LOCAL $W^{2,m(\cdot)}_{loc}$ REGULARITY FOR $p(.)$-LAPLACE EQUATIONS

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ABSTRACT. We study $W^{2,m(\cdot)}_{loc}$ regularity for local weak solutions of $p(\cdot)$-Laplace equations where $p \in C^1(\Omega) \cap C(\overline{\Omega})$ and $\min_{x\in\overline{\Omega}} p(x) > 1$.

1. Introduction

We consider the following $p(\cdot)$-Laplace equation

$$-\operatorname{div}(|Du|^{p(x)-2}Du) = f \quad \text{ in } \Omega, \tag{1.1}$$

where Ω is an open bounded domain in $\mathbb{R}^N$. Since in this paper we will prove *local* regularity results, we do not require any assumption on the regularity of the boundary of Ω.
Set $C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}) \mid \min_{x\in\overline{\Omega}} h(x) > 1\}$. For $h \in C(\overline{\Omega})$ we denote

$$h_- = \min_{x\in\overline{\Omega}} h(x) \quad \text{ and } \quad h_+ = \max_{x\in\overline{\Omega}} h(x).$$

Throughout this paper, we always assume that $u \in W^{1,p(\cdot)}_{loc}(\Omega)$ is a local weak solution to (1.1), that is

$$\int_\Omega |Du|^{p(x)-2}(Du, D\varphi)dx = \int_\Omega f\varphi dx, \quad \forall \varphi \in C^1_c(\Omega)$$

and the following conditions are fulfilled:

(P_0) $p \in C^1(\Omega) \cap C_+(\overline{\Omega})$,

(F_0) $f \in L^{h(\cdot)}(\Omega)$ where $h \in C_+(\overline{\Omega})$ and $h(x) > N/p(x)$ for all $x \in \overline{\Omega}$.

We deal with the study of the local regularity of the weak solutions to (1.1). More precisely, since in general solutions are not of class $C^2_{loc}(\Omega)$, then it is an important issue the study of the summability of the second derivatives of the solutions. To state our main result we need some notation:

We denote

$$\Omega^1 = \{x \in \Omega \mid 1 < p(x) < 2\} \ ; \ \Omega^2 = \{x \in \Omega \mid 2 \le p(x) < 3\} \ ; \ \Omega^3 = \{x \in \Omega \mid p(x) \ge 3\},$$

and for $\delta > 0$, we also set:

$$\Omega^1(\delta) = \{x \in \Omega \mid 1 < p(x) < 2 + \delta\} \text{ and } \Omega^2(\delta) = \{x \in \Omega \mid 2 + \delta \le p(x) < 3\}.$$

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Moreover for future use for $k = 1, 2, 3$, given $\Omega_0 \subset \Omega$, we define $\Omega_0^k = \Omega^k \cap \Omega_0$ and for $k = 1, 2$, we define $\Omega_0^k(\delta) = \Omega^k(\delta) \cap \Omega_0$.

Consider the additional conditions on f:

(F_1) $f \in W^{1,q(\cdot)}(\Omega)$ where $q \in C_+(\overline{\Omega})$,

(F_2) f is continuous and $f > 0$ in a neighborhood of $\overline{\Omega_3}$

and let $m \in C^1(\Omega) \cap C_+(\overline{\Omega})$ such that:

(i) $m(x) \leq 2$ in Ω_3^c (the complement of Ω_3 with respect to Ω)

(ii) $m(x) \leq \frac{p(x)-1}{p(x)-2} - \delta$ in Ω^3 for some $\delta > 0$.

Here we prove the following:

Theorem 1.1. *Suppose that conditions (P_0), (F_0), (F_1) and (F_2) are satisfied and assume that $q(x)$ in (F_1) is such that $q(x) \geq \frac{p(x)}{p(x)-1}$ in a neighborhood of $\overline{\Omega^1}$ and $q(x) \geq \frac{p(x)}{2p(x)-3}$ in Ω^2, then*

$$u \in W^{2,m(\cdot)}_{loc}(\Omega).$$

If else we assume that p is a constant such that $1 < p < 2$, and condition (F_0) are satisfied with $h(x)$ in (F_0) such that $h(x) \geq \frac{p}{p-1}$ in Ω, then

$$u \in W^{2,2}_{loc}(\Omega).$$

The proof will be given in two separate theorems: Theorem (3.6) and Theorem (3.9) in section 3.

If we consider the case when p is constant and $u \in C^1(\Omega)$ (see [6, 9, 15, 8] and [1, 2, 7]), our results reduce to the ones obtained in [5] (see also [14] for a local version). The interested reader may find very useful reading the survey of G. Mingione [11], and the references therein.

Note that, even in the case when p is constant, the regularity of the second derivatives proved in Theorem 1.1 is optimal, at least in the case $p \geq 3$. To show this it is sufficient to consider the function $u(x_1, \ldots, x_N) = \frac{|x_1|^{p'}}{p'}$, which solves

$$\Delta_p u = 1 \, .$$

It is easy to see that, for $p \geq 3$, the regularity of this solution is no more than the one proved here and in [5].

Previous regularity results on the second derivatives of solutions to p-Laplace equations are known in the case $1 < p \leq 2$, see [15].The achievement of results regarding summability of the second derivatives in our context, and in the context of p-Laplace equations is a hard task. The nonlinear Calderón-Zygmund theory does not work simply, even if important results have been obtained in [12], see also the references therein (and [11]).

It would be interesting to study regularity result, following the approach presented here, in the setting described in the interesting paper [13]. We point out however that in the case $p(x) \equiv p \geq 3$, the condition $f \in L^q(\Omega)$ is not enough to get the regularity result in Theorem 1.1:

To show this, let $\Omega = B(0,1)$, $u = -(1/r)|x_1|^{r-1}x_1$ for $1 < r < p'$ and set $\sigma = 1-(r-1)(p-1)$. Note that $r < 2$ and $0 < \sigma < 1$. We have $\frac{\partial u}{\partial x_1} = -|x_1|^{r-1}$, $\frac{\partial^2 u}{\partial x_1^2} = (1-r)|x_1|^{r-3}x_1$ and $\frac{\partial u}{\partial x_i} = \frac{\partial^2 u}{\partial x_i^2} = 0$ for all $i \neq 1$.

Therefore, $\frac{\partial^2 u}{\partial x_1^2} \in L^s(\Omega)$ if and only if $s < \frac{1}{2-r}$. Since $\frac{1}{2-r} < \frac{1}{2-p'} = \frac{p-1}{p-2}$, u does not satisfy regularity property in Theorem 1.1.

However u is a solution of (1.1) where

$$f = -\text{div}(|Du|^{p-2}Du) = \frac{\partial}{\partial x_1}(|x_1|^{(r-1)(p-1)-1}x_1) = \frac{\partial}{\partial x_1}(|x_1|^{-\sigma}x_1)$$
$$= (1-\varepsilon)|x_1|^{-\sigma}$$

belongs to $L^t(\Omega)$ for $1 < t < 1/\sigma$ and $1/\sigma \approx \infty$ if $r \approx p'$.

The proof of Theorem 1.1 will be carried out in two different theorems: Theorem 3.6 and Theorem 3.9 below. Following [5] we linearize the equation and then we exploit it via the right choice of test functions. However we deal with solutions that may not be of class C^1_{loc}, consequently we need to construct a regularized problem. We consequently prove some *uniform* estimates on the second derivatives, that provides the desired result by passing to the limiting problem.

2. Preliminaries

It is standard to see that, under our assumptions, we can assume without loss of generality that the problem can be locally regularized as follows:
for an open set B such that $\overline{B} \subset\subset \Omega$ and $\varepsilon \in (0,1)$, let $f_\varepsilon \in C^\infty(B)$ satisfy

$$f_\varepsilon \geq f \text{ a.e. in } B, f_\varepsilon \to f \text{ strongly in } L^{h(\cdot)}(B) \text{ and a.e. in } B. \tag{2.1}$$

Then there exists a unique solution $u_\varepsilon \in W^{1,p(\cdot)}(B)$ to the local regularized problem

$$\begin{cases} -\text{div}\left((\varepsilon^2 + |Du_\varepsilon|^2)^{\frac{p(x)-2}{2}} Du_\varepsilon\right) = f_\varepsilon & \text{in } B, \\ u_\varepsilon|_{\partial B} = u. \end{cases} \tag{2.2}$$

Moreover, $u_\varepsilon \in C^2(B)$ and

$$\begin{cases} \int_B (\varepsilon^2 + |Du_\varepsilon|^2)^{\frac{p(x)}{2}}\, dx \leq C, \\ u_\varepsilon \to u \text{ strongly in } W^{1,p(\cdot)}(B). \end{cases} \tag{2.3}$$

Remark 2.1. The construction of the regularized problem (2.2) has been used by many authors. We refer the readers in particular to [6] where this approach was used in order to deal with the study of $C^{1,\alpha}_{loc}$ regularity of the solutions.
Let us point out for the reader's convenience that the existence of the solution u_ε to (2.2) follows in a standard way by minimizing the energy functional (note that $f_\varepsilon = f_\varepsilon(x)$ is fixed). The fact that $u_\varepsilon \in C^2(B)$ is now a consequence of standard elliptic regularity, since the operator in (2.2) in strictly-uniformly elliptic.
Finally it follows that u_ε converges to u (that is (2.3) holds) by uniqueness and exploiting the results in [3] (see also [4]).

Here and hereafter, $C > 0$ denotes a constant independent of $\varepsilon \in (0,1)$, $q' = q/(q-1)$ is the conjugate of q, $q^* = Nq/(N-q)$ is the Sobolev exponent of q, Ω_0 is an open set such that $\overline{\Omega_0} \subset\subset B$ and $\eta \in C_c^\infty(B)$ denotes a test function such that

$$\begin{cases} 0 \le \eta \le 1 & \text{in } B, \\ \eta = 1 & \text{in } \Omega_0. \end{cases} \tag{2.4}$$

For a function $u : \Omega \to \mathbb{R}$, we also denote its partial derivatives $u_i = D_i u = \frac{\partial}{\partial x_i} u$ and $\|D^2 u\| = (\sum_{i,j} |u_{ij}|^2)^{\frac{1}{2}}$.

If f satisfies (F_1), we will choose $f_\varepsilon \in C^\infty(B)$ such that $f_\varepsilon \ge f$ a.e. in B, $f_\varepsilon \to f$ strongly in $W^{1,q(\cdot)}(B)$ and a.e. in B. Then u_ε is constructed as above.

We will use the following inequality

$$|\log t|^2 + |\log t| \le C_\delta + t^\delta + t^{-\delta} \tag{2.5}$$

for $t > 0$ and $\delta > 0$ and C_δ is a constant depending on δ. It is important to note that the constant C_δ becomes arbitrary large when δ approaches zero. In our applications, see Lemma 3.1, we will in any case use (2.5) with $\delta > 0$ fixed.

3. Local regularity

Lemma 3.1. *Let u_ε be a solution to (2.2) and assume that f satisfies (F_1). Let $\beta \in C(\overline{B})$ such that $0 < \beta_- \le \beta_+ < 1$ and $1 - p(x)/q'(x) \le \beta(x)$ for all $x \in B$, then we have*

$$\int_{\Omega_0} (\varepsilon^2 + |Du_\varepsilon|^2)^{\frac{p(x)-2-\beta(x)}{2}} \|D^2 u_\varepsilon\|^2 \, dx < C$$

where C depends only on B, p, f and β.

Moreover, if p is constant in B, the conclusion is still valid for constant β such that $0 \le \beta < 1$ and $1 - p/q'(x) \le \beta$ for all $x \in B$.

Proof. The main ingredient in this proof is the linearized equation, which involves the second derivatives of the solutions. Then, everything is reduced to the right choice of test functions.

For simplicity in notation, we denote $w = u_\varepsilon$. Choosing test function $\varphi = D_i \psi$, with $\psi \in C_c^1(B)$, for the regularized problem (2.2), we get:

$$\int_B (\varepsilon^2 + |Dw|^2)^{\frac{p(x)-2}{2}} (Dw, D(D_i\psi)) - f_\varepsilon D_i \psi \, dx = 0.$$

Integrating by parts we obtain

$$\int_B (\varepsilon^2 + |Dw|^2)^{\frac{p(x)-2}{2}} (Dw_i, D\psi) + (p(x)-2)(\varepsilon^2 + |Dw|^2)^{\frac{p(x)-4}{2}} (Dw, Dw_i)(Dw, D\psi)$$
$$+ \frac{1}{2} p_i(x) \log(\varepsilon^2 + |Dw|^2)(\varepsilon^2 + |Dw|^2)^{\frac{p(x)-2}{2}} (Dw, D\psi) - \psi D_i f_\varepsilon \, dx = 0.$$

Choose $\psi = w_i(\varepsilon^2+|w_i|^2)^{-\frac{\beta(x)}{2}}\eta^2$ where η satisfies (2.4), then

$$D\psi = \eta^2(\varepsilon^2+|w_i|^2)^{-\frac{\beta(x)}{2}}\left(1-\frac{\beta(x)|w_i|^2}{\varepsilon^2+|w_i|^2}\right)Dw_i$$
$$+ 2w_i(\varepsilon^2+|w_i|^2)^{-\frac{\beta(x)}{2}}\eta D\eta - \frac{1}{2}w_i\eta^2(\varepsilon^2+|w_i|^2)^{-\frac{\beta(x)}{2}}\log(\varepsilon^2+|w_i|^2)D\beta.$$

Hence we have

$$\begin{aligned}
0 = &\\
&\int_B \frac{(\varepsilon^2+|Dw|^2)^{\frac{p(x)-2}{2}}}{(\varepsilon^2+|w_i|^2)^{\frac{\beta(x)}{2}}}|Dw_i|^2\eta^2\left(1-\frac{\beta(x)|w_i|^2}{\varepsilon^2+|w_i|^2}\right)dx\\
&+2\int_B \frac{(\varepsilon^2+|Dw|^2)^{\frac{p(x)-2}{2}}}{(\varepsilon^2+|w_i|^2)^{\frac{\beta(x)}{2}}}w_i\eta(Dw_i,D\eta)\,dx\\
&-\frac{1}{2}\int_B \frac{(\varepsilon^2+|Dw|^2)^{\frac{p(x)-2}{2}}}{(\varepsilon^2+|w_i|^2)^{\frac{\beta(x)}{2}}}w_i\eta^2\log(\varepsilon^2+|w_i|^2)(Dw_i,D\beta)\,dx\\
&+\int_B (p(x)-2)\frac{(\varepsilon^2+|Dw|^2)^{\frac{p(x)-4}{2}}}{(\varepsilon^2+|w_i|^2)^{\frac{\beta(x)}{2}}}(Dw,Dw_i)^2\eta^2\left(1-\frac{\beta(x)|w_i|^2}{\varepsilon^2+|w_i|^2}\right)dx\\
&+2\int_B (p(x)-2)\frac{(\varepsilon^2+|Dw|^2)^{\frac{p(x)-4}{2}}}{(\varepsilon^2+|w_i|^2)^{\frac{\beta(x)}{2}}}w_i\eta(Dw,Dw_i)(Dw,D\eta)\,dx\\
&-\frac{1}{2}\int_B (p(x)-2)\frac{(\varepsilon^2+|Dw|^2)^{\frac{p(x)-4}{2}}}{(\varepsilon^2+|w_i|^2)^{\frac{\beta(x)}{2}}}w_i\eta^2\log(\varepsilon^2+|w_i|^2)(Dw,Dw_i)(Dw,D\beta)\,dx\\
&+\frac{1}{2}\int_B p_i(x)\log(\varepsilon^2+|Dw|^2)\frac{(\varepsilon^2+|Dw|^2)^{\frac{p(x)-2}{2}}}{(\varepsilon^2+|w_i|^2)^{\frac{\beta(x)}{2}}}(Dw,Dw_i)\eta^2\left(1-\frac{\beta(x)|w_i|^2}{\varepsilon^2+|w_i|^2}\right)dx\\
&+\int_B p_i(x)\log(\varepsilon^2+|Dw|^2)\frac{(\varepsilon^2+|Dw|^2)^{\frac{p(x)-2}{2}}}{(\varepsilon^2+|w_i|^2)^{\frac{\beta(x)}{2}}}w_i\eta(Dw,D\eta)\,dx\\
&-\frac{1}{4}\int_B p_i(x)\log(\varepsilon^2+|Dw|^2)\frac{(\varepsilon^2+|Dw|^2)^{\frac{p(x)-2}{2}}}{(\varepsilon^2+|w_i|^2)^{\frac{\beta(x)}{2}}}w_i\eta^2\log(\varepsilon^2+|w_i|^2)(Dw,D\beta)\,dx\\
&-\int_B w_i(\varepsilon^2+|w_i|^2)^{-\frac{\beta(x)}{2}}\eta^2 D_i f_\varepsilon\,dx\\
&\equiv I_1+I_2+I_3+I_4+I_5+I_6+I_7+I_8+I_9+I_{10}.
\end{aligned}$$

Here I_k denotes each term in the previous expression, consecutively.
Since

$$0 < 1-\beta_+ \le 1-\frac{\beta(x)|w_i|^2}{\varepsilon^2+|w_i|^2}\le 1,$$

we have

$$
\begin{aligned}
I_1+I_4 &= \int_B \frac{(\varepsilon^2+|Dw|^2)^{\frac{p(x)-2}{2}}}{(\varepsilon^2+|w_i|^2)^{\frac{\beta(x)}{2}}}|Dw_i|^2\eta^2\left(1-\frac{\beta(x)|w_i|^2}{\varepsilon^2+|w_i|^2}\right)dx \\
&+\int_B (p(x)-2)\frac{(\varepsilon^2+|Dw|^2)^{\frac{p(x)-4}{2}}}{(\varepsilon^2+|w_i|^2)^{\frac{\beta(x)}{2}}}(Dw,Dw_i)^2\eta^2\left(1-\frac{\beta(x)|w_i|^2}{\varepsilon^2+|w_i|^2}\right)dx \\
&\geq \inf_{x\in\Omega}(\min\{p(x)-1,1\})(1-\beta_+)I_1 \geq C_0\int_B \frac{(\varepsilon^2+|Dw|^2)^{\frac{p(x)-2}{2}}}{(\varepsilon^2+|w_i|^2)^{\frac{\beta(x)}{2}}}|Dw_i|^2\eta^2\,dx.
\end{aligned}
\tag{3.1}
$$

On the other hand, for all $\delta > 0$

$$
\begin{aligned}
|I_2|+|I_5| &= \Big|2\int_B \frac{(\varepsilon^2+|Dw|^2)^{\frac{p(x)-2}{2}}}{(\varepsilon^2+|w_i|^2)^{\frac{\beta(x)}{2}}}w_i\eta(Dw_i,D\eta)\,dx\Big| \\
&+\Big|2\int_B (p(x)-2)\frac{(\varepsilon^2+|Dw|^2)^{\frac{p(x)-4}{2}}}{(\varepsilon^2+|w_i|^2)^{\frac{\beta(x)}{2}}}w_i\eta(Dw,Dw_i)(Dw,D\eta)\,dx\Big| \\
&\leq \sup_{x\in\Omega}(2+2|p(x)-2|)\int_B \frac{(\varepsilon^2+|Dw|^2)^{\frac{p(x)-2}{2}}}{(\varepsilon^2+|w_i|^2)^{\frac{\beta(x)}{2}}}|w_i||\eta||Dw_i||D\eta|\,dx \\
&\leq \delta\int_B \frac{(\varepsilon^2+|Dw|^2)^{\frac{p(x)-2}{2}}}{(\varepsilon^2+|w_i|^2)^{\frac{\beta(x)}{2}}}|Dw_i|^2\eta^2\,dx + C(\delta)\int_B \frac{(\varepsilon^2+|Dw|^2)^{\frac{p(x)-2}{2}}}{(\varepsilon^2+|w_i|^2)^{\frac{\beta(x)}{2}}}|w_i|^2|D\eta|^2\,dx \\
&\leq \delta\int_B \frac{(\varepsilon^2+|Dw|^2)^{\frac{p(x)-2}{2}}}{(\varepsilon^2+|w_i|^2)^{\frac{\beta(x)}{2}}}|Dw_i|^2\eta^2\,dx + C(\delta)\int_B (\varepsilon^2+|Dw|^2)^{\frac{p(x)-\beta(x)}{2}}\,dx \\
&\leq \delta\int_B \frac{(\varepsilon^2+|Dw|^2)^{\frac{p(x)-2}{2}}}{(\varepsilon^2+|w_i|^2)^{\frac{\beta(x)}{2}}}|Dw_i|^2\eta^2\,dx + C(\delta)
\end{aligned}
\tag{3.2}
$$

where the second estimate is an application of Young inequality and the last estimate is derived from (2.3).
For future use we point out that later δ will be a fixed (small) parameter. Consequently $C(\delta)$ will be a fixed (large) constant.

By Young inequality and (2.5) we get

$$
\begin{aligned}
|I_7| &= \Big|\frac{1}{2}\int_B (p(x)-2)\frac{(\varepsilon^2+|Dw|^2)^{\frac{p(x)-4}{2}}}{(\varepsilon^2+|w_i|^2)^{\frac{\beta(x)}{2}}} w_i\eta^2\log(\varepsilon^2+|w_i|^2)(Dw,Dw_i)(Dw,D\beta)\,dx\Big| \\
&\leq C\int_B \big|\log(\varepsilon^2+|Dw|^2)\big|\frac{(\varepsilon^2+|Dw|^2)^{\frac{p(x)-1}{2}}}{(\varepsilon^2+|w_i|^2)^{\frac{\beta(x)}{2}}}|Dw_i|\eta\,dx \\
&\leq \delta\int_B \frac{(\varepsilon^2+|Dw|^2)^{\frac{p(x)-2}{2}}}{(\varepsilon^2+|w_i|^2)^{\frac{\beta(x)}{2}}}|Dw_i|^2\eta^2\,dx + C(\delta)\int_B \big|\log(\varepsilon^2+|Dw|^2)\big|^2\frac{(\varepsilon^2+|Dw|^2)^{\frac{p(x)}{2}}}{(\varepsilon^2+|w_i|^2)^{\frac{\beta(x)}{2}}}\,dx \\
&\leq \delta\int_B \frac{(\varepsilon^2+|Dw|^2)^{\frac{p(x)-2}{2}}}{(\varepsilon^2+|w_i|^2)^{\frac{\beta(x)}{2}}}|Dw_i|^2\eta^2\,dx \\
&+ C(\delta)\int_B \frac{(\varepsilon^2+|Dw|^2)^{\frac{p(x)}{2}}}{(\varepsilon^2+|w_i|^2)^{\frac{\beta(x)}{2}}} + \frac{(\varepsilon^2+|Dw|^2)^{\frac{p(x)+\delta}{2}}}{(\varepsilon^2+|w_i|^2)^{\frac{\beta(x)}{2}}} + \frac{(\varepsilon^2+|Dw|^2)^{\frac{p(x)-\delta}{2}}}{(\varepsilon^2+|w_i|^2)^{\frac{\beta(x)}{2}}}\,dx \\
&\leq \delta\int_B \frac{(\varepsilon^2+|Dw|^2)^{\frac{p(x)-2}{2}}}{(\varepsilon^2+|w_i|^2)^{\frac{\beta(x)}{2}}}|Dw_i|^2\eta^2\,dx \\
&+ C(\delta)\int_B (\varepsilon^2+|Dw|^2)^{\frac{p(x)-\beta(x)}{2}} + (\varepsilon^2+|Dw|^2)^{\frac{p(x)-\beta(x)+\delta}{2}} + (\varepsilon^2+|Dw|^2)^{\frac{p(x)-\beta(x)-\delta}{2}}\,dx.
\end{aligned}
$$

Note that $C(\delta)$ depends on δ because of the Young inequality and also includes C_δ given by(2.5). Also in the following $C(\delta)$ may be relabeled.
Similarly,

$$
\begin{aligned}
|I_8|+|I_9| &= \Big|\int_B p_i(x)\log(\varepsilon^2+|Dw|^2)\frac{(\varepsilon^2+|Dw|^2)^{\frac{p(x)-2}{2}}}{(\varepsilon^2+|w_i|^2)^{\frac{\beta(x)}{2}}} w_i\eta(Dw,D\eta)\,dx\Big| \\
&+ \Big|\frac{1}{4}\int_B p_i(x)\log(\varepsilon^2+|Dw|^2)\frac{(\varepsilon^2+|Dw|^2)^{\frac{p(x)-2}{2}}}{(\varepsilon^2+|w_i|^2)^{\frac{\beta(x)}{2}}} w_i\eta^2\log(\varepsilon^2+|w_i|^2)(Dw,D\beta)\,dx\Big| \\
&\leq C\int_B (\varepsilon^2+|Dw|^2)^{\frac{p(x)-\beta(x)}{2}} + (\varepsilon^2+|Dw|^2)^{\frac{p(x)-\beta(x)+\delta}{2}} + (\varepsilon^2+|Dw|^2)^{\frac{p(x)-\beta(x)-\delta}{2}}\,dx
\end{aligned}
$$

and

$$
\begin{aligned}
&|I_3|+|I_6| = \Big|\frac{1}{2}\int_B \frac{(\varepsilon^2+|Dw|^2)^{\frac{p(x)-2}{2}}}{(\varepsilon^2+|w_i|^2)^{\frac{\beta(x)}{2}}} w_i\eta^2 \log(\varepsilon^2+|w_i|^2)(Dw_i, D\beta)\,dx\Big| \\
&+\Big|\frac{1}{2}\int_B (p(x)-2)\frac{(\varepsilon^2+|Dw|^2)^{\frac{p(x)-4}{2}}}{(\varepsilon^2+|w_i|^2)^{\frac{\beta(x)}{2}}} w_i\eta^2 \log(\varepsilon^2+|w_i|^2)(Dw, Dw_i)(Dw, D\beta)\,dx\Big| \\
&\leq \delta\int_B \frac{(\varepsilon^2+|Dw|^2)^{\frac{p(x)-2}{2}}}{(\varepsilon^2+|w_i|^2)^{\frac{\beta(x)}{2}}}|Dw_i|^2\eta^2\,dx \\
&+C(\delta)\int_B (\varepsilon^2+|Dw|^2)^{\frac{p(x)-\beta(x)}{2}} + (\varepsilon^2+|Dw|^2)^{\frac{p(x)-\beta(x)+\delta}{2}} + (\varepsilon^2+|Dw|^2)^{\frac{p(x)-\beta(x)-\delta}{2}} dx.
\end{aligned}
$$

If p and β are constants then $I_3 = I_6 = I_7 = I_8 = I_9 = 0$. Otherwise, recalling the assumption $0 < \beta_- \leq \beta_+ < 1$, we can choose δ *fixed* small enough such that $\delta < C_0/4$ and $\delta < \beta(x) < p_- - \delta$ for all $x \in B$, from the previous estimates and (2.3) we have

$$
|I_3|+|I_6|+|I_7|+|I_8|+|I_9| \leq 2\delta\int_B \frac{(\varepsilon^2+|Dw|^2)^{\frac{p(x)-2}{2}}}{(\varepsilon^2+|w_i|^2)^{\frac{\beta(x)}{2}}}|Dw_i|^2\eta^2\,dx + C(\delta). \tag{3.3}
$$

On the other hand,

$$
\begin{aligned}
|I_{10}| &= \Big|\int_B w_i(\varepsilon^2+|w_i|^2)^{-\frac{\beta(x)}{2}}\eta^2 D_i f_\varepsilon\,dx\Big| \\
&\leq \int_B (\varepsilon^2+|w_i|^2)^{\frac{1-\beta(x)}{2}}|D_i f_\varepsilon|\,dx \leq C\|D_i f_\varepsilon\|_{L^{q(\cdot)}(B)}\Big\|(\varepsilon^2+|w_i|^2)^{\frac{1-\beta(x)}{2}}\Big\|_{L^{q'(\cdot)}(B)}.
\end{aligned}
\tag{3.4}
$$

Since $f_\varepsilon \to f$ in $W^{1,q(\cdot)}(B)$, we have $\|D_i f_\varepsilon\|_{L^{q(\cdot)}(B)} < C$. Moreover, from $0 < \frac{1-\beta(x)}{2}q'(x) \leq \frac{p(x)}{2}$ in B and (2.3) we have $\|(\varepsilon^2+|w_i|^2)^{\frac{1-\beta(x)}{2}}\|_{L^{q'(\cdot)}(B)} < C$, and then $|I_{10}| < C$.

Taking into account (3.1), and exploiting (3.2), (3.3) and (3.4), we conclude that

$$
\int_B \frac{(\varepsilon^2+|Dw|^2)^{\frac{p(x)-2}{2}}}{(\varepsilon^2+|w_i|^2)^{\frac{\beta(x)}{2}}}|Dw_i|^2\eta^2\,dx \leq C.
$$

Since $\eta = 1$ in Ω_0, the desired result follows by taking the sum of the previous inequality over i.

□

Lemma 3.2. *Let u_ε be a solution to (2.2) and assume that f satisfies (F_1). Let $l \in C^1(B) \cap C_+(\overline{B})$ such that $p(x) \leq l(x)$ in B and $l(x) \leq 2$ if $1 < p(x) < 2$. We assume that*

$$
\int_B (\varepsilon^2+|Du_\varepsilon|^2)^{\frac{l(x)}{2}}\,dx \leq C. \tag{3.5}
$$

Let $\beta \in C(\overline{B})$ such that $\sup_B(p-l) < \beta_- \leq \beta_+ < 1$ and $1 - l(x)/q'(x) \leq \beta(x)$ for all $x \in B$. Then we have

$$
\int_{\Omega_0} (\varepsilon^2+|Du_\varepsilon|^2)^{\frac{p(x)-2-\beta(x)}{2}}\|D^2 u_\varepsilon\|^2\,dx < C
$$

where C depends only on B, p, l, f and β.

Moreover, if p and l are constant in B, the conclusion is still valid for constant β such that $p-l\le\beta<1$ and $1-l/q'(x)\le\beta$ for all $x\in B$.

Proof. The proof is rather similar to that of Lemma 3.1 but (3.5) is used instead of (2.3). □

Lemma 3.3. *Let u_ε be a solution to (2.2). Consider $\overline{\Omega_0}\subset\subset B$ as above and assume that f satisfies (F_1) and $f>0$ in a neighborhood of $\overline{B}$ and $0\le r<1$. Then we have*

$$\int_{\Omega_0}\frac{dx}{(\varepsilon^2+|Du_\varepsilon|^2)^{\frac{(p(x)-1)r}{2}}}<C.$$

Proof. As before, denote $w=u_\varepsilon$.

It is sufficient to prove the lemma for

$$\sup_{x\in B}\max\{1-\frac{p(x)}{(p(x)-1)q'(x)},1-\frac{1-s_-}{p_+-1},0\}\le r<1-\frac{1-s_+}{p_--1}$$

where $0<s_-\le s_+<1$ and $1-s_+$ is small enough.

By hypothesis, $f_\varepsilon(x)\ge C_1$ in B for some $C_1>0$ and ε small enough.

Using test function $\psi^i_\varepsilon=(\varepsilon^2+|Dw|^2)^{\frac{(1-p(x))r}{2}}\eta$ in (2.2) where η satisfies (2.4), we have

$$\begin{aligned}
&C_1\int_B(\varepsilon^2+|Dw|^2)^{\frac{(1-p(x))r}{2}}\eta\,dx\\
&\le\int_B f_\varepsilon\psi^i_\varepsilon\,dx\\
&=\int_B(\varepsilon^2+|Dw|^2)^{\frac{p(x)-2}{2}}(Dw,D\psi^i_\varepsilon)\,dx\\
&=\int_B(\varepsilon^2+|Dw|^2)^{\frac{p(x)-2}{2}}(\varepsilon^2+|Dw|^2)^{\frac{(1-p(x))r}{2}}(Dw,D\eta)\,dx\\
&+\int_B(1-p(x))r(\varepsilon^2+|Dw|^2)^{\frac{p(x)-2}{2}}|Dw|(\varepsilon^2+|Dw|^2)^{\frac{(1-p(x))r-2}{2}}\eta(Dw,D|Dw|)\,dx\\
&-\frac{r}{2}\int_B\log(\varepsilon^2+|Dw|^2)(\varepsilon^2+|Dw|^2)^{\frac{p(x)-2}{2}}(\varepsilon^2+|Dw|^2)^{\frac{(1-p(x))r}{2}}\eta(Dw,Dp)\,dx\\
&\equiv J_1+J_2+J_3.
\end{aligned}$$

Exploiting (2.5) and (2.3), we have the estimates:

$$\begin{aligned}
&J_1+J_3\\
&\le C\int_B\left(1+|\log(\varepsilon^2+|Dw|^2)|\right)(\varepsilon^2+|Dw|^2)^{\frac{(p(x)-1)(1-r)}{2}}\,dx\\
&\le C\int_B(\varepsilon^2+|Dw|^2)^{\frac{(p(x)-1)(1-r)}{2}}+(\varepsilon^2+|Dw|^2)^{\frac{(p(x)-1)(1-r)+\delta}{2}}+(\varepsilon^2+|Dw|^2)^{\frac{(p(x)-1)(1-r)-\delta}{2}}\,dx\\
&\le C
\end{aligned}$$

where $0<\frac{(p(x)-1)(1-r)}{2}\le\frac{1}{2}<\frac{p(x)}{2}$ in B and $0<\delta<p_--1$ can be taken sufficiently small.

On the other hand,

$$\begin{aligned} J_2 &\le (p_+ - 1)r \int_B (\varepsilon^2 + |Dw|^2)^{\frac{(p(x)-1)(1-r)-1}{2}} |D^2 w||\eta|\,dx \\ &\le \delta \int_B (\varepsilon^2 + |Dw|^2)^{\frac{(1-p(x))r}{2}} \eta\,dx + \frac{C}{\delta}\int_B (\varepsilon^2 + |Dw|^2)^{\frac{p(x)-2-\beta(x)}{2}} |D^2 w|^2\,dx, \end{aligned}$$

where $\delta < C_1$ and $\beta \in C(\overline{B})$ such that

$$\frac{(1-p(x))r}{2} + \frac{p(x)-2-\beta(x)}{2} = (p(x)-1)(1-r) - 1$$

or equivalently, $\beta(x) = 1 - (p(x)-1)(1-r)$. Note that $s_- \le \beta(x) \le s_+$ for every $x \in B$ and β satisfies conditions in Lemma 3.1. Hence the proof is completed. □

Lemma 3.4. *Let u_ε be a solution to (2.2) and $l \in C^1(\Omega)\cap C_+(\overline{\Omega})$ such that $l(x) \le p(x)-\delta_1$ in $\Omega^1(\delta)$, $l(x) \le 2$ in $\Omega^2(\delta)$ and $l(x) \le \frac{p(x)-1}{p(x)-2} - \delta_2$ in Ω^3 for some positive numbers δ, δ_1 and δ_2. Suppose that f satisfies (F_1), (F_2), $q(x) \ge \frac{p(x)}{p(x)-1}$ in a neighborhood of $\overline{\Omega^1(\delta)}$ and $q(x) \ge \frac{p(x)}{2p(x)-3}$ in Ω^2. Then*

$$\int_{\Omega_0} \|D^2 u_\varepsilon\|^{l(x)}\,dx < C.$$

Proof. Recall that we define $\Omega_0^k = \Omega^k \cap \Omega_0$ and $\Omega_0^k(\delta) = \Omega^k(\delta)\cap\Omega_0$. Note that it suffices to prove the lemma for δ, δ_1 and δ_2 small enough.

We can choose $\delta, \delta_1 \in (0, p_- - 1)$ so that $\delta < \delta_1$ and δ_1 is small enough, there exists a bounded neighborhood B^1 of $\overline{\Omega_0^1(\delta)}$ such that for $x \in B^1$ we have $q(x) \ge \frac{p(x)}{p(x)-1}$, which implies $q'(x) \le p(x)$. We can choose $\beta \in C(\overline{B^1})$ such that

$$\frac{(2-p(x)+\beta(x))(p(x)-\delta_1)}{4}\,\frac{2}{2-p(x)+\delta_1} = \frac{p(x)}{2},$$

then $0 < \beta_- \le \beta_+ < 1$ and we can apply Lemma 3.1 to obtain

$$\int_{\Omega_0^1(\delta)} \frac{\|D^2 u_\varepsilon\|^2}{(\varepsilon^2 + |Du_\varepsilon|^2)^{\frac{2-p(x)+\beta(x)}{2}}}\,dx < C.$$

We write $\|D^2 u_\varepsilon\|^{p(x)-\delta_1}$ as a product of two functions as follows

$$\|D^2 u_\varepsilon\|^{p(x)-\delta_1} = \left(\frac{\|D^2 u_\varepsilon\|^2}{(\varepsilon^2 + |Du_\varepsilon|^2)^{\frac{2-p(x)+\beta(x)}{2}}}\right)^{\frac{p(x)-\delta_1}{2}} (\varepsilon^2 + |Du_\varepsilon|^2)^{\frac{(2-p(x)+\beta(x))(p(x)-\delta_1)}{4}}.$$

The argument above and (2.3) yield $\left(\frac{\|D^2 u_\varepsilon\|^2}{(\varepsilon^2+|Du_\varepsilon|^2)^{\frac{2-p(x)+\beta(x)}{2}}}\right)^{\frac{p(x)-\delta_1}{2}} \in L^{\frac{2}{p(x)-\delta_1}}(\Omega_0^1(\delta))$ and $(\varepsilon^2 + |Du_\varepsilon|^2)^{\frac{(2-p(x)+\beta(x))(p(x)-\delta_1)}{4}} \in L^{\frac{2}{2-p(x)+\delta_1}}(\Omega_0^1(\delta))$. Consequently, $\int_{\Omega_0^1(\delta)} \|D^2 u_\varepsilon\|^{p(x)-\delta_1}\,dx < C$, then

$$\int_{\Omega_0^1(\delta)} \|D^2 u_\varepsilon\|^{l(x)}\,dx < C. \tag{3.6}$$

Next, from $q(x) \geq \frac{p(x)}{2p(x)-3}$ in Ω^2 we have $q'(x) \leq \frac{p(x)}{3-p(x)}$ for $x \in \Omega^2_0(\delta)$, then

$$\int_{\Omega^2_0(\delta)} \|D^2 u_\varepsilon\|^2 \, dx < C \tag{3.7}$$

follows directly from Lemma 3.1 by choosing $\beta(x) = p(x) - 2$.

Finally, for $x \in \Omega^3_0$, we write $\|D^2 u_\varepsilon\|^{l(x)}$ as a product of two functions as follows

$$\|D^2 u_\varepsilon\|^{l(x)} = \Big((\varepsilon^2 + |Du_\varepsilon|^2)^{\frac{p(x)-2-\beta(x)}{2}} \|D^2 u_\varepsilon\|^2 \Big)^{\frac{l(x)}{2}} \frac{1}{(\varepsilon^2 + |Du_\varepsilon|^2)^{\frac{(p(x)-2-\beta(x))l(x)}{4}}}.$$

Let B^3 be a neighborhood of $\overline{\Omega^3_0}$ restricted in Ω. Choose $\beta \in (1 - p_-/q_-, 1)$ such that $1 - \beta$ is small enough. We have

$$\frac{(p(x)-2-\beta)l(x)}{4} \frac{2}{2-l(x)} = \frac{(p(x)-1)r(x)}{2}$$

for $r \in C(\overline{B^3})$, $r_+ < 1$ and $1 - r_-$ is small enough. From Lemma 3.1 and Lemma 3.3 we get $\Big((\varepsilon^2 + |Du_\varepsilon|^2)^{\frac{p(x)-2-\beta}{2}} \|D^2 u_\varepsilon\|^2 \Big)^{\frac{l(x)}{2}} \in L^{\frac{2}{l(x)}}(\Omega^3_0)$ and $\frac{1}{(\varepsilon^2 + |Du_\varepsilon|^2)^{\frac{(p(x)-2-\beta(x))l(x)}{4}}} \in L^{\frac{2}{2-l(x)}}(\Omega^3_0)$, respectively. Hence

$$\int_{\Omega^3_0} \|D^2 u_\varepsilon\|^{l(x)} \, dx < C. \tag{3.8}$$

The conclusion follows now from (3.6), (3.7) and (3.8). □

Lemma 3.5. *Let u_ε be a solution to* (2.2) *and $m \in C^1(\Omega) \cap C_+(\overline{\Omega})$ such that $m(x) \leq 2$ in $(\Omega^3)^c$ and $m(x) \leq \frac{p(x)-1}{p(x)-2} - \delta$ in Ω^3 for some $\delta > 0$. Suppose that f satisfies (F_1), (F_2), $q(x) \geq \frac{p(x)}{p(x)-1}$ in a neighborhood of $\overline{\Omega^1}$ and $q(x) \geq \frac{p(x)}{2p(x)-3}$ in Ω^2. Then*

$$\int_{\Omega_0} \|D^2 u_\varepsilon\|^{m(x)} \, dx < C.$$

Proof. From Lemma 3.4, it is sufficient to consider δ sufficiently small and to prove that

$$\int_{\Omega_0 \cap (\Omega^3)^c} \|D^2 u_\varepsilon\|^{m(x)} \, dx < C. \tag{3.9}$$

Moreover, we can assume that $q(x) \geq \frac{p(x)}{p(x)-1}$ in a neighborhood of $\overline{\Omega^1(\delta)}$. Let $l \in C_+(\overline{\Omega})$ such that $l(x) \leq p(x) - \delta$ in $\Omega^1(\delta)$, $l(x) \leq 2$ in $\Omega^2(\delta)$ and $l(x) \leq \frac{p(x)-1}{p(x)-2} - \delta$ in Ω^3. Since δ is small, we can assume $l^*(x) > p(x)$ in $(\Omega^3)^c$. From Lemma 3.4, we have

$$\int_{\Omega_0 \cap (\Omega^3)^c} \|D^2 u_\varepsilon\|^{l(x)} \, dx < C, \tag{3.10}$$

and then by Sobolev embedding

$$\int_{\Omega_0 \cap (\Omega^3)^c} |Du_\varepsilon|^{l^*(x)} \, dx < C. \tag{3.11}$$

Since this is true for all Ω_0 such that $\overline{\Omega_0} \subset \Omega$, by Lemma 3.2 we have

$$\int_{\Omega_0 \cap (\Omega^3)^c} (\varepsilon^2 + |Du_\varepsilon|^2)^{\frac{p(x)-2-\beta(x)}{2}} \|D^2 u_\varepsilon\|^2 \, dx < C,$$

where $\sup_B (p - l^*) < \beta_- \leq \beta_+ < 1$ and $1 - l^*(x)/q'(x) \leq \beta(x)$.

We can, therefore, repeat the proof in the first and second part of that of Lemma 3.4 with finer choice of β, allowed by (3.11), and get

$$\int_{\Omega_0 \cap (\Omega^3)^c} \|D^2 u_\varepsilon\|^{l_1(x)} \, dx < C, \tag{3.12}$$

where $l_1(x) = 2$ if $l^*(x) \geq 2 + \delta_1$ and $l_1(x) \leq l^*(x) - \delta_1$ if $l^*(x) < 2 + \delta_1$ for some δ_1 small enough.

Since the sequence $l^{k+1} = (l^k)^*$ converges to ∞ uniformly, by repeating the previous argument we finally get

$$\int_{\Omega_0 \cap (\Omega^3)^c} \|D^2 u_\varepsilon\|^{m(x)} \, dx < C. \tag{3.13}$$

This proof also points out that

$$\int_{\Omega_0} |Du_\varepsilon|^{m(x)} \, dx < C. \tag{3.14}$$

□

Theorem 3.6. *Let $m \in C^1(\Omega) \cap C_+(\overline{\Omega})$ such that $m(x) \leq 2$ in $(\Omega^3)^c$ and $m(x) \leq \frac{p(x)-1}{p(x)-2} - \delta$ in Ω^3 for some $\delta > 0$. Suppose that f satisfies (F_1), (F_2), $q(x) \geq \frac{p(x)}{p(x)-1}$ in a neighborhood of $\overline{\Omega^1}$ and $q(x) \geq \frac{p(x)}{2p(x)-3}$ in Ω^2. Then $u \in W^{2,m(\cdot)}_{loc}(\Omega)$.*

Proof. We have, $\int_{\Omega_0} \|D^2 u_\varepsilon\|^{m(x)} \, dx < C$, by Lemma 3.5. Moreover, from $\int_{\Omega_0} |Du_\varepsilon|^{m(x)} \, dx < C$, we obtain $\|(\varepsilon^2 + |Du_\varepsilon|^2)^{\frac{1}{2}}\|_{W^{1,m(\cdot)}(\Omega_0)} < C$. Thus, we can suppose that

$$(\varepsilon^2 + |Du_\varepsilon|^2)^{\frac{1}{2}} \to h$$

weakly in $W^{1,m(\cdot)}(\Omega_0)$, strongly in $L^{m(\cdot)}(\Omega_0)$ and almost everywhere in Ω_0.

Then, by (2.3), we must have $h = |Du|$. Therefore,

$$|Du| \in W^{1,m(\cdot)}(\Omega_0).$$

Consequently, $|Du| \in W^{1,m(\cdot)}_{loc}(\Omega)$, and then $u \in W^{2,m(\cdot)}_{loc}(\Omega)$. □

Now we prove a regularity result for the constant exponent case where the assumption that f belongs to a variable exponent Sobolev space is relaxed.

Lemma 3.7. *Let u_ε be a solution to (2.2) and assume that p is a constant such that $1 < p < 2$ and $h(x) \geq \frac{p}{p-1}$ in Ω, then we have*

$$\int_{\Omega_0} (\varepsilon^2 + |Du_\varepsilon|^2)^{\frac{p-2}{2}} \|D^2 u_\varepsilon\|^2 \, dx < C.$$

Proof. As before, we denote $w = u_\varepsilon$.

Working similarly to Lemma 3.1 with $\beta = 0$, we obtain the same estimates for I_1, ... I_9. Now we give a new estimate for I_{10}.

$$\begin{aligned} I_{10} &= -\int_B w_i \eta^2 D_i f_\varepsilon \, dx \\ &= \int_B D_i(w_i\eta^2) f_\varepsilon \, dx \\ &= \int_B \eta^2 w_{ii} f_\varepsilon \, dx + 2\int_B w_i \eta \eta_i f_\varepsilon \, dx \\ &= J_1 + J_2. \end{aligned}$$

By Young inequality, we have

$$\begin{aligned} J_1 &\le \delta \int_B (\varepsilon^2 + |Dw|^2)^{\frac{p-2}{2}} |w_{ii}|^2 \eta^2 \, dx + \frac{C}{\delta}\int_B (\varepsilon^2 + |Dw|^2)^{\frac{2-p}{2}} |f_\varepsilon|^2 \eta^2 \, dx \\ &\le \delta \int_B (\varepsilon^2 + |Dw|^2)^{\frac{p-2}{2}} |w_{ii}|^2 \eta^2 \, dx + \frac{C}{\delta}\int_B (\varepsilon^2 + |Dw|^2)^{\frac{2-p}{2}} |f_\varepsilon|^2 \, dx. \end{aligned}$$

Moreover,

$$\begin{aligned} \int_B (\varepsilon^2 + |Dw|^2)^{\frac{2-p}{2}} |f_\varepsilon|^2 \, dx &\le \left\| (\varepsilon^2 + |Dw|^2)^{\frac{2-p}{2}} \right\|_{L^{\frac{p}{2-p}}(B)} \left\| |f_\varepsilon|^2 \right\|_{L^{\frac{p}{2p-2}}(B)} \\ &= \|(\varepsilon^2 + |Dw|^2)\|^{(2-p)/2}_{L^{p/2}(B)} \|f_\varepsilon\|^2_{L^{\frac{p}{p-1}}(B)}, \end{aligned}$$

which is bounded due to (2.3), (2.1) and $h(x) \ge \frac{p}{p-1}$. Therefore,

$$J_1 \le \delta \int_B (\varepsilon^2 + |Dw|^2)^{\frac{p-2}{2}} |Dw_i|^2 \eta^2 \, dx + \frac{C}{\delta}.$$

On the other hand,

$$J_2 \le C \int_B (\varepsilon^2 + |w_i|^2)^{\frac{1}{2}} |f_\varepsilon| \, dx \le C \|(\varepsilon^2 + |w_i|^2)^{\frac{1}{2}}\|_{L^p(B)} \|f_\varepsilon\|_{L^{\frac{p}{p-1}}(B)},$$

which is bounded by the same reasons.

Now proceed as in Lemma 3.1, we obtain

$$\int_B (\varepsilon^2 + |Dw|^2)^{\frac{p-2}{2}} |Dw_i|^2 \eta^2 \, dx \le C,$$

as desired.

□

Lemma 3.8. *Let u_ε be a solution to* (2.2) *and assume that p and l are constants such that $1 < p \le l < 2$ and $h(x) \ge \frac{p}{p-1}$ in Ω. Moreover, we assume that*

$$\int_B (\varepsilon^2 + |Du_\varepsilon|^2)^{\frac{l}{2}}\, dx \le C. \tag{3.15}$$

Then we have

$$\int_{\Omega_0} (\varepsilon^2 + |Du_\varepsilon|^2)^{\frac{l-2}{2}} \|D^2 u_\varepsilon\|^2\, dx < C.$$

Proof. The proof is rather similar to that of Lemma 3.7 but (3.15) is used instead of (2.3). □

Theorem 3.9. *Suppose that p is a constant such that $1 < p < 2$ and $h(x) \ge \frac{p}{p-1}$ in Ω. Then $u \in W^{2,2}_{loc}(\Omega)$.*

Proof. From Lemma 3.7 we have

$$\int_{\Omega_0} \frac{\|D^2 u_\varepsilon\|^2}{(\varepsilon^2 + |Du_\varepsilon|^2)^{\frac{2-p}{2}}}\, dx < C.$$

We write $\|D^2 u_\varepsilon\|^p$ as product of two functions as follows

$$\|D^2 u_\varepsilon\|^p = \left(\frac{\|D^2 u_\varepsilon\|^2}{(\varepsilon^2 + |Du_\varepsilon|^2)^{\frac{2-p}{2}}} \right)^{\frac{p}{2}} (\varepsilon^2 + |Du_\varepsilon|^2)^{\frac{(2-p)p}{4}}.$$

The argument above and (2.3) yield $\left(\frac{\|D^2 u_\varepsilon\|^2}{(\varepsilon^2 + |Du_\varepsilon|^2)^{\frac{2-p}{2}}} \right)^{\frac{p}{2}} \in L^{\frac{2}{p}}(\Omega_0)$ and $(\varepsilon^2 + |Du_\varepsilon|^2)^{\frac{(2-p)p}{4}} \in L^{\frac{2}{2-p}}(\Omega_0)$. From Hölder's inequality we obtain

$$\int_{\Omega_0} \|D^2 u_\varepsilon\|^p\, dx < C.$$

The remaining proof is similar to that of Theorem 3.6 with the aid of Lemma 3.8. □

References

[1] E. Acerbi and G. Mingione, Regularity results for a class of functionals with non-standard growth, *Arch. Ration. Mech. Anal.*, 156 (2), (2001), 121–140.

[2] E. Acerbi and G. Mingione, Gradient estimates for the $p(x)$-Laplacean system, *J. Reine Angew. Math.*, 584, (2005), 117–148.

[3] L. Boccardo and F. Murat, Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations, *Nonlinear Anal.*, 19(6), (1992), 581–597.

[4] G. Dal Maso and F. Murat, Almost everywhere convergence of gradients of solutions to nonlinear elliptic systems, *Nonlinear Anal.*, 31(3-4), (1998), 405–412.

[5] L. Damascelli, B. Sciunzi, Regularity, monotonicity and symmetry of positive solutions of m-Laplace equations, *J. Differential Equations*, 206 (2), (2004), 483–515.

[6] E. DiBenedetto, $C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations, *Nonlinear Anal.*, 7, (1983), 827–850.

[7] F. Duzaar and G. Mingione, Gradient continuity estimates, *Calc. Var. Partial Differential Equations*, 39 (3-4), (2010), 379–418.

[8] X. Fan, Global $C^{1,\alpha}$ regularity for variable exponent elliptic equations in divergence form, *J. Differential Equations*, 235, (2007), 397–417.
[9] G.M. Lieberman, Boundary regularity for solutions of degenerate elliptic equations, *Nonlinear Anal.*, 12, (1988), 1203–1219.
[10] H. Lou, On singular sets of local solutions to p-Laplace equations, *Chinese Annals of Mathematics, Series B*, 29B (5), (2008), 521–530.
[11] G. Mingione, Regularity of minima: an invitation to the dark side of the calculus of variations, *Applications of Mathematics*, 51 (4), (2006), 355–426.
[12] G. Mingione, The Calderon-Zygmund theory for elliptic problems with measure data, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5)*, 6, (2007), 195–261.
[13] M. Sanchón and J. M. Urbano, Entropy solutions for the $p(x)$-Laplace equation, *Trans. Amer. Math. Soc.*, 361(12), (2009), 6387–6405.
[14] B. Sciunzi, Some results on the qualitative properties of positive solutions of quasilinear elliptic equations,*NoDEA Nonlinear Differential Equations Appl.* 14, (2007), 315–334.
[15] P. Tolksdorf, Regularity for a more general class of quasilinear elliptic equations, *J. Differential Equations*, 51, (1984), 126–150.

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