{\Omega})$ be a weak solution of (2.2) with h satisfying (*) and $h(s) > 0$ for $s > 0$, $1 < m < +\infty$. Then, for any $x \in \Omega$ and for every $r < 1$, we have that ($|Z_u| = 0$ and)*

$$\int_\Omega \frac{1}{|Du|^{(m-1)r}} \frac{1}{|x-y|^\gamma}\, dy \leqslant C$$

where C does not depend on x, $\gamma < N-2$ if $N \geqslant 3$ and $\gamma = 0$ if $N = 2$.

Proof. Since h is positive, by Hopf's Lemma, there exists E such that $Z_u \subset\subset E \subset\subset \Omega$. Moreover we can suppose dist $(Z_u, \partial E) > 0$. Since $(\Omega \setminus E) \cap Z_u = \emptyset$, it follows that

$$\int_{\Omega\setminus E} \frac{1}{|Du|^{(m-1)r}} \frac{1}{|x-y|^\gamma}\, dy \leqslant \frac{1}{\min_{\Omega\setminus E} |Du|^{(m-1)r}} \int_{\Omega\setminus E} \frac{1}{|x-y|^\gamma}\, dy \leqslant C$$

and therefore to prove the theorem it is sufficient to show that for every $x \in \Omega$ we have that

$$\int_E \frac{1}{|Du|^{(m-1)r}} \frac{1}{|x-y|^\gamma}\, dy \leqslant C,$$

where C does not depend on x. Finally the same arguments in the proof of Theorem 2.1 allow us to reduce to proving that, considering only $x \in E$,

$$\int_E \frac{1}{|Du|^{(m-1)r}} \frac{1}{|x-y|^\gamma}\, dy \leqslant C,$$

where C does not depend on $x \in E$.

Now let $\varphi_{\epsilon,x}$ be defined as in Theorem 2.1 and define

$$\psi_{\epsilon,x} = \frac{1}{(|Du|^{m-1}+\epsilon)^r} \frac{\varphi_{\epsilon,x}}{|x-y|^\gamma}.$$

Since $|Du|^{m-1} \in W^{1,2}(\Omega)$, its gradient vanishes a.e. in the critical set Z_u and $\psi_{\epsilon,x}$ can be used as test function in (2.2). By the positivity hypothesis on h, we have $h(y) \geqslant \frac{1}{C_1} > 0$ for any $y \in E$, so that we get

$$\begin{aligned}\int_E \psi_{\epsilon,x}\, dy \leqslant C_1 \int_E \psi_{\epsilon,x} h\, dy \leqslant C_1 \int_\Omega \psi_{\epsilon,x} h\, dy \leqslant \\ \leqslant C_1 \int_\Omega |Du|^{m-2}(Du, D\psi_{\epsilon,x}) dy.\end{aligned}$$

The proof follows now, as in Theorem 2.3 of [13], exploiting Theorem 2.1.

3 Comparison results

We begin here the study of the properties of the solutions of (1.1). In the sequel, as in [25], if $\rho \in L^1(\Omega)$, the space $H^{1,p}_\rho(\Omega)$ is defined as the completion of $C^1(\overline{\Omega})$ (or $C^\infty(\overline{\Omega})$) under the norm

$$\|v\|_{H^{1,p}_\rho} = \|v\|_{L^p(\Omega)} + \|Dv\|_{L^p(\Omega,\rho)} \tag{3.1}$$

and $\|Dv\|^p_{L^p(\Omega,\rho)} = \int_\Omega |Dv|^p \rho\, dx$. Thus, $H^{1,p}_\rho(\Omega)$ is a Banach space and $H^{1,2}_\rho(\Omega)$ is a Hilbert space. Moreover we define $H^{1,p}_{0,\rho}(\Omega)$ as the closure of $C^1_c(\overline{\Omega})$ (or $C^\infty_c(\overline{\Omega})$) in $H^{1,p}_\rho(\Omega)$. We also recall that in [32], $H^{1,p}_{o,\rho}$ is defined as the space of functions having a distributional derivatives represented by a function for which the norm defined in (3.1) is bounded. These two definitions are equivalent if the domain has piecewise regular boundary.

From now on, given $(u,v) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$ a fixed solution of (1.1), we will consider

$$\rho_u \equiv |Du|^{m_1-2}, \qquad \rho_v \equiv |Dv|^{m_2-2}. \tag{3.2}$$

With these definition, using Theorem 2.1 and Theorem 2.2 with $h = f(v)$ or $h = g(u)$, we have the following:

Theorem 3.1 *Let $(u,v) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$ be a weak solution of (1.1) where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geqslant 2$, $1 < m_1, m_2 < \infty$ and the nonlinearities f, g are locally Lipschitz continuous. Then, for any $E \subset\subset \Omega$ and for every $i, j = 1, \ldots, N$, we have, for every $x \in \Omega$,*

$$\int_{E \setminus \{u_{x_i}=0\}} \frac{|Du|^{m_1-2}}{|u_{x_i}|^{\beta}|x-y|^{\gamma}} |Du_{x_i}|^2 \, dy < C_1$$

and

$$\int_{E \setminus \{v_{x_i}=0\}} \frac{|Dv|^{m_2-2}}{|v_{x_i}|^{\beta}|x-y|^{\gamma}} |Dv_{x_i}|^2 \, dy < C_2$$

where $\beta < 1$, $\gamma < N-2$ if $N \geqslant 3$, $\gamma = 0$ if $N = 2$ and C_1, C_2 depend on γ, β, E and on the solution (u,v), but not on $x \in \Omega$. Moreover

$$\int_{E \setminus Z_u} \frac{|Du|^{m_1-2-\beta}}{|x-y|^{\gamma}} \|D^2 u\|^2 \, dy < C_1,$$

and

$$\int_{E \setminus Z_v} \frac{|Dv|^{m_2-2-\beta}}{|x-y|^{\gamma}} \|D^2 v\|^2 \, dy < C_2,$$

where $Z_u = \{x \in \Omega : Du(x) = 0\}$ is the critical set of u and $Z_v = \{x \in \Omega : Dv(x) = 0\}$ is the critical set of v.

Finally, if Ω is smooth and f, g are positive($f(s), g(s) > 0$ for $s > 0$), then $|Z_u| = |Z_v| = 0$ and, for any $x \in \Omega$ and for every $r < 1$, we have

$$\int_{\Omega} \frac{1}{|Du|^{(m_1-1)r}} \frac{1}{|x-y|^{\gamma}} \, dy \leqslant C_1$$

and

$$\int_{\Omega} \frac{1}{|Dv|^{(m_2-1)r}} \frac{1}{|x-y|^{\gamma}} \, dy \leqslant C_2$$

where C_1 and C_2 do not depend on x, $\gamma < N-2$ if $N \geqslant 3$ and $\gamma = 0$ if $N = 2$.

Therefore $\rho_u, \rho_v \in L^{\infty}(\Omega)$ if $m_1, m_2 > 2$ since $(u,v) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$. If instead $\frac{2N+2}{N+2} < m < 2$, then $\rho_u, \rho_v \in L^1(\Omega)$, which follows easily from Theorem 3.1.

In particular, summability properties of $\frac{1}{\rho_u}$ and $\frac{1}{\rho_v}$ of Theorem 3.1 are exactly those needed in [13] to prove weighted Sobolev inequality and consequently weighted Poincaré inequality. Referring to [13] for the proof, we can state the following:

Theorem 3.2 (Weighted Poincaré inequality) *Let $(u,v) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$ be a weak solution of (1.1) where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geqslant 2$, $1 < m_1, m_2 < \infty$ and the nonlinearities f, g are positive($f(s), g(s) > 0$ for $s > 0$) and locally Lipschitz continuous. Then, if we consider $\rho_u = |Du|^{m_1-2}$ and $\rho_v = |Dv|^{m_2-2}$, we get, for every $p \geqslant 2$*

$$\|\xi\|_{L^p(\Omega)} \leqslant C_1(|\Omega|) \|D\xi\|_{L^p(\Omega, \rho_u)} \quad \text{for every } \xi \in H^{1,p}_{0,\rho_u}(\Omega) \tag{3.3}$$

and

$$\|\eta\|_{L^p(\Omega)} \leqslant C_2(|\Omega|)\|D\eta\|_{L^p(\Omega,\rho_v)} \quad \text{for every } \eta \in H^{1,p}_{0,\rho_v}(\Omega) \tag{3.4}$$

where $C_1(|\Omega|), C_1(|\Omega|) \to 0$ *if* $|\Omega| \to 0$. *In particular,* (3.3) *and* (3.4) *hold for every* $\xi \in H^{1,2}_{0,\rho_u}(\Omega)$ *or* $\eta \in H^{1,2}_{0,\rho_v}(\Omega)$.

If $(u,v) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$ is a weak solution of (1.1), then by Corollary 2.1 (exploited with $h = f(v)$ or $h = g(u)$) it follows that $|Du|^{m_1-2}Du \in W^{1,2}(\Omega, \mathbb{R}^N)$ and $|Dv|^{m_2-2}Du \in W^{1,2}(\Omega, \mathbb{R}^N)$, and we can define

$$L_{(u,v)}((u_{x_i}, v_{x_j}), (\varphi, \psi)) \equiv (L^1_{(u,v)}((u_{x_i}, v_{x_j}), (\varphi, \psi)), L^2_{(u,v)}((u_{x_i}, v_{x_j}), (\varphi, \psi))$$

where

$$L^1_{(u,v)}((u_{x_i}, v_{x_j}), (\varphi, \psi)) \equiv$$

$$\int_\Omega [|Du|^{m_1-2}(Du_{x_i}, D\varphi) + (m_1-2)|Du|^{m_1-4}(Du, Du_{x_i})(Du, D\varphi) - f'(v)v_{x_i}\varphi]dx$$

and

$$L^2_{(u,v)}((u_{x_i}, v_{x_j}), (\varphi, \psi)) \equiv$$

$$\int_\Omega [|Dv|^{m_2-2}(Dv_{x_i}, D\psi) + (m_2-2)|Dv|^{m_2-4}(Dv, Dv_{x_i})(Dv, D\psi) - g'(u)u_{x_i}\psi]dx$$

for any $\varphi \in C^1_0(\Omega)$ and $1 < m_1, m_2 < \infty$. Moreover, the following equation holds:

$$L_{(u,v)}((u_{x_i}, v_{x_j}), (\varphi, \psi)) = 0 \qquad \forall (\varphi, \psi) \in C^1_0(\Omega) \times C^1_0(\Omega), \quad i,j = 1, \ldots, N. \tag{3.5}$$

More generally, if $(w,h) \in H^{1,2}_{\rho_u}(\Omega) \times H^{1,2}_{\rho_v}(\Omega)$ we can define $L_{(u,v)}((w,h),(\varphi,\psi))$ as above. In this case we say that $(w,h) \in H^{1,2}_{\rho_u}(\Omega) \times H^{1,2}_{\rho_v}(\Omega)$ is a weak solution of the Linearized Operator $L_{(u,v)}$ if

$$L_{(u,v)}((w,h),(\varphi,\psi)) \equiv (L^1_{(u,v)}((w,h),(\varphi,\psi)), L^2_{(u,v)}((w,h),(\varphi,\psi)) \equiv (0,0). \tag{3.6}$$

In particular, by density arguments we can assume $(\varphi, \psi) \in H^{1,2}_{0,\rho_u}(\Omega) \times H^{1,2}_{0,\rho_v}(\Omega)$.

In [14] the authors showed that by a weighted Sobolev inequality, a Harnack inequality follows for solutions of the Linearized Operator of the problem $-\Delta_m(u) = f(u)$. The same arguments apply to our case and allow to prove the following

Theorem 3.3 *Let* $(w,h) \in H^{1,2}_{\rho_u}(\Omega) \times H^{1,2}_{\rho_v}(\Omega)$ *be nonnegative weak solutions of* (3.6) *in a bounded smooth domain* Ω *of* $\mathbb{R}^N$, $N \geqslant 2$, $2 < m_1, m_2 < \infty$, *and suppose that the nonlinearities* f, g *are positive* ($f(s), g(s) > 0$ *for* $s > 0$), *nondecreasing and locally Lipschitz continuous.*

Suppose that $\overline{B(x, 5\delta)} \subset \Omega$. *Let us set*

$$\frac{1}{2^*} = \frac{1}{2} - \frac{1}{N} + \frac{1}{N}\left(\frac{m-2}{m-1}\right)$$

(consequently $\overline{2}^* > 2$ for $m > 2$) and let 2^* be any real number such that $2 < 2^* < \overline{2}^*$. Then for every $0 < s < \chi$, $\chi \equiv \frac{2^*}{2}$, there exists $C_1, C_2 > 0$ such that

$$\|w\|_{L^s(B(x,2\delta))} \leqslant C_1 \inf_{B(x,\delta)} w \tag{3.7}$$

and

$$\|h\|_{L^s(B(x,2\delta))} \leqslant C_2 \inf_{B(x,\delta)} h \tag{3.8}$$

where C_1, C_2 are constants depending on x, s, N, u, v, m, f. If $\frac{2N+2}{N+2} < m_1 < 2$ or $\frac{2N+2}{N+2} < m_2 < 2$, the same result holds with χ replaced by $\chi' \equiv \frac{2^\sharp}{s^\sharp}$ where $2^\sharp$ is the classical Sobolev exponent, $\frac{2}{s^\sharp} \equiv 1 - \frac{1}{s}$ and $s < \frac{m_1-1}{2-m_1}$ or $s < \frac{m_2-1}{2-m_2}$ respectively.

Proof. The proof follows directly from [14] once we note that, since f, g are nondecreasing and w and h are nonnegative, w weakly solves

$$\int_\Omega [|Du|^{m_1-2}(Du_{x_i}, D\varphi) + (m_1-2)|Du|^{m_1-4}(Du, Du_{x_i})(Du, D\varphi)]dx \geqslant 0$$

and h weakly solves

$$\int_\Omega [|Dv|^{m_2-2}(Dv_{x_i}, D\psi) + (m_2-2)|Dv|^{m_2-4}(Dv, Dv_{x_i})(Dv, D\psi)]dx \geqslant 0.$$

Therefore we can apply the results of [14] to w and to h separately and the thesis follows. An immediate consequence is the following

Theorem 3.4 (Strong Maximum Principle) *Let $(w,h) \in H^{1,2}_{\rho_u}(\Omega) \times H^{1,2}_{\rho_v}(\Omega) \cap C^0(\Omega) \times C^0(\Omega)$ be nonnegative weak solutions of (3.6) in a bounded smooth domain Ω of $\mathbb{R}^N$, $N \geqslant 2$, $2 < m_1, m_2 < \infty$, and suppose that the nonlinearities f, g are positive($f(s), g(s) > 0$ for $s > 0$), nondecreasing and locally Lipschitz continuous. Then, for any domain $\Omega' \subset \Omega$ with $w \geqslant 0$ in Ω' and $h \geqslant 0$ in Ω', we have $w \equiv 0$ in Ω' or $w > 0$ in Ω' and $h \equiv 0$ in Ω' or $h > 0$ in Ω'.*

Proof. Let us define $K_w = \{x \in \Omega' \,|\, w(x) = 0\}$. By the continuity of w, then K_w is closed. Moreover by Theorem 3.3, for any $x \in K_w$, there exists a ball $B(x)$ centered in x, and contained in K_w. Therefore K_w is also open and the thesis follows. The same arguments apply to h.

4 Weak comparison principle

In what follows, we will use the following standard estimates for the m-Laplace operator(see e.g. Lemma 2.1 of [11]):

$$||\eta|^{m-2}\eta - |\eta'|^{m-2}\eta'| \leqslant c_1(|\eta| + |\eta'|)^{m-2}|\eta - \eta'| \tag{4.1}$$

$$[|\eta|^{m-2}\eta - |\eta'|^{m-2}\eta'][\eta - \eta'] \geqslant c_2(|\eta| + |\eta'|)^{m-2}|\eta - \eta'|^2. \tag{4.2}$$

Theorem 4.1 (Weak Comparison Principle) *Let $(u,v) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$ and $(\bar{u},\bar{v}) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$ be weak solutions of (1.1) where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geqslant 2$, $1 < m_1, m_2 < \infty$, and the nonlinearities f,g are positive ($f(s), g(s) > 0$ for $s > 0$), nondecreasing and locally Lipschitz continuous.*

Let $\Omega' \subseteq \Omega$ be open and suppose $u \leqslant \bar{u}$ on $\partial\Omega'$ and $v \leqslant \bar{v}$ on $\partial\Omega'$. Then there exists $\delta > 0$ such that, if $|\Omega'| \leqslant \delta$, then $u \leqslant \bar{u}$ in Ω' and $v \leqslant \bar{v}$ in Ω'.

Proof. Let us first consider the case $1 < m_1 < 2$ and $m_2 > 2$. Let Ω' be defined as in the statement of the theorem and consider $(u-\bar{u})^+ \in H_0^{1,m_1}(\Omega')$ and $(v-\bar{v})^+ \in H_0^{1,m_2}(\Omega')$. Using $(u-\bar{u})^+$ as test function for $-\Delta_{m_1}(u) = f(v)$ and $(v-\bar{v})^+$ as test function for $-\Delta_{m_2}(v) = g(u)$, we get

$$\int_{\Omega'} \big[(|Du|^{m_1-2}Du - |D\bar{u}|^{m_1-2}D\bar{u}), D(u-\bar{u})^+)\big]\, dx = \int_{\Omega'} (f(v) - f(\bar{v}))(u-\bar{u})^+\, dx. \tag{4.3}$$

and

$$\int_{\Omega'} \big[[|Dv|^{m_2-2}Dv - |D\bar{v}|^{m_2-2}D\bar{v}), D(v-\bar{v})^+)\, dx = \int_{\Omega'} (g(u) - g(\bar{u}))(v-\bar{v})^+\, dx. \tag{4.4}$$

By (4.1) and (4.2), since f,g are locally Lipschitz continuous and nondecreasing we get that there exists $\Lambda > 0$ such that $f(v) - f(\bar{v}) \leqslant \Lambda(v-\bar{v})^+$. Therefore

$$c_2 \int_{\Omega'} ((|Du| + |D\bar{u}|)^{m_1-2}|D(u-\bar{u})^+)|^2\, dx \leqslant C_1\Lambda \int_{\Omega'} (v-\bar{v})^+(u-\bar{u})^+\, dx \tag{4.5}$$

and, analogously

$$c_2 \int_{\Omega'} ((|Dv| + |D\bar{v}|)^{m_2-2}|D(v-\bar{v})^+)|^2\, dx \leqslant C_2\Lambda \int_{\Omega'} (v-\bar{v})^+(u-\bar{u})^+\, dx. \tag{4.6}$$

In particular, we have used the fact that, since f,g are nondecreasing, then $(f(v) - f(\bar{v})) \leqslant 0$ iff $v \leqslant \bar{v}$ and $(g(u) - g(\bar{u})) \leqslant 0$ iff $u \leqslant \bar{u}$.

Adding (4.5)and (4.6), and using Young's inequality, we get

$$\begin{aligned} \int_{\Omega'} |D(u-\bar{u})^+)|^2\, dx + \int_{\Omega'} |Dv|^{m_2-2}|D(v-\bar{v})^+|^2\, dx \leqslant \\ \leqslant C_2 \int_{\Omega'} [(u-\bar{u})^+]^2\, dx + C_3 \int_{\Omega'} [(v-\bar{v})^+]^2\, dx \end{aligned} \tag{4.7}$$

where we have also used the fact that, since $m_1 < 2$ then $(|Du| + |D\bar{u}|)^{m_1-2} > 0$. We can now apply the classic Poincaré inequality to $(u-\bar{u})^+$ and weighted Poincaré inequality (see Theorem 3.2) with weight $\rho_v \equiv |Dv|^{m_2-2}$ to $(v-\bar{v})^+$ and get

$$\begin{aligned} \int_{\Omega'} |D(u-\bar{u})^+)|^2\, dx + \int_{\Omega'} |Dv|^{m_2-2}|D(v-\bar{v})^+|^2\, dx \leqslant \\ \leqslant C_1(|\Omega'|) \int_{\Omega'} |D(u-\bar{u})^+)|^2\, dx + C_2(|\Omega'|) \int_{\Omega'} |Dv|^{m_2-2}|D(v-\bar{v})^+|^2\, dx \end{aligned} \tag{4.8}$$

where $C_1(|\Omega'|) \to 0$ if $|\Omega'| \to 0$ and $C_2(|\Omega'|) \to 0$ if $|\Omega'| \to 0$.
Now, if $|\Omega'| \leqslant \delta$ and δ is sufficiently small so that $C_1(|\Omega'|) < 1$ and $C_2(|\Omega'|) < 1$, then we get an absurdity unless

$$\int_{\Omega'} |D(u-\bar{u})^+)|^2\,dx + \int_{\Omega'} |Dv|^{m_2-2}|D(v-\bar{v})^+|^2\,dx = 0 \tag{4.9}$$

which implies $(u-\bar{u})^+ = (v-\bar{v})^+ = 0$ and therefore $u \leqslant \bar{u}$ and $v \leqslant \bar{v}$ in Ω'.

To deal with the general case we note that if $m_1 > 2$ and $1 < m_2 < 2$ then we will apply weighted Poincaré inequality(see Theorem 3.2) with weight $\rho_u \equiv |Du|^{m_1-2}$ to $(u-\bar{u})^+$ and the classic Poincaré inequality to $(v-\bar{v})^+$. We will otherwise use only the classic Poincaré inequality if $1 < m_1, m_2 < 2$ or only weighted Poincaré inequality if $m_1, m_2 > 2$.

In the proofs of our results we will also use a strong comparison principle proved in [11]. For the readers convenience we recall the statement:

Theorem 4.2 (Strong Comparison Principle) *Let $1 < m < \infty$, and $u, v \in C^1(\Omega)$ satisfy*

$$-div(|Du|^{m-2}Du) + \Lambda u \leqslant -div(|Dv|^{m-2}Dv) + \Lambda v, \qquad u \leqslant v \ \text{in}\ \Omega. \tag{4.10}$$

Define $Z_{u,v} = \{x \in \Omega : |Du(x)| + |Dv(x)| = 0\}$ if $m \neq 2$, $Z_{u,v} = \emptyset$ if $m = 2$. If $x_0 \in \Omega \setminus Z_{u,v}$ and $u_{x_0} = v_{x_0}$ then $u \equiv v$ in the connected component of $\Omega \setminus Z_{u,v}$ containing x_o.

5 Qualitative properties of the solutions

To state our monotonicity and symmetry result we need some notations.
Let ν be a direction in $\mathbb{R}^N$. For a real number λ we define

$$T_\lambda^\nu = \{x \in \mathbb{R} : x \cdot \nu = \lambda\} \tag{5.1}$$

$$\Omega_\lambda^\nu = \{x \in \Omega : x \cdot \nu < \lambda\} \tag{5.2}$$

$$x_\lambda^\nu = R_\lambda^\nu(x) = x + 2(\lambda - x\cdot\nu)\nu, \qquad x \in \mathbb{R}^N \tag{5.3}$$

and

$$a(\nu) = \inf_{x\in\Omega} x \cdot \nu. \tag{5.4}$$

If $\lambda > a(\nu)$ then Ω_λ^ν is nonempty, thus we set

$$(\Omega_\lambda^\nu)' = R_\lambda^\nu(\Omega_\lambda^\nu). \tag{5.5}$$

Following [27, 18] we observe that for $\lambda - a(\nu)$ small then $(\Omega_\lambda^\nu)'$ is contained in Ω and will remain in it, at least until one of the following occurs:

(i) $(\Omega_\lambda^\nu)'$ becomes internally tangent to $\partial\Omega$.

(ii) T^ν_λ is orthogonal to $\partial\Omega$.

Let $\Lambda_1(\nu)$ be the set of those $\lambda > a(\nu)$ such that for each $\mu < \lambda$ neither one of the conditions (i) and (ii) holds, and define

$$\lambda_1(\nu) = \sup \Lambda_1(\nu). \tag{5.6}$$

Moreover, let

$$\Lambda_2(\nu) = \{\lambda > a(\nu) : (\Omega^\nu_\mu)' \subseteq \Omega \quad \forall \mu \in (a(\nu), \lambda]\} \tag{5.7}$$

and

$$\lambda_2(\nu) = \sup \Lambda_2(\nu). \tag{5.8}$$

Note that since Ω is supposed to be smooth, neither $\Lambda_1(\nu)$ nor $\Lambda_2(\nu)$ are empty, and $\Lambda_1(\nu) \subseteq \Lambda_2(\nu)$ so that $\lambda_1(\nu) \leqslant \lambda_2(\nu)$ (in the terminology of [18] $\Omega^\nu_{\lambda_1(\nu)}$ and $\Omega^\nu_{\lambda_2(\nu)}$ correspond to the 'maximal cap', respectively to the 'optimal cap'). Finally define

$$\Lambda^{uv}_0(\nu) = \{\lambda > a(\nu) : u \leqslant u^\nu_\lambda \text{ and } v \leqslant v^\nu_\lambda \quad \forall \mu \in (a(\nu), \lambda]\} \tag{5.9}$$

and

$$\lambda^{uv}_0(\nu) = \sup \Lambda_0(\nu). \tag{5.10}$$

Here below we prove a useful result regarding the geometric properties of the critical set of the solutions. The result we prove has been already proved in [13] for solutions of $\Delta_m(u) = f(u)$. Anyway, for future use, we give here the details of the proof for the general case of solution of (2.2).

Theorem 5.1 *Let $u \in C^1(\overline{\Omega})$ be a weak solution of* (2.2) *where Ω is a general bounded domain, and suppose that h satisfies (*) with $h(s) > 0$ if $s > 0$. Then $\Omega \setminus Z_u$ does not contain any connected component C such that $\overline{C} \subset \Omega$. Moreover, if we assume that Ω is a smooth bounded domain with connected boundary, it follows that $\Omega \setminus Z_u$ is connected.*

Proof. Let C be a connected component of $\Omega \setminus Z_u$ such that $C \subset\subset \Omega$. Then

$$Du(x) = 0 \quad \forall x \in \partial C. \tag{5.11}$$

By Corollary 2.1, since $|Du|^{m-2}Du$ is continuous and identically zero on ∂C, we get $|Du|^{m-2}Du \in W^{1,2}_0(C, \mathbb{R}^N)$. Then there exists a vector field $A_n \in C^\infty_0(C, \mathbb{R}^N)$ which approximates $|Du|^{m-2}Du$ in the norm of $W^{1,2}_0(C, \mathbb{R}^N)$. If now $E \subset C$ is a smooth subset such that

$$\text{supp } (A_n) \subset\subset E \subset\subset C$$

by the Divergence Theorem applied to A_n in E, it follows, for every $\phi \in W^{1,2}$

$$\begin{aligned} &\int_C div(A_n)\phi + (A_n, D\phi)dx = \int_E div(A_n)\phi + (A_n, D\phi)dx = \\ &= \int_{\partial E} \phi(A_n, \eta)d\sigma = 0. \end{aligned} \tag{5.12}$$

Moreover, since when h is positive, $|Z_u| = 0$, by (1.4) we get

$$-div(|Du|^{m-2}Du) = h \quad \text{almost everywhere in } C.$$

If now we choose $\phi \equiv k \neq 0$ then we get

$$\begin{aligned} &\int_C k \cdot h\, dx = \int_C -div(|Du|^{m-2}Du) \cdot k\, dx = \\ &\lim_{n\to\infty} \int_C -div(A_n) \cdot k\, dx = \lim_{n\to\infty} \int_C (A_n, Dk)\, dx = 0 \end{aligned} \tag{5.13}$$

and by (5.13)

$$\int_C h\, dx = 0 \tag{5.14}$$

which is impossible when h is positive.

If Ω is smooth, since h is positive, by Hopf's Lemma a neighborhood of the boundary belongs to a component C of $\Omega \setminus Z_u$. A second component C' would be compactly contained in Ω, which is impossible by what we have just proved. So $\Omega \setminus Z_u$ is connected.

If we consider solutions of (1.1), exploiting Theorem 5.1 with $h = f(v)$ or $h = g(u)$, we immediately get:

Corollary 5.1 *Let $(u, v) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$ a weak solution of* (1.1) *where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geqslant 2$, $1 < m_1, m_2 < \infty$, and the nonlinearities f, g are positive($f(s), g(s) > 0$ for $s > 0$) and locally Lipschitz continuous. Then we have that $\Omega \setminus Z_u$ and $\Omega \setminus Z_v$ are connected. Here $Z_u = \{x \in \Omega : Du(x) = 0\}$ is the critical set of u and $Z_v = \{x \in \Omega : Dv(x) = 0\}$ is the critical set of v.*

We now prove our symmetry and monotonicity result:

Theorem 5.2 *Let $(u, v) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$ be a weak solution of* (1.1) *where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geqslant 2$, $1 < m_1, m_2 < \infty$ and the nonlinearities f, g are positive($f(s), g(s) > 0$ for $s > 0$), nondecreasing and locally Lipschitz continuous. For any direction ν and for λ in the interval $(a(\nu), \lambda_2(\nu)]$, we have*

$$u(x) \leqslant u(x_\lambda^\nu) \quad \textit{and} \quad v(x) \leqslant v(x_\lambda^\nu) \quad \forall x \in \Omega_\lambda^\nu. \tag{5.15}$$

Moreover, for any λ with $a(\nu) < \lambda < \lambda_2(\nu)$ we have

$$u(x) < u(x_\lambda^\nu) \quad \forall x \in \Omega_\lambda^\nu \setminus Z_{u\lambda}^\nu \tag{5.16}$$

where $Z_{u\lambda}^\nu \equiv \{x \in \Omega_\lambda^\nu \,:\, Du(x) = Du_\lambda^\nu(x) = 0\}$, and

$$v(x) < v(x_\lambda^\nu) \quad \forall x \in \Omega_\lambda^\nu \setminus Z_{v\lambda}^\nu \tag{5.17}$$

where $Z_{v\lambda}^\nu \equiv \{x \in \Omega_\lambda^\nu \,:\, Du(x) = Du_\lambda^\nu(x) = 0\}$. Finally

$$\frac{\partial u}{\partial \nu}(x) > 0 \quad \forall x \in \Omega_{\lambda_2(\nu)}^\nu \setminus Z_u \tag{5.18}$$

where $Z_u = \{x \in \Omega : Du(x) = 0\}$*, and*

$$\frac{\partial v}{\partial \nu}(x) > 0 \quad \forall x \in \Omega^{\nu}_{\lambda_2(\nu)} \setminus Z_v \tag{5.19}$$

where $Z_v = \{x \in \Omega : Dv(x) = 0\}$*.*

Proof. Since Ω is smooth, $\Lambda_2(\nu)$ is nonempty for any direction ν. For $a(\nu) < \lambda < \lambda_2(\nu)$ we can compare (u, v) and $(u^{\nu}_{\lambda}, v^{\nu}_{\lambda})$, using Theorem 4.1, since $(u^{\nu}_{\lambda}, v^{\nu}_{\lambda})$ still satisfies (1.1). In particular, if $\lambda - a(\nu)$ is small, then $|\Omega^{\nu}_{\lambda}|$ is small. Hence, by the Weak Comparison Principle in small domains (see Theorem 4.1), since $u \leqslant u^{\nu}_{\lambda}$ and $v \leqslant v^{\nu}_{\lambda}$ on $\partial\Omega^{\nu}_{\lambda}$, it follows that $u \leqslant u^{\nu}_{\lambda}$ and $v \leqslant v^{\nu}_{\lambda}$ in Ω^{ν}_{λ} if $\lambda - a(\nu)$ is small, so that $\Lambda^{uv}_0(\nu) \neq \emptyset$.

Suppose now, by contradiction, that $\lambda^{uv}_0(\nu) < \lambda_2(\nu)$. By continuity it follows $u^{\nu}_{\lambda^{uv}_0(\nu)} \geqslant u$ and $v^{\nu}_{\lambda^{uv}_0(\nu)} \geqslant v$ in $\Omega^{\nu}_{\lambda^{uv}_0(\nu)}$. By the Strong Comparison Principle (see Theorem 4.2) we get that, if C^u and C^v are connected components of $\Omega^{\nu}_{\lambda^{uv}_0(\nu)} \setminus Z_u$ and $\Omega^{\nu}_{\lambda^{uv}_0(\nu)} \setminus Z_v$ respectively, then $u^{\nu}_{\lambda^{uv}_0} > u$ unless $u^{\nu}_{\lambda_0(\nu)} \equiv u$ in C^u and $v^{\nu}_{\lambda^{uv}_0} > v$ unless $v^{\nu}_{\lambda^{uv}_0(\nu)} \equiv v$ in C^v.

The case $u^{\nu}_{\lambda_0(\nu)} \equiv u$ in C^u would imply $\Omega \setminus Z_u$ to be not connected against Corollary 5.1 and therefore we have $u^{\nu}_{\lambda^{uv}_0} > u$. In the same way we also get $v^{\nu}_{\lambda^{uv}_0} > v$.

Now Let A be an open set such that $Z_u \cap \Omega^{\nu}_{\lambda^{uv}_0(\nu)} \subset A \subset \Omega^{\nu}_{\lambda^{uv}_0(\nu)}$ and $Z_v \cap \Omega^{\nu}_{\lambda^{uv}_0(\nu)} \subset A \subset \Omega^{\nu}_{\lambda^{uv}_0(\nu)}$. Note that since $|Z_u| = |Z_v| = 0$, we can choose A as small as we like. Consider a compact set K in $\Omega^{\nu}_{\lambda^{uv}_0(\nu)}$ such that $|\Omega^{\nu}_{\lambda^{uv}_0(\nu)} \setminus K|$ is sufficiently small in order to guarantee the applicability of Theorem 4.1. By what we proved before, $u^{\nu}_{\lambda^{uv}_0(\nu)} - u$ and $v^{\nu}_{\lambda^{uv}_0(\nu)} - v$ are positive in $K \setminus A$ which is compact. Thus $\min_{K\setminus A}(u^{\nu}_{\lambda^{uv}_0(\nu)} - u) \geqslant m > 0$ and $\min_{K\setminus A}(v^{\nu}_{\lambda^{uv}_0(\nu)} - v) \geqslant m > 0$. By continuity there exists $\epsilon > 0$ such that $\lambda^{uv}_0(\nu) + \epsilon < \lambda_2(\nu)$ and for $\lambda^{uv}_0(\nu) < \lambda < \lambda^{uv}_0(\nu) + \epsilon$ we have that $|\Omega^{\nu}_{\lambda} \setminus K|$ is still sufficiently small as before and $u^{\nu}_{\lambda} - u > m/2 > 0$ in $K \setminus A$, $v^{\nu}_{\lambda} - v > m/2 > 0$ in $K \setminus A$. In particular $u^{\nu}_{\lambda} - u > 0$ and $v^{\nu}_{\lambda} - v > 0$ on $\partial(K \setminus A)$. Moreover, for such values of λ we have that $u \leqslant u^{\nu}_{\lambda}$ and $v \leqslant v^{\nu}_{\lambda}$ on $\partial(\Omega^{\nu}_{\lambda} \setminus (K \setminus A))$. By the Weak Comparison Principle(Theorem 4.1) applied to $\Omega^{\nu}_{\lambda} \setminus (K \setminus A)$(which may be taken as small as we like), we get $u \leqslant u^{\nu}_{\lambda}$ and $v \leqslant v^{\nu}_{\lambda}$ in Ω^{ν}_{λ}, which contradicts the assumption $\lambda^{uv}_0(\nu) < \lambda_2(\nu)$. Therefore $\lambda^{uv}_0(\nu) \equiv \lambda_2(\nu)$ and the thesis is proved.

The proof of (5.16) and (5.17) follow immediately by Theorem 4.2 and the first part of this Theorem. In fact if (5.16) (5.17) were not true, by the Strong Comparison Principle, there would exist components of local symmetry, contrary to what we have just proved.

Finally, to prove (5.18) and (5.19), let us note that, by the linearity of L_{uv}, we get that $(\frac{\partial u}{\partial \nu}, \frac{\partial v}{\partial \nu})$ weakly solves (3.6). Therefore, by the strong maximum principle for uniformly elliptic operators, we have that (5.18) and (5.19) hold unless $\frac{\partial u}{\partial \nu} \equiv 0$ or $\frac{\partial v}{\partial \nu} \equiv 0$ in some connected components of $\Omega \setminus Z_u$ and $\Omega \setminus Z_v$ respectively. Since this is not possible by (5.16) and (5.17), the thesis follows.

An immediate consequence is the following:

Corollary 5.2 *If the domain Ω is convex with respect to a direction ν and symmetric with respect to the hyperplane $T_0^\nu = \{x \in \mathbb{R}^N : x \cdot \nu = 0\}$, then u and v are symmetric, i. e. $u(x) = u(x_0^\nu)$ and $v(x) = v(x_0^\nu)$, and nondecreasing in the ν–direction in Ω_0^ν with $\frac{\partial u}{\partial \nu}(x) > 0$ in $\Omega_0^\nu \setminus Z_u$ and $\frac{\partial v}{\partial \nu}(x) > 0$ in $\Omega_0^\nu \setminus Z_v$.*

In particular if Ω is a ball then u and v are radially symmetric and $\frac{\partial u}{\partial r} < 0$, $\frac{\partial v}{\partial r} < 0$.

Proof. It is immediate from the previous theorem. Let us only note that in the case of a ball, since the level sets of the solutions are spheres, an application of Hopf's Lemma (recall that f and g are positive) shows that 0 is the only critical point and that the derivative in the radial direction is negative in all the other points.

Theorem 5.3 *Let $(u, v) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$ be a weak solution of (1.1) where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geqslant 2$, $\frac{2N+2}{N+2} < m_1, m_2 < \infty$ and the nonlinearities f, g are positive($f(s), g(s) > 0$ for $s > 0$), nondecreasing and locally Lipschitz continuous. For any direction ν and for λ in the interval $(a(\nu), \lambda_2(\nu)]$ we have*

$$u(x) \leqslant u(x_\lambda^\nu) \quad \text{and} \quad v(x) \leqslant v(x_\lambda^\nu) \quad \forall x \in \Omega_\lambda^\nu. \tag{5.20}$$

Moreover, for any λ with $a(\nu) < \lambda < \lambda_2(\nu)$, we have

$$u(x) < u(x_\lambda^\nu) \quad \forall x \in \Omega_\lambda^\nu \setminus Z_{u\lambda}^\nu \tag{5.21}$$

where $Z_{u\lambda}^\nu \equiv \{x \in \Omega_\lambda^\nu : Du(x) = Du_\lambda^\nu(x) = 0\}$, and

$$v(x) < v(x_\lambda^\nu) \quad \forall x \in \Omega_\lambda^\nu \setminus Z_{v\lambda}^\nu. \tag{5.22}$$

where $Z_{v\lambda}^\nu \equiv \{x \in \Omega_\lambda^\nu : Dv(x) = Dv_\lambda^\nu(x) = 0\}$. Finally,

$$\frac{\partial u}{\partial \nu}(x) > 0 \quad \forall x \in \Omega_{\lambda_2(\nu)}^\nu \tag{5.23}$$

where $Z_u = \{x \in \Omega : Du(x) = 0\}$, and

$$\frac{\partial v}{\partial \nu}(x) > 0 \quad \forall x \in \Omega_{\lambda_2(\nu)}^\nu \tag{5.24}$$

where $Z_v = \{x \in \Omega : Dv(x) = 0\}$. Consequently, $Z_u \cap \Omega_{\lambda_2(\nu)}^\nu \equiv \emptyset$ and $Z_v \cap \Omega_{\lambda_2(\nu)}^\nu \equiv \emptyset$.

Proof. By Theorem 5.2, we get (5.20), (5.21) and (5.22). Let us now prove (5.23) ((5.24) follows in the same way). To prove that

$$\frac{\partial u}{\partial \nu}(x) > 0 \quad \forall x \in \Omega_{\lambda_2(\nu)}^\nu$$

assume on the contrary that $\frac{\partial u}{\partial \nu}(x_0)$ for some $x_0 \in \Omega_{\lambda_2(\nu)}^\nu$. Then, since $\frac{\partial u}{\partial \nu}$ is a nonnegative solution of the linearized equation, by Theorem 3.4 we find $\rho > 0$ such that

$$\frac{\partial u}{\partial \nu} = 0 \qquad \text{in} \quad B_\rho(x_0)$$

and $B_\rho(x_0) \subset \Omega^\nu_{\lambda_2(\nu)}$. This is a contradiction to (5.18) and the fact that $|Z_u| = 0$, and therefore (5.23) follows.

We point out an immediate consequences of Theorem 5.3, which may be very useful:

Corollary 5.3 *If the domain Ω is convex with respect to a direction ν and symmetric with respect to the hyperplane $T^\nu_0 = \left\{x \in \mathbb{R}^N : x \cdot \nu = 0\right\}$ then u and v are symmetric, i. e. $u(x) = u(x^\nu_0)$ and $v(x) = v(x^\nu_0)$, and nondecreasing in the ν–direction in Ω^ν_0 with $\frac{\partial u}{\partial \nu}(x) > 0$ in Ω^ν_0 and $\frac{\partial v}{\partial \nu}(x) > 0$ in Ω^ν_0.*

In particular $Z_u \subset T^\nu_0$ and $Z_v \subset T^\nu_0$. Therefore if for N orthogonal directions e_i the domain Ω is symmetric with respect to any hyperplane $T^{e_i}_0$ and $\lambda_2(e_i) = \lambda_2(-e_i) = 0$, then

$$Z_u \equiv \{x \in \Omega \,|\, D(u)(x) = 0\} = \{0\} = Z_v \equiv \{x \in \Omega \,|\, D(v)(x) = 0\} \tag{5.25}$$

assuming that 0 is the center of symmetry.

Finally, since the m-Laplace operator in not degenerate in $\Omega \setminus \{0\}$, we get

$$u \in C^2(\Omega \setminus \{0\}) \quad \textit{and} \quad v \in C^2(\Omega \setminus \{0\}).$$

References

[1] A. Aftalion and F. Pacella, *Morse index and uniqueness for positive solutions of radial p-Laplace equations*, T.A.M.S. **356** (2004), 4225–4272.

[2] C. Azizieh, *Monotonicity and symmetry results for p-Laplace systems*, Rendiconti Università di Trieste **34** (2002), 67–98.

[3] C. Azizieh and P. Clément, *A priori estimates and continuation methods for positive solutions of p-Laplace equations*, J. Diff. Equations **179** (2002), 213–245.

[4] C. Azizieh, P. Clément, and E. Mitidieri, *Existence and a priori estimates for positive solutions of p-Laplace systems*, J. Diff. Equations **184** (2002), 422–442.

[5] H. Berestycki and L. Niremberg, *On the method of moving planes and the sliding methods*, Bol. Soc. Brasil. Mat. **22** (1991), 1–37.

[6] P. Clément, D.G. de Figueiredo, and E. Mitidieri, *Positive solutions of semilinear elliptic systems*, Comm. P.D.E. **17** (1992), 923–940.

[7] P. Clément, J. Fleckinger, E. Mitidieri, and F. de Thelin, *Existence of positive solutions for a nonvariational quasilinear elliptic systems*, J. Doff. Equations **166** (2000), 455–477.

[8] P. Clément, R. Manasevich, and E. Mitidieri, *Positive solutions for quasilinear systems via blow up*, Comm. P.D.E. **18** (1993), 2071–2106.

[9] M. Cuesta and P. Takàc, *A Strong Comparison Principle for the Dirichlet p-Laplacian* Proceedings of the Conference on Reaction Diffusion Equations, Trieste 1995 Marcell Dekker, Inc., New York-Basel, 1997.

[10] M. Cuesta and P. Takàc, *A Strong Comparison Principle for Positive Solutions of Degenerate Elliptic Equations* Differ. Integral Equ. **4**(6) (2000), 721–746.

[11] L. Damascelli, *Comparison theorems for some quasilinear degenerate elliptic operators and applications to symmetry and monotonicity results*, Ann. Inst. H. Poincaré. Analyse non linéaire **15**(4) (1998), 493–516.

[12] L. Damascelli and F. Pacella, *Monotonicity and symmetry of solutions of p-Laplace equations, $1 < p < 2$, via the moving plane method.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. **26**(4) (1998), 689–707.

[13] L. Damascelli and B. Sciunzi, *Regularity, Monotonicity and Symmetry of positive solutions of m-Laplace equations.* J. Diff. Equations **206** (2004), 483–515.

[14] L. Damascelli and B. Sciunzi, *Harnack Inequalities, Maximum and Comparison Principles, and Regularity of positive solutions of m-Laplace equations* Calculus of Variations and P.D.E. (to appear).

[15] D.G. de Figueiredo, P.L. Lions, and R.D. Nussbaum, *A priori estimates and existence of positive solutions of semilinear elliptic equations.* J. Math. Pures Appl., IX Ser. **61** (1982), 41–63.

[16] E. DiBenedetto, $C^{1+\alpha}$ *local regularity of weak solutions of degenerate elliptic equations*, Nonlinear Anal. **7**(8) (1983), 827–850.

[17] J. Douglas, T. Doupont , and J. Serrin, *Uniqueness and Comparison Theorems for nonlinear elliptic equations in divergence form.* Arch. Rational Mech. Anal. **42**, (1971) 157–168.

[18] B. Gidas, W. M. Ni, and L. Niremberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979), 209–243.

[19] D. Gilbarg and N.S. Trudinger, Elliptic partial differential equations of second order, Springer, second edition, 1983.

[20] M. Grossi, S. Kesavan, and F. Pacella, *Symmetry of positive solutions of some nonlinear equations*, Topol. Methods Nonlinear Analysis **12**(1) (1998), 47–59.

[21] H. Hopf, Lectures on differential geometry in the large, Stanford University, 1956.

[22] S. Kesavan and F. Pacella, *Symmetry of positive solutions of a quasilinear elliptic equation via isoperimetric inequality*, Appl. Anal. **54** (1994), 27–37.

[23] O.A. Ladyzhenskaya and N. N. Ural'tseva, Linear and quasilinear elliptic equations, Academic Press, 1968.

[24] J.K. Moser, *On Harnack's theorem for elliptic differential equations*, Comm. on Pure and Applied Math. **14** (1961), 577–591.

[25] M.K.V. Murthy and G. Stampacchia, *Boundary value problems for some degenerate-elliptic operators*, Ann. Mat. Pura Appl. **80** (1968), 1–122.

[26] J. Nash, *Continuity of solutions of elliptic and parabolic equations*, Amer. J. Math., **80** (1958), 931–954.

[27] J. Serrin, *A symmetry problem in potential theory*, Arch. Rational Mech. Anal. **43** (1971), 304–318.

[28] J. Serrin, and M. Tang *Uniqueness of ground states for quasilinear elliptic equations*, Indiana Univ. Math. J. **49**(3) (2000), 897–923.

[29] J. Serrin, and H. Zou *Symmetry of Ground states of quasilinear elliptic equations*, Arch. Rational Mech. Anal. **148** (1999), 265–290.

[30] P. Tolksdorf, *Regularity for a more general class of quasilinear elliptic equations*, J. Diff. Equations **51** (1984), 126–150.

[31] G.M. Troianiello, Elliptic differential equations and obstacle problems, Plenum 1987.

[32] N.S. Trudinger, *Linear elliptic operators with measurable coefficients*, Ann. Scuola Norm. Sup. Pisa. **27**(3) (1973), 265–308.

[33] N.S. Trudinger, *On Harnack type inequalities and their application to quasilinear elliptic equations*, Comm. on Pure and Applied Math. **27** (1967), 721–747.

[34] N.S. Trudinger, *On the regularity of generalized solutions of linear non-uniformly elliptic equations*, Arch. Rational Mech. Anal. **42** (1971), 51–62.

[35] J. L. Vazquez, *A strong maximum principle for some quasilinear elliptic equations*, Appl. Math. Optim. (1984) 191–202.