One dimensional symmetry of solutions to some anisotropic quasilinear elliptic equations in the plane.

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Abstract

We prove one-dimensional symmetry of monotone solutions for some anisotropic quasilinear elliptic equations in the plane.

1 Introduction

Let us first recall the following striking conjecture that was posed by De Giorgi in [5]: Let $u \in C^2(\mathbb{R}^n, [-1, 1])$ satisfy $\Delta u + u - u^3 = 0$ and $\partial_{x_n} u > 0$ in the whole $\mathbb{R}^n$.

Is it true that all the level sets of u are hyperplanes, at least if $n \leqslant 8$?

Many outstanding mathematicians contributed to this important issue, which is related to many physical and mathematical applications. Let us only mention the papers [1, 2, 3, 6, 7, 8, 10, 11, 12, 13, 14, 15, 17, 18, 21, 27, 28, 29], and refer the reader to [16] for a survey on this topic and a nice and complete description of recent developments. We only remark here that the conjecture has been completely understood in dimension $n = 2, 3$ in both the semilinear and quasilinear case, see [1, 2, 21] and [10, 12, 13]. In particular by [1] and [10, 12, 13] it follows that in low dimension $n = 2, 3$ the conjecture holds true actually for any smooth nonlinearity. In higher dimensions the conjecture is still open in spite of the important contribution in [27] (see also [29]), where the conjecture is solved under the additional assumption that the limiting profiles are constants. A remarkable improvement in this direction has been recently obtained in [17] where, up to dimension eight, the validity of the De Giorgi conjecture has been proved under more general assumptions on the limiting profiles. The case of other settings and operators have been considered in [3, 11, 12, 13, 14, 15, 18, 28, 29], while an important

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contribution in [6] provides a counterexample in dimension $n \geqslant 9$.

In this paper, following the techniques in [12, 13], that go back to [9], we address the validity of the conjecture of De Giorgi for some anisotropic quasilinear elliptic operators in the plane.
More precisely we consider in $\mathbb{R}^2$ the anisotropic quasilinear degenerate elliptic equation:

$$\frac{\partial}{\partial x_1}\left(|u_1|^{p_1-2}u_1\right) + \frac{\partial}{\partial x_2}\left(|u_2|^{p_2-2}u_2\right) = f(u) \quad x \in \mathbb{R}^2 . \tag{1}$$

with $p = (p_1, p_2) \in \mathbb{R}^2$, $p_i \in [2, +\infty)$ and $f \in C^1(\mathbb{R})$. Note that the operator in (1) does not reduce to the p-Laplace operator even if $p_1 = p_2$.

Let us set

$$\mathcal{Z}_u = \{x \in \mathbb{R}^2 \; s.t. \; |u_1(x)| \cdot |u_2(x)| = 0\} \tag{2}$$

with the notation $u_i := \frac{\partial u}{\partial x_i}$. We assume that $|\nabla u|$ is bounded and

$$u \in C^1(\mathbb{R}^2) \cap C^2(\mathbb{R}^2 \setminus \mathcal{Z}_u) \quad \text{and} \quad |u_i|^{p_i-2}u_i \in W^{1,2}_{loc}(\mathbb{R}^2)$$

Remark 1.1. Let us remark that the assumption $u \in C^2(\mathbb{R}^2 \setminus \mathcal{Z}_u)$ is not restrictive because of the fact that outside $\mathcal{Z}_u$ the solutions turn out to be smooth by standard regularity results. The $C^{1,\alpha}$ regularity of the solutions in this setting is an hard task not and yet well understood as is in the case of the p-laplacian. Nevertheless this assumption is actually necessary, for technical reasons but also because of some counterexamples (see [12]) that and show that the result is not valid in general. The assumption $|u_i|^{p_i-2}u_i \in W^{1,2}_{loc}(\mathbb{R}^2)$ is a natural assumption. This is by now standard in the case of the p-laplacian, see for example [4] and also [22].

We say that $u \in C^1(\mathbb{R}^2) \cap C^2(\mathbb{R}^2 \setminus \mathcal{Z}_u)$ is a *weak solution* of (1) if u satisfies:

$$\int_{\mathbb{R}^2} \left\{|u_1|^{p_1-2}u_1v_1 + |u_2|^{p_2-2}u_2v_2\right\} dx = \int_{\mathbb{R}^2} f(u)v \; dx \quad \forall v \in C^\infty_c(\mathbb{R}^2) . \tag{3}$$

It is easy to see that by density arguments we may assume that (3) is actually fulfilled for every $v \in C^1_c(\mathbb{R}^2)$.

We are now in position to state our main result:

Theorem 1.2. *Let $u \in C^1(\mathbb{R}^2) \cap C^2(\mathbb{R}^2 \setminus \mathcal{Z}_u)$ be a weak solution of* (3) *with $|u_i|^{p_i-2}u_i \in W^{1,2}_{loc}(\mathbb{R}^2)$. Let $p_i \in [2, +\infty)$ and let $f \in C^1(\mathbb{R})$. Assume that $|\nabla u|$ is bounded and that*

$$u_2(x) > 0 \text{ for every } x \in \mathbb{R}^2 , \tag{4}$$

then the level sets of u are flat, and there exists $\nu \in S^1$ and $w : \mathbb{R} \to \mathbb{R}$ such that

$$u(x) = w(x \cdot \nu) \quad \forall x \in \mathbb{R}^2 . \tag{5}$$

Remark 1.3. In the case of isotropic quasilinear elliptic equations, if u is a solution, we can rotate it (that means we consider the function $v(x) = u(Rx)$ for some orthogonal matrix R) to get a new solution. This means that, if u varies only in one direction (and hence its level sets are flat), we can obtain infinitely many solutions with flat level sets. On the contrary equation (1) is not invariant up to rotations. Nevertheless we can look for other solutions of (1), which have a one dimensional profile, and flat level sets that are not parallel to the axis. More precisely we can look for solutions of the following form:
$u(x) = w(a \cdot x)$ for $a = (a_1, a_2) \in \mathbb{R}^2$ and $w : \mathbb{R} \to \mathbb{R}$. Then u satisfy (1) with $u_2 > 0$ if w is a solution of:

$$w'' = \frac{f(w)}{(p_1 - 1)a_1^{p_1} w'^{p_1-2} + (p_2 - 1)a_2^{p_2} w'^{p_2-2}} \,. \tag{6}$$

with $w' > 0$.

2 Preliminary results

For completeness let us first remark that equation (3) is well defined in anisotropic Sobolev spaces. More precisely, for $\Omega \subseteq \mathbb{R}^n$ and $p = (p_1, \cdots, p_n) \in \mathbb{R}^n$ consider

$$\|u\|_{1,p} = \sum_{i=1}^{n} \|u_i\|_{p_i} \,, \tag{7}$$

and denote by $W^{1,p}(\Omega)$ the set of those functions having distributional derivative for which the norm in (7) is bounded. It is customary to define $W_0^{1,p}(\Omega)$ as the closure of $C_c^\infty(\Omega)$ in $W^{1,p}(\Omega)$ with respect to the norm

$$\|u\|_{1,p} = \sum_{i=1}^{n} \|u_i\|_{p_i} \,,$$

where $\|v\|_{p_i} := \left(\int_\Omega |v|^{p_i} dx \right)^{\frac{1}{p_i}}$. An elegant and useful description of this approach may be found in [24], where previous founding papers [19, 25, 26, 32, 33] are also resumed. In the above cited papers it is shown that the use of density arguments in order to assume that (3) is actually fulfilled for every $v \in W_0^{1,p}(\Omega)$ is delicate and requires embeddings in some Lebesgue spaces which generally holds true only in some cases.

For a given solution $u \in C^1(\mathbb{R}^2) \cap C^2(\mathbb{R}^2 \setminus \mathcal{Z}_u)$ to (3) such that $|u_i|^{p_i-2}u_i \in W^{1,2}_{loc}(\mathbb{R}^2)$ we have the following:

Definition 2.1. We say that u is *stable* if:

$$\int_{\mathbb{R}^2} (p_1 - 1)|u_1|^{p_1-2}\varphi_1^2 + (p_2 - 1)|u_2|^{p_2-2}\varphi_2^2 - f'(u)\varphi^2 dx \geq 0 \tag{8}$$

for every $\varphi \in C_c^\infty(\mathbb{R}^2)$ (or $\varphi \in C_c^1(\mathbb{R}^2)$).

Let us consider a domain $\Omega \subseteq \mathbb{R}^2$ and two positive weights ρ_1 and ρ_2 and set

$$\|v\|_{\rho_1,\rho_2} = \|v\|_{L^2(\Omega)} + \|u_1\|_{L^2(\Omega,\rho_1)} + \|u_2\|_{L^2(\Omega,\rho_2)} \tag{9}$$

where $\|u_i\|_{L^2(\Omega,\rho_i)} = \big(\int_\Omega u_i^2\rho_i\big)^{\frac{1}{2}}$.
The anisotropic weighted Sobolev space $W^{1,2}(\Omega,\rho_1,\ \rho_2)$ can be defined as the set of those functions having distributional derivative for which the norm in (9) is bounded. Consequently we can also define the anisotropic weighted Sobolev space $H^{1,2}(\Omega,\rho_1,\ \rho_2)$ as the closure of $C^\infty(\Omega)\cap W^{1,2}(\Omega,\rho_1,\ \rho_2)$ in $W^{1,2}(\Omega,\rho_1,\ \rho_2)$ w.r.t. the norm in (9). Analogously $H_0^{1,2}(\Omega,\rho_1,\ \rho_2)$ as the closure of $C_c^\infty(\Omega)\cap W^{1,2}(\Omega,\rho_1,\ \rho_2)$ in $W^{1,2}(\Omega,\rho_1,\ \rho_2)$ w.r.t. the norm in (9). It follows that $H \subseteq W$. In the isotropic standard case $\rho_1 = \rho_2 = 1$, by the paper of Meyers and Serrin [23] it is known that in any domain actually $H = W$.
There is a large literature dealing with the isotropic weighted case $\rho_1 = \rho_2 = \rho$. And generally sufficient conditions which guarantees that W is a Banach space and $H = W$ are studied. Generally we may resume that summability of the weight ρ and summability of the inverse of the weight $\frac{1}{\rho}$ are requested. For example, see [4], if the weight is bounded and $\frac{1}{\rho}$ is in L^1 it follows that actually $W^{1,2}(\Omega,\rho)$ is a Banach space and $H^{1,2}(\Omega,\rho) = W^{1,2}(\Omega,\rho)$. This has been used for example in [4] in the study of the linearized equation corresponding to $-\Delta_p u = f(u)$ $(p > 2)$ with positive nonlinearity f. In this case the weight which naturally is associated to the problem is $\rho = |\nabla u|^{p-2}$ and summability of $\frac{1}{\rho}$ is proved in the case of positive nonlinearities. We guess one can try to extend this theory also to the weighted anisotropic case $\rho_1 \neq \rho_2$. It is not our intent since in our applications summability of the inverse of the weights are not in general expected since we do not assume f to be positive.
In our context the weights that are naturally associated to the problem are

$$\rho_1 = |u_1|^{p_1-2} \qquad \text{and} \qquad \rho_2 = |u_2|^{p_2-2}.$$

We therefore consider the space $H_0^{1,2}(\Omega,\rho_1,\ \rho_2)$ defined as above. In the proof of our main result this is sufficient.

Remark 2.2. With the notations above it is now clear that, if the solution u is *stable* according to Definition (2.1), it follows also by density arguments that:

$$\int_{\mathbb{R}^2}(p_1-1)|u_1|^{p_1-2}\varphi_1^2 + (p_2-1)|u_2|^{p_2-2}\varphi_2^2 - f'(u)\varphi^2 dx \geq 0 \tag{10}$$

for every $\varphi \in H_0^{1,2}(\mathbb{R}^2,\rho_1,\ \rho_2)$.

Let us now describe a linearization argument, that will be useful in the sequel. Given $\varphi \in C_c^\infty(\mathbb{R}^2)$, let us put $v = \varphi_1$ and $v = \varphi_2$ in (3). By the fact that

$$|u_1|^{p_1-2}u_1 \in W^{1,2}_{loc} \qquad \text{and} \qquad |u_2|^{p_2-2}u_2 \in W^{1,2}_{loc}$$

we can integrate by parts either with respect to x_1 or with respect to x_2 and we get the *linearized equations*:

$$\int_{\mathbb{R}^2} \left\{(p_1-1)|u_1|^{p_1-2}u_{11}\varphi_1 + (p_2-1)|u_2|^{p_2-2}u_{22}\varphi_1\right\} dx = \int_{\mathbb{R}^2} f'(u)u_1\varphi \, dx \quad (11)$$

$$\int_{\mathbb{R}^2} \left\{(p_1-1)|u_1|^{p_1-2}u_{11}\varphi_1 + (p_2-1)|u_2|^{p_2-2}u_{12}\varphi_2\right\} dx = \int_{\mathbb{R}^2} f'(u)u_1\varphi \, dx \quad (12)$$

$$\int_{\mathbb{R}^2} \left\{(p_1-1)|u_1|^{p_1-2}u_{11}\varphi_2 + (p_2-1)|u_2|^{p_2-2}u_{22}\varphi_2\right\} dx = \int_{\mathbb{R}^2} f'(u)u_2\varphi \, dx \quad (13)$$

$$\int_{\mathbb{R}^2} \left\{(p_1-1)|u_1|^{p_1-2}u_{12}\varphi_1 + (p_2-1)|u_2|^{p_2-2}u_{22}\varphi_1\right\} dx = \int_{\mathbb{R}^2} f'(u)u_2\varphi \, dx. \quad (14)$$

Lemma 2.3. *Let $u \in C^1(\mathbb{R}^2) \cap C^2(\mathbb{R}^2 \setminus \mathcal{Z}_u)$ be a solution to* (3) *and assume that $|u_i|^{p_i-2}u_i \in W^{1,2}_{loc}$ for $i = 1, 2$ and that $|\nabla u|$ is bounded.*
If $u_1 > 0$ in $\mathbb{R}^2$ or $u_2 > 0$ in $\mathbb{R}^2$, then it follows that u is stable.

Proof. Let us assume for example that $u_1 > 0$ in $\mathbb{R}^2$.
Given $\psi \in C^\infty_c(\mathbb{R}^2)$, since u_1 is strictly positive, we can take $\varphi = \frac{\psi^2}{u_1}$ as test function in equation (12) and we get:

$$\begin{aligned} &\int_{\mathbb{R}^2} (p_1-1)\frac{|u_1|^{p_1-2}}{|u_1|^2} u_{11}(2\psi\psi_1 u_1 - \psi^2 u_{11}) + \\ &+ \int_{\mathbb{R}^2} (p_2-1)\frac{|u_2|^{p_2-2}}{|u_1|^2} u_{12}(2\psi\psi_2 u_1 - \psi^2 u_{12}) - \int_{\mathbb{R}^2} f'(u)\psi^2 = 0. \end{aligned} \quad (15)$$

Noting that $2\psi\psi_1 u_1 u_{11} - \psi^2 u_{11}^2 \le \psi_1^2 u_1^2$ and $2\psi\psi_2 u_1 u_{12} - \psi^2 u_{12}^2 \le \psi_2^2 u_1^2$, we immediately get the thesis. □

Proposition 2.4. *Let u be a solution of* (3) *with $|\nabla u|$ bounded and $u_1(x) > 0$ in $\mathbb{R}^2$ or $u_2(x) > 0$ in $\mathbb{R}^2$.*
Then u is stable *and there exists $M > 0$ such that the following inequality holds:*

$$\int_{\mathbb{R}^2} \left\{|u_1|^{p_1-2}\left[|\nabla u_1|^2 - (|\nabla u|_1)^2\right] + |u_2|^{p_2-2}\left[|\nabla u_2|^2 - (|\nabla u|_2)^2\right]\right\}\psi^2 dx \le M\int_{\mathbb{R}^2} |\nabla\psi|^2 dx \quad (16)$$

for every $\psi \in C^\infty_c(\mathbb{R}^2)$.

Proof. Assume for example that $u_2(x) > 0$ in $\mathbb{R}^2$. The case $u_1(x) > 0$ in $\mathbb{R}^2$ is analogous. Let us first note that by Lemma 2.3 it follows that the solution is *stable* (accordingly to (8) and (10)), because of the assumption $u_2(x) > 0$. Consider now the real Lipschitz continuous function $G_\varepsilon(t) = (2t - 2\varepsilon)\chi_{[\varepsilon\,,\,2\varepsilon]}(t) + t\chi_{[2\varepsilon\,,\,\infty)}(t)$ for $t \geqslant 0$, while $G_\varepsilon(t) = -G_\varepsilon(-t)$ for $t \leqslant 0$ ($\chi_{[a,b]}(\cdot)$ denoting the characteristic function of a set).
We will use the stability condition (8) (see (10)) taking $\varphi = \psi\sqrt{(G_\varepsilon(u_1))^2 + u_2^2}$, where $\psi \in C^\infty_c(\mathbb{R}^2)$.

We claim that φ can be plugged into (8) (see (10)). More precisely, it follows that $\varphi \in W_0^{1,2}(\mathbb{R}^2) \subseteq H_0^{1,2}(\mathbb{R}^2\,,\,\rho_1\,,\,\rho_2)$. In fact, in order to prove this, note that we are assuming $u_2 > 0$. Consequently φ is smooth in the set $|u_1| \geqslant \varepsilon$ by the assumption $u \in C^1(\mathbb{R}^2) \cap C^2(\mathbb{R}^2 \setminus \mathcal{Z}_u)$. Also by construction $\varphi = \psi|u_2|$ in the set $|u_1| \leqslant \varepsilon$. Note now that $|u_2|^{p_2-2}u_2 \in W^{1,2}_{loc}$ which gives $u_2 \in W^{1,2}_{loc}$ since we already assumed $u_2 > 0$. Consequently we have $\varphi \in W_0^{1,2}(\mathbb{R}^2)$. Finally, since we have $p_1 \geqslant 2$ and $p_2 \geqslant 2$, it follows $W_0^{1,2}(\mathbb{R}^2) \subseteq H_0^{1,2}(\mathbb{R}^2\,,\,\rho_1\,,\,\rho_2)$ and therefore $\varphi \in H_0^{1,2}(\mathbb{R}^2\,,\,\rho_1\,,\,\rho_2)$ can be used as test function in (10) by Remark 2.2.
Since

$$\frac{\partial\,\varphi}{\partial x_1} = \psi \cdot \frac{G_\varepsilon(u_1)G'_\varepsilon(u_1)u_{1\,1} + u_2u_{1\,2}}{\sqrt{(G_\varepsilon(u_1))^2 + u_2^2}} + \psi_1\sqrt{(G_\varepsilon(u_1))^2 + u_2^2}$$

and

$$\frac{\partial\,\varphi}{\partial x_2} = \psi \cdot \frac{G_\varepsilon(u_1)G'_\varepsilon(u_1)u_{1\,2} + u_2u_{2\,2}}{\sqrt{(G_\varepsilon(u_1))^2 + u_2^2}} + \psi_2\sqrt{(G_\varepsilon(u_1))^2 + u_2^2}$$

by the stability condition (10) it follows

$$\begin{aligned}
&\int_{\mathbb{R}^2}(p_1-1)|u_1|^{p_1-2}\left[\psi \cdot \frac{G_\varepsilon(u_1)G'_\varepsilon(u_1)u_{1\,1} + u_2u_{1\,2}}{\sqrt{(G_\varepsilon(u_1))^2 + u_2^2}} + \psi_1\sqrt{(G_\varepsilon(u_1))^2 + u_2^2}\right]^2 dx\, + \\
+ \;\; &\int_{\mathbb{R}^2}(p_2-1)|u_2|^{p_2-2}\left[\psi \cdot \frac{G_\varepsilon(u_1)G'_\varepsilon(u_1)u_{2\,1} + u_2u_{2\,2}}{\sqrt{(G_\varepsilon(u_1))^2 + u_2^2}} + \psi_2\sqrt{(G_\varepsilon(u_1))^2 + u_2^2}\right]^2 dx \geq \\
\geq \;\; &\int_{\mathbb{R}^2} f'(u)\psi^2((G_\varepsilon(u_1))^2 + u_2^2) \qquad\qquad (17)
\end{aligned}$$

where we also used that $u_{1\,2} = u_{2\,1}$ in the set $|u_1| \geqslant \varepsilon$, since the solution is smooth there by assumption. Considering now the fact that $|G_\varepsilon(t)| \leqslant t$ and $|G'_\varepsilon(t)| \leqslant 2$, and $\psi \in C_c^\infty(\mathbb{R}^2)$, it follows that

$$\left[\psi \cdot \frac{G_\varepsilon(u_1)G'_\varepsilon(u_1)u_{1\,1} + u_2u_{1\,2}}{\sqrt{(G_\varepsilon(u_1))^2 + u_2^2}} + \psi_1\sqrt{(G_\varepsilon(u_1))^2 + u_2^2}\right]^2 \leqslant \mathrm{const}(|u_{1\,1}|^2 + |u_{1\,2}|^2 + 1)$$

and

$$\left[\psi \cdot \frac{G_\varepsilon(u_1)G'_\varepsilon(u_1)u_{2\,1} + u_2u_{2\,2}}{\sqrt{(G_\varepsilon(u_1))^2 + u_2^2}} + \psi_2\sqrt{(G_\varepsilon(u_1))^2 + u_2^2}\right]^2 \leqslant \mathrm{const}(|u_{2\,1}|^2 + |u_{2\,2}|^2 + 1)$$

This, via the regularity assumption on u, allows us to exploit the dominated convergence theorem and pass to the limit in (17). Therefore, letting $\varepsilon \to 0$, observing that $G_\varepsilon(t)$ converges to t and $G'_\varepsilon(t)$ converges to 1, we get:

$$\begin{aligned}
&\int_{\mathbb{R}^2} (p_1-1)|u_1|^{p_1-2}\left[\psi_1|\nabla u| + \psi\frac{\nabla u\cdot\nabla u_1}{|\nabla u|}\right]^2 dx + \\
+ &\int_{\mathbb{R}^2} (p_2-1)|u_2|^{p_2-2}\left[\psi_2|\nabla u| + \psi\frac{\nabla u\cdot\nabla u_2}{|\nabla u|}\right]^2 dx \\
\geq &\int_{\mathbb{R}^2} f'(u)\psi^2|\nabla u|^2
\end{aligned}$$

and hence:

$$\begin{aligned}
&(p_1-1)\int_{\mathbb{R}^2}\left\{|u_1|^{p_1-2}\psi_1^2|\nabla u|^2 + \frac{|u_1|^{p_1-2}}{|\nabla u|^2}\psi^2(\nabla u\cdot\nabla u_1)^2 + 2|u_1|^{p_1-2}\psi\psi_1(\nabla u\cdot\nabla u_1)\right\}dx + \\
&(p_2-1)\int_{\mathbb{R}^2}\left\{|u_2|^{p_2-2}\psi_2^2|\nabla u|^2 + \frac{|u_2|^{p_2-2}}{|\nabla u|^2}\psi^2(\nabla u\cdot\nabla u_2)^2 + 2|u_2|^{p_2-2}\psi\psi_2(\nabla u\cdot\nabla u_2)\right\}dx \geq \\
&\geq \int_{\mathbb{R}^2} f'(u)\psi^2|\nabla u|^2 dx\ . \qquad (18)
\end{aligned}$$

For $\psi \in C_c^\infty(\mathbb{R}^2)$ we choose $\varphi = G_\varepsilon(u_1)\psi^2$ in equation (12) and we get:

$$\begin{aligned}
\int_{\mathbb{R}^2} f'(u)\psi^2 u_1^2 \;=\; &(p_1-1)\int_{\mathbb{R}^2}\left\{|u_1|^{p_1-2}u_{11}^2 G'_\varepsilon(u_1)\psi^2 + 2|u_1|^{p_1-2}u_{11}G_\varepsilon(u_1)\psi\psi_1\right\}dx + \\
+\; &(p_2-1)\int_{\mathbb{R}^2}\left\{|u_2|^{p_2-2}u_{12}^2 G'_\varepsilon(u_1)\psi^2 + 2|u_2|^{p_2-2}u_{12}G_\varepsilon(u_1)\psi\psi_2\right\}dx\ .
\end{aligned}$$

Passing to the limit for $\varepsilon \to 0$ as above and exploiting the dominated convergence theorem, it follows:

$$\begin{aligned}
\int_{\mathbb{R}^2} f'(u)\psi^2 u_1^2 = (p_1-1)&\int_{\mathbb{R}^2}\left\{|u_1|^{p_1-2}u_{11}^2\psi^2 + 2|u_1|^{p_1-2}u_{11}u_1\psi\psi_1\right\}dx + \\
+ (p_2-1)&\int_{\mathbb{R}^2}\left\{|u_2|^{p_2-2}u_{12}^2\psi^2 + 2|u_2|^{p_2-2}u_{12}u_1\psi\psi_2\right\}dx\ .
\end{aligned} \qquad (19)$$

For $\psi \in C_c^\infty(\mathbb{R}^2)$ we choose now $\varphi = u_2\psi^2$ in[1] equation (14) and we get:

$$\begin{aligned}
\int_{\mathbb{R}^2} f'(u)\psi^2 u_2^2 = (p_1-1)&\int_{\mathbb{R}^2}\left\{|u_1|^{p_1-2}u_{12}^2\psi^2 + 2|u_1|^{p_1-2}u_{12}u_2\psi\psi_1\right\}dx + \\
+ (p_2-1)&\int_{\mathbb{R}^2}\left\{|u_2|^{p_2-2}u_{22}^2\psi^2 + 2|u_2|^{p_2-2}u_{22}u_2\psi\psi_2\right\}dx\ ,
\end{aligned} \qquad (20)$$

[1]Note that here it is not necessary to consider the smoothing given by G_ε since $u_2 > 0$ by assumption.

where we also use that $|u_1|^{p_1-2}u_{12}^2 = |u_1|^{p_1-2}u_{21}^2$, since u is smooth outside the set $u_1 = 0$, while $|u_1|^{p_1-2}u_{12}^2 = 0 = |u_1|^{p_1-2}u_{21}^2$ in the set $u_1 = 0$.
Recalling that for $i = 1, 2$ $|\nabla u|_i = \frac{\nabla u \cdot \nabla u_i}{|\nabla u|}$, by (18), (19), (20) it follows:

$$
\begin{aligned}
& (p_1-1)\int_{\mathbb{R}^2}\left\{|u_1|^{p_1-2}\psi_1^2|\nabla u|^2 + |u_1|^{p_1-2}\psi^2(|\nabla u|_1)^2 + 2|u_1|^{p_1-2}\psi\psi_1|\nabla u|_1|\nabla u|\right\}dx + \\
+ \quad & (p_2-1)\int_{\mathbb{R}^2}\left\{|u_2|^{p_2-2}\psi_2^2|\nabla u|^2 + |u_2|^{p_2-2}\psi^2(|\nabla u|_2)^2 + 2|u_2|^{p_2-2}\psi\psi_2|\nabla u|_2|\nabla u|\right\}dx \geq \\
\geq \quad & \int_{\mathbb{R}^2} f'(u)\psi^2|\nabla u|^2 dx = \qquad (21) \\
= \quad & (p_1-1)\int_{\mathbb{R}^2}\left\{|u_1|^{p_1-2}u_{11}^2\psi^2 + 2|u_1|^{p_1-2}u_{11}u_1\psi\psi_1 + |u_1|^{p_1-2}u_{12}^2\psi^2 + 2|u_1|^{p_1-2}u_{12}u_2\psi\psi_1\right\}dx + \\
+ \quad & (p_2-1)\int_{\mathbb{R}^2}\left\{|u_2|^{p_2-2}u_{12}^2\psi^2 + 2|u_2|^{p_2-2}u_{12}u_1\psi\psi_2 + |u_2|^{p_2-2}u_{22}^2\psi^2 + 2|u_2|^{p_2-2}u_{22}u_2\psi\psi_2\right\}dx\ .
\end{aligned}
$$

After simplifications inequality (21) becomes:

$$
\begin{aligned}
& \int_{\mathbb{R}^2}\left\{(p_1-1)|u_1|^{p_1-2}\left[|\nabla u_1|^2 - (|\nabla u|_1)^2\right] + (p_2-2)|u_2|^{p_2-2}\left[|\nabla u_2|^2 - (|\nabla u|_2)^2\right]\right\}\psi^2 dx \leq \\
\leq \quad & \int_{\mathbb{R}^2}\left\{(p_1-1)|u_1|^{p_1-2}|\nabla u|^2\psi_1^2 + (p_2-1)|u_2|^{p_2-2}|\nabla u|^2\psi_2^2\right\}dx\ , \qquad (22)
\end{aligned}
$$

which gives:

$$
\begin{aligned}
& \int_{\mathbb{R}^2}\left\{|u_1|^{p_1-2}\left[|\nabla u_1|^2 - (|\nabla u|_1)^2\right] + |u_2|^{p_2-2}\left[|\nabla u_2|^2 - (|\nabla u|_2)^2\right]\right\}\psi^2 dx \leq \\
\leq \quad & \frac{\max\{(p_1-1)\,,\,(p_2-2)\}}{\min\{(p_1-1)\,,\,(p_2-2)\}}\|\nabla u\|_\infty^{\max\{p_1\,,\,p_2\}}\int_{\mathbb{R}^2}|\nabla\psi|^2 dx \qquad (23)
\end{aligned}
$$

and the thesis follows with $M = M(u, p_1, p_2) = \frac{\max\{(p_1-1)\,,\,(p_2-2)\}}{\min\{(p_1-1)\,,\,(p_2-2)\}}\|\nabla u\|_\infty^{\max\{p_1\,,\,p_2\}}$. □

3 Proof of Theorem 1.2

Let $u \in C^1(\mathbb{R}^2) \cap C^2(\mathbb{R}^2 \setminus \mathcal{Z}_u)$, such that $|u_i|^{p_i-2}u_i \in W^{1,2}_{loc}(\mathbb{R}^2)$, be a weak solution of (3). Assume that $|\nabla u|$ is bounded and:

$$u_2(x) > 0 \text{ for every } x \in \mathbb{R}^2. \qquad (24)$$

It follows by Lemma 2.3 that u is *stable* and by Proposition 2.4 we get that there exists $M > 0$ such that:

$$\int_{\mathbb{R}^2}\left\{|u_1|^{p_1-2}\left[|\nabla u_1|^2 - (|\nabla u|_1)^2\right] + |u_2|^{p_2-2}\left[|\nabla u_2|^2 - (|\nabla u|_2)^2\right]\right\}\psi^2 dx \leq M\int_{\mathbb{R}^2}|\nabla\psi|^2 dx \qquad (25)$$

for every $\psi \in C_c^\infty(\mathbb{R}^2)$. Following [12] we now choose

$$\psi = \psi_R := \max\{0\,,\,\min\{1\,,\,\frac{\ln(R^2/|x|)}{\ln R}\}\}$$

and letting $R \to \infty$ it occurs that $M\int_{\mathbb{R}^2} |\nabla\psi_R|^2 dx$ goes to zero and consequently

$$|u_1|^{p_1-2}\left[|\nabla u_1|^2 - (|\nabla u|_1)^2\right] + |u_2|^{p_2-2}\left[|\nabla u_2|^2 - (|\nabla u|_2)^2\right] = 0\ . \tag{26}$$

Note now that the quantities $\alpha_i := |\nabla u_i|^2 - (|\nabla u|_i)^2 (i = 1, 2)$, are nonnegative. In fact explicit calculation shows that $\alpha_1 = \frac{(u_{11}u_2 - u_{12}u_1)^2}{u_1^2+u_2^2}$ and $\alpha_2 = \frac{(u_{22}u_1-u_{12}u_2)^2}{u_1^2+u_2^2}$.
Equation (26) consequently implies that

$$\sum_{i=1}^{2} |\nabla u_i|^2 - |\nabla|\nabla u||^2 = 0\ , \qquad \text{outside} \quad \{u_1 = 0\}. \tag{27}$$

For $u, g \in C^1(\mathbb{R}^n)$, we now set $\mathcal{L}_{u,x} := \{y \in \mathbb{R}^n :\ u(y) = u(x)\}$ and we denote by $\nabla_{\tau(u,x)}g$ the tangential gradient of g along $\mathcal{L}_{u,x}$, that means:

$$\nabla_{\tau(u,x)}g = \nabla g - \nabla g \cdot \frac{\nabla u}{|\nabla u|}\ . \tag{28}$$

For $i = 1, \ldots, n-1$, $\kappa^i_{u,x}$ denotes the i-th principal curvature of $\mathcal{L}_{u,x}$ at point x. $\kappa_{u,x}(y)$ denotes the mean curvature of $\mathcal{L}_{u,x}$ at y. We recall that for $n = 2$ there is one only principal curvature, which therefore coincides with the mean curvature.
By formula (2.1) in [31] we have:

$$\sum_{i=1}^{2} |\nabla u_i|^2 - |\nabla|\nabla u||^2 = |\nabla u|^2\kappa^2_{u,x} + |\nabla_{\tau(u,x)}|\nabla u||^2\ . \tag{29}$$

By (27) and (29) it follows that

$$\kappa_{u,x} = 0 \quad \text{and} \quad \nabla_{\tau(u,x)}|\nabla u| = 0 \tag{30}$$

along $\mathcal{L}_{u,x}$.
Using (30) and arguing as in Section 2.4 of [12], it follows that the level sets are flat and the thesis. For the readers convenience we recall some details.
Note that, if $u_1(x) = 0$ for every $x \in \mathbb{R}^2$, then the thesis trivially follows and $\mathcal{L}_{u,x}$ are lines parallel to the x_1-axis. If u_1 is not identically equal to zero, let $\overline{x} \in \mathbb{R}^2$ be such that $u_1(\overline{x}) \neq 0$ and set $L := \mathcal{L}_{u,\overline{x}}$ and $\tilde{L} := L \cap \{x \in \mathbb{R}^2 :\ u_1 \neq 0\}$. Since u is continuous and strictly monotone increasing w.r.t. the x_2-direction, it follows that L is a graph. Arguing as in Lemma 2.7 in [12], we also infer that $|\nabla u|$ is constant on every connected component of $\tilde{L}$.

By (30) we know that the curvature of L is zero at $\overline{x}$ and hence L is flat near $\overline{x}$, that means that there exist $v_0, v \in \mathbb{R}^2$ such that $\gamma(t) = v_0 + tv$ is a local parametrization of L for $t \in I \subseteq \mathbb{R}$ and for some interval $I = (a, b)$. We show that I must be equal to $\mathbb{R}$ and hence the whole level set is a line. Let us fix a and set

$$\mathcal{B}_a = \{b \,|\, b > a \,\text{and}\, \{\gamma(t)\,,\, t \in (a,b)\} \subseteq L\}$$

and set $\bar{b} = \sup \mathcal{B}_a$. It follows that $\gamma(\bar{b})$ does not lies in $\mathcal{Z}_u$. In fact, to prove this, let us first note that $u_2(\gamma(\bar{b})) \neq 0$ by assumption. Also $u_1(\gamma(t)) \neq 0$ on $\gamma(t) = v_0 + tv$ with $t \in (a, b)$ since we have that $|\nabla u|$ and $\frac{\nabla u}{|\nabla u|}$ are constant there as remarked above (see Lemma 2.7 in [12]). By the continuity of ∇u it follows now that actually $u_1(\gamma(\bar{b})) \neq 0$. Therefore L is flat near $\gamma(\bar{b})$. This is a contradiction with the definition of $\bar{b}$ and shows that $\bar{b} = \infty$. Analogously we can fix some b and set

$$\mathcal{A}_b = \{a \,|\, a < b \,\text{and}\, \{\gamma(t)\,,\, t \in (a,b)\} \subseteq L\}$$

and $\bar{a} = \inf \mathcal{A}_b$. In the same way it follows that $\bar{a} = -\infty$. Therefore, $\{\gamma(t)\,,\, t \in \mathbb{R}\} \subseteq L$. And this shows that actually L is flat with $\{\gamma(t)\,,\, t \in \mathbb{R}\} = L$, since L is a graph w.r.t. the x_2-direction by the monotonicity of u w.r.t. the x_2-direction. This also shows that every level set is flat, and this follows exactly in the same way as in Lemma 2.11 of [12]. We therefore conclude that every level set of u is flat, and it is standard now to see that this is equivalent to say that there exists $\nu \in S^1$ and $w : \mathbb{R} \to \mathbb{R}$ such that

$$u(x) = w(x \cdot \nu) \quad \forall x \in \mathbb{R}^2 \,. \tag{31}$$

This concludes the proof of Theorem 1.2.

References

[1] Alberti G., Ambrosio L, and Cabré X., *On a long-standing conjecture of E. De Giorgi: symmetry in 3D for general nonlinearities and a local minimality property.* Acta Appl. Math., 65 (1-3), (2001), pp. 9-33, Special issue dedicated to Antonio Avantaggiati on the occasion of his 70th birthday.

[2] Ambrosio L. and Cabré X. *Entire solutions of semilinear elliptic equations in* $\mathbb{R}^3$ *and a conjecture of De Giorgi.* J. Amer. Math. Soc., 13(4), (2000), pp. 725-739

[3] Birindelli I., Valdinoci E., *The Ginzburg-Landau equation in the Heisenberg group*, Commun. Contemp. Math., 10(5), (2008), pp. 671-719

[4] Damascelli L., Sciunzi B., *Regularity, monotonicity and symmetry of positive solutions of m-Laplace equations*, J. Differential Equations, 206(2), (2004), pp. 483-515

[5] De Giorgi E., *Convergence problems for functionals and operators*, Proceedings of the International Meeting on Recent Methods in Nonlinear Analysis (Rome, 1978), Pitagora (1979), pp. 131-188

[6] Del Pino M., Kowalczyk M., Wei J., *A counterexample to a conjecture by De Giorgi in large dimensions*, C. R. Math. Acad. Sci. Paris, 346(23-24), (2008), pp. 12611266.

[7] De Silva D., Savin O., *Symmetry of global solutions to a class of fully nonlinear elliptic equations in* $2D$, Indiana Univ. Math. J., 58(1), (2009), pp. 301-315

[8] Farina A., *Symmetry for solutions of semilinear elliptic equations in* $\mathbb{R}^N$ *and related conjectures*, Ricerche Mat., 48 (suppl.), Papers in memory of Ennio De Giorgi (1999), pp. 129–154

[9] Farina A., *Propriétés qualitatives de solutions d'équations et systèmes d'équations non-linéaires*. 2002. Habilitation à diriger des recherches, Paris VI.

[10] Farina A., *One-dimensional symmetry for solutions of quasilinear equations in* $\mathbb{R}^2$. Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8), 6(3) (2003), pp. 685-692

[11] Farina A., *Liouville-type theorems for elliptic problems. Handbook of differential equations: stationary partial differential equations. vol. IV*, (2003), pp. 61-116 Elsevier North-Holland, Amsterdam,

[12] Farina A., Sciunzi B., Valdinoci E., *Bernstein and De Giorgi type problems: new results via a geometric approach*, Ann. Sc. Norm. Super. Pisa, Cl. Sci. (5), 7(4), (2008), pp. 741–791.

[13] Farina A., Sciunzi B., Valdinoci E., *On a Poincaré type formula for solutions of singular and degenerate elliptic equations*, Manuscripta Math., 132(3-4), (2010), pp. 335-342.

[14] Farina A., Sire Y., Valdinoci E., *Stable solutions of elliptic equations on riemannian manifolds*, Preprint, 2008.

[15] Farina A., Valdinoci E., *Geometry of quasiminimal phase transitions*, Calc. Var. Partial Differential Equations, 33(1), (2008), pp. 1–35

[16] Farina A., Valdinoci E., *The state of the art for a conjecture of De Giorgi and related problems*, Ser. Adv. Math. Appl. Sci. World Sci. Publ., Hackensack, NJ (2008)

[17] Farina A. and Valdinoci E., *1D symmetry for solutions of semilinear and quasilinear elliptic equations*, Trans. Amer Math. Soc **363** (2) (2011), pp. 579-609

[18] Ferrari F., Valdinoci E., *A geometric inequality in the Heisenberg group and its applications to stable solutions of semilinear problems*, Math. Ann., 343(2), (2009), pp. 351-370

[19] Fragalà I. , Gazzola F., Kawohl B., *Existence and nonexistence results for anisotropic quasilinear equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire 21 (2004) 715-734.

[20] Gilbarg D., Trudinger N. S., *Elliptic partial differential equations of second order*, Springer, Berlin (1997)

[21] Ghoussoub N., Gui C., *On a conjecture of De Giorgi and some related problems*, Math. Ann., 311(3), (1998), pp. 481–491

[22] Le P., Sciunzi B., *Regularity of solutions of degenerate quasilinear elliptic equations*, In preparation.

[23] Meyers N. G., Serrin J. $H = W$, Proc. Nat. Acad. Sci. U.S.A. **51** (1964), pp. 1055-1056

[24] Mihâilescu M., Pucci P., Radulescu V., *Eigenvalue problems for anisotropic quasilinear elliptic equations with variable exponent*, J. Math. Anal. Appl., 340 (1), 2008, pp. 687–698

[25] Rácosník J. , *Some remarks to anisotropic Sobolev Spaces I*, Beitrage Anal. 13 (1979), pp. 55-68

[26] Rácosník J. , *Some remarks to anisotropic Sobolev Spaces II*, Beitrage Anal. 15 (1981), pp. 127-140

[27] Savin O., *Regularity of flat level sets in phase transitions*, Ann. of Math. 169 (1) (2009), pp. 41-78

[28] Sciunzi B., Valdinoci E., *Mean curvature properties for p-Laplace phase transitions*, J. Eur. Math. Soc. (JEMS), 7(3), (2005), pp. 319–359

[29] Valdinoci E, Sciunzi B., Savin O., *Flat level set regularity of p-Laplace phase transitions*, Mem. Amer. Math. Soc. , 182(858), (2006),

[30] Sire Y., Valdinoci E., *Fractional Laplacian phase transitions and boundary reactions: a geometric inequality and a symmetry result*, J. Funct. Anal. **256** (2009), no. 6, pp. 18421864

[31] Sternberg P., Zumbrun K., *Connectivity of phase boundaries in strictly convex domains*, Arch. Rational. Mech. Anal., 141(4), (1998), pp. 375–400

[32] Troisi M., *Teoremi di inclusioni per spazi di Sobolev non isotropi*, Ricerche Mat. **18** (1969), pp. 3–24

[33] Ven'tuan L., *On embedding theorems for spaces of functions with partial derivatives of various degree of summability*, Vestnik Leingrad. Univ. **16** (1961), pp. 23–37