REGULARITY AND COMPARISON PRINCIPLES FOR p-LAPLACE EQUATIONS WITH VANISHING SOURCE TERM

BERARDINO SCIUNZI

Abstract. We prove some sharp estimates on the summability properties of the second derivatives of solutions to the equation $-\Delta_p u = f(x)$, under suitable assumptions on the source term. As an application we deduce some strong comparison principles for the p-Laplacian, in the case of vanishing source terms.

Contents

1. Introduction

In this paper we study the *regularity* of weak solutions to

$$-\Delta_p u = f(x) \tag{1.1}$$

in a domain $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$. The issue of regularity, as we will see, is also related to the validity of the *strong comparison principle*.

A solution u to (1.1) can be defined e.g. assuming that $u \in W^{1,p}(\Omega)$ in the weak distributional meaning. This is also the space where it is natural to prove the existence of the solutions under suitable assumptions. Nevertheless, under our assumptions on the source term f, it follows

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by [8, 29] that $u \in C^{1,\alpha}(\Omega)$ for some $0 < \alpha < 1$, see also [14] for the regularity up to the boundary. On the contrary, solutions to p-Laplace equations generally are not of class $C^2(\Omega)$. We recall in Section 5 two leading examples (Example 5.1 and Example 5.2) for the reader's convenience, that show the existence of solutions that are not in $C^2(\Omega)$. Nevertheless it is possible to show that, in the case $p > 2$ and strictly positive (or negative) source terms, any solution is not of class C^2 at its critical points. We add the details of this fact in Proposition 5.4.

Therefore two crucial issues arises from this fact:

- the study of integrability properties of the second derivatives of the solutions,
- the study of the maximal exponent for the Hölder continuity of the gradients.

One of the first regularity results regarding the second derivatives of the solutions states that, under suitable assumptions on f, we have:

$$|\nabla u|^{p-2}\nabla u \in W^{1,2}_{loc}(\Omega)\,.$$

This result can be found in [15] for the general case $f \in L^s(\Omega)$ with $s > \max\{2, \frac{n}{p}\}$. Let us mention that this is also implicit in [8, 13, 29]. We also refer the readers to [24] for $W^{2,2}$ estimates in the case of singular $(1 < p < 2)$ quasilinear elliptic equations involving singular potentials. In [5, 25] it has been shown that: if $u \in C^1(\Omega)$ is a solution to (1.1), then

i) $u \in W^{2,2}_{loc}(\Omega)$ if $1 < p < 3$.
ii) $u \in W^{2,q}_{loc}(\Omega)$ for any $q < \frac{p-1}{p-2}$ if $p \geq 3$ and if f is strictly bounded away from zero.

Actually in [5] positive solutions to $-\Delta_p u = f(u)$ were considered. Anyway the arguments of [5] apply also in our context providing *i)* and *ii)* that are in any case a consequence of the results in this paper. Note that the degenerate nonlinear nature of the p-laplacian causes that the Calderón-Zygmund theory can not be extended simply to the case of p-Laplace equations. We refer the readers to [17, 18] for a very interesting extension of the Calderón-Zygmund theory to the quasilinear case.
It is worth emphasizing that, as it follows observing the 1-D solution in Example 5.2, the exponent q in *ii)* here above is optimal. On the contrary the *radial* solutions in Example 5.1 are more regular. This phenomenon (see Remark 5.3) highlights the important role played by the critical set $\mathcal{Z}_u$ of the solution u:

$$\mathcal{Z}_u := \{\nabla u = 0\}\,. \tag{1.2}$$

The critical set $\mathcal{Z}_u$ is in fact the set of points where the p-Laplace operator is singular $(1 < p < 2)$ or degenerate $(p > 2)$. Away from the critical set standard regularity theory applies. It is an open problem to understand if the positivity assumption on the source term in *ii*) can be removed. We will show here that in general it is not necessary.

One of the purposes of this paper is to point out the strong relation between the study of the integrability properties of the second derivatives of the solutions and the study of the maximal exponent for the Hölder continuity of the gradient, starting from a recent important result of Eduardo V. Teixeira [27] which is based on previous estimates obtained in [11]. Namely, exploiting the results in [11] and also some techniques from [1], it has been shown in [27] that, if $u \in W^{1,p}(\Omega)$ is a solution to (1.1) and

$$f \in L^s(\Omega) \qquad \text{with } s > n\,, \tag{1.3}$$

then

$$u \in C^{1,\min\{\alpha_M^-, \frac{s-n}{(p-1)s}\}}\,, \tag{1.4}$$

that is $u \in C^{1,\alpha}$ for any $\alpha \in (0, \alpha_M) \cap (0, \frac{s-n}{(p-1)s}]$ where α_M is the maximal exponent for the $C^{1,\alpha}$ regularity of p-harmonic functions. We also refer the interested readers to [9, 11, 19, 20, 28]. Let us point out that (1.3) is in particular implied by stronger assumptions that we will consider in our results. It is in any case important to state explicitly (1.3) since the parameter s will appear in our statements and in some cases it could be different by the one obtained by embedding theorems. Let us also mention that in [27] more general problems are considered including in particular operators with variable coefficients.
Improving the technique in [5] and exploiting (1.4), we get some weighted estimates that in particular show that the positivity of f is not necessary to get *ii*). Let us set

$$0 < \mu_* := \min\{\alpha_M^-, \frac{s-n}{(p-1)s}\}(p-1) \le 1\,, \tag{1.5}$$

with $s > n$ as in (1.3). Also set

$$0 < \mu_*^\infty := \min\{\alpha_M^-(p-1), 1\} \le 1\,, \tag{1.6}$$

that is the limiting case for $s \to \infty$.

We have the following:

Theorem 1.1. *Let $u \in W^{1,p}(\Omega)$ be a solution to* (1.1)*. Assume that* (1.3) *holds and $f \in W^{1,n}(\Omega)$. Consider a critical point $x_0 \in \mathcal{Z}_u$ with $B_{2\rho}(x_0) \subset \Omega$. Then:*

if

$$1 < p < 3$$

we have that

$$\int_{B_\rho(x_0)} \frac{\|D^2u\|^2}{|x-x_0|^\eta} \leqslant \mathcal{C} \tag{1.7}$$

with $\mathcal{C} = \mathcal{C}(p\,,\, n\,,\, f\,,\, u\,,\, x_0)$ *and for any* η *such that*

$$\eta < \eta_* \,:=\, n + \big(2\, min\{\alpha_M^-, \frac{s-n}{(p-1)s}\} - 2\big)\,,$$

with $s > n$ *given by* (1.3)*. In particular* $u \in W^{2,2}_{loc}(\Omega)$*. If else*

$$p \geq 3$$

and f *satisfies* (I_{μ_*}) *(see* (1.12) *and* (1.13)*), then, for any* $q < \frac{p-1}{p-2}$*, we have:*

$$\int_{B_\rho(x_0)} \frac{\|D^2u\|^q}{|x-x_0|^\tau} \leqslant \mathcal{C}\,, \tag{1.8}$$

for any

$$\tau < \tau_* \,:=\, n - 2 + \frac{q}{2}\mu_*\,, \tag{1.9}$$

with μ_* *defined in* (1.5)*.*

Theorem 1.1 is a consequence of Corollary 2.2 and Corollary 3.2 that improves the results in [5]. In fact it is not required the strict positivity of f as in [5] and the exponents to the singular weights in (1.7) and (1.8) are optimal, as it can be deduced from the examples in Section 5. To guess this, the reader should evaluate the integral in (1.8) in the case of the solution of Example 5.2.
As an application it is interesting to state Theorem 1.1 for the case of Hénon-type equations:

$$-\Delta_p u \,=\, |x|^\sigma g(u)\,, \qquad\qquad \sigma \geq 0\,. \tag{1.10}$$

We have the following:

Theorem 1.2. *Let* $u \in C^1(\overline{\Omega})$ *be a solution to* (1.10) *with* $0 \in \Omega$ *and* $0 \leq \sigma < \mu_*^\infty$*. Let* $x_0 \in \mathcal{Z}_u$ *with* $B_{2\rho}(x_0) \subset \Omega$*. Then, if* $1 < p < 3$ *we have that* (1.7) *holds for* u *for any* η *such that*

$$\eta < \eta_*^\infty \,:=\, n + \big(2\, min\{\alpha_M^-, \frac{1}{(p-1)}\} - 2\big)\,.$$

In particular $u \in W^{2,2}_{loc}(\Omega)$*. If else* $p \geq 3$ *then, for any* $q < \frac{p-1}{p-2}$*, we have that* (1.8) *holds true for* u *for any* τ *such that:*

$$\tau < \tau_*^\infty \,:=\, n - 2 + \frac{q}{2}\mu_*^\infty\,, \tag{1.11}$$

with μ_*^∞ *defined in* (1.5)*.*

Theorem 1.2 follows by applying Theorem 1.1 with $f(x) := |x|^\sigma g(u(x)) \in L^\infty(\Omega)$.
The regularity theory developed in Section 2 is very much related to study of the summability properties of $|\nabla u|^{-1}$. This is a consequence of regularity estimates at first and later it allows to improve the regularity results, as it is the case of Corollary 3.2.
The summability of $|\nabla u|^{-1}$ near the critical set $\mathcal{Z}_u$ is in some sense a measure of the degeneracy of our equation at its critical points. This is an information that allows to get weighted Sobolev inequalities and that consequently turns out to be crucial in many applications. This is the case in particular when dealing with the *strong comparison principle* for p-Laplacian.

Consider a differential operator

$$\mathcal{L} : \mathcal{D} \to \mathbb{R} \qquad \text{(to be understood in the weak sense if necessary)},$$

defined on its domain $\mathcal{D} = \mathcal{D}(\Omega)$ (a function space over some domain Ω). We say that the *strong comparison principle* holds for $\mathcal{L}$ in Ω if, for any (connected) subdomain $\Omega' \subseteq \Omega$ and any $u, v \in \mathcal{D}$ such that

$$\mathcal{L}u \leq \mathcal{L}v,$$

then the assumption $u \leq v$ in Ω' implies the alternative $u < v$ or $u \equiv v$ in Ω'.

The semilinear case is well understood and we refer the readers to [10]. Note that, for example, in the case $\mathcal{L}u := \Delta u + g(x, u)$ with g locally Lipschitz continuous w.r.t. the second variable, *the strong comparison principle* reduces to the *maximum principle.* It is crucial here the fact that the Laplace operator is linear. Nevertheless, also in the quasilinear case, the situation is well understood far away from the critical set $\mathcal{Z}_u$. Namely, in the case $\mathcal{L}u := \Delta_p u + g(x, u)$ and $p > 1$ with g locally Lipschitz continuous w.r.t. the second variable, *the strong comparison principle* holds in any connected component of $\Omega \setminus \mathcal{Z}_u$ (see [3, 23]).

Near the critical set, the nonlinear nature of the p-Laplacian is in addition to the degeneracy of the operator and very few is known, even in the p-harmonic case $\mathcal{L}u := \Delta_p u$ and assuming furthermore that both u and v are p-harmonic. Let us mention [12] for some results in the p-harmonic case in dimension two.
In the general case $\mathcal{L}u := \Delta_p u + g(x, u)$ it has been proved in [6] that *the strong comparison principle* holds (also over the critical set) provided that u or v are a solution of the equation, $\frac{2n+2}{n+2} < p < \infty$ and the right hand side $g(x, u)$ is *strictly positive.*

It is a common feeling that, at least in the case $p > 2$, the strong comparison principle could fail if the source term change sign. However, as we will see here, the strict positivity of the source term is not necessary. Namely we will provide some conditions on the (possibly vanishing) source term under which the strong comparison principle holds. We will in particular assume that f satisfies the condition (I_{μ_*}) here below. Namely, for μ_* defined as in (1.5) we state:

Condition (I_{μ_*}): for some $0 < \mu < \mu_*$, we have:

$$f \in W^{1,m}(\Omega) \quad \text{for some} \quad m > \frac{n}{2(1-\mu)} \tag{1.12}$$

and, given any $x_0 \in \Omega$, there exist $C(x_0, \mu) > 0$ and $\rho(x_0, \mu) > 0$ such that, $B_{\rho(x_0,\mu)}(x_0) \subset \Omega$ and

$$|f(x)| \geq C(x_0, \mu)|x - x_0|^\mu \qquad \text{in} \quad B_{\rho(x_0,\mu)}(x_0)\,. \tag{1.13}$$

We have the following:

Theorem 1.3. *Let $u, v \in C^1(\overline{\Omega})$ such that*

$$-\Delta_p u - f(x) \leq -\Delta_p v - f(x) \qquad \text{in } \Omega\,. \tag{1.14}$$

Assume that u or v is a solution to (1.1) *with $\frac{2n+2}{n+2} < p < \infty$ and assume that f satisfies* (1.3) *and* (I_{μ_*}). *Then, if $u \leq v$ in a* (*connected*) *subdomain $\Omega' \subseteq \Omega$, it follows that*

$$u < v \text{ in } \Omega', \text{ unless } u \equiv v \text{ in } \Omega'.$$

Let us point out that, even if f may be deleted in (1.14), we need in any case that one of the functions we are comparing is a solution to the equation. As an example, we state here below a *strong comparison principle* for Hénon type problems. We have the following:

Theorem 1.4. *Let $u, v \in C^1(\overline{\Omega})$ such that*

$$-\Delta_p u - |x|^\sigma g(u) \leq -\Delta_p v - |x|^\sigma g(v) \qquad \text{in } \Omega\,, \tag{1.15}$$

with $0 \in \Omega$, $0 \leq \sigma < \mu_*^\infty$ (see (1.6)*) and $g(\cdot)$ nonnegative locally Lipschitz continuous with $g(s) > 0$ for $s > 0$. Assume that u or v is a* (*nontrivial*) *non-negative solution to* (1.10) *with $\frac{2n+2}{n+2} < p < \infty$.*

Then, if $u \leq v$ in a (*connected*) *subdomain $\Omega' \subseteq \Omega$, it follows that*

$$u < v \text{ in } \Omega', \text{ unless } u \equiv v \text{ in } \Omega'.$$

Theorem 1.3 and Theorem 1.4 follow by a Harnack-type comparison inequality, see Theorem 4.2, that can be proved exploiting the iterative technique in [30] which goes back to [7, 21, 22] and was first used to prove Hölder continuity properties of solutions of some strictly elliptic linear operators. To apply this technique in our case the key tool is the weighted Sobolev inequality in Theorem 4.1 that can be obtained once the regularity results of Theorem 1.1 are available.

The paper is organized as follows. In Section 2 we prove some weighted regularity estimates for the solutions to (1.1), that we exploit in Section 3 to prove summability properties of $|\nabla u|^{-1}$ avoiding the assumption that f is strictly bounded away from zero needed in [5]. In Section 4 we prove Theorem 1.3 and Theorem 1.4. Finally in Section 5 we collect some examples showing which is the best regularity expected for the solutions to (1.1).

2. Local regularity

Let $u \in W^{1,p}(\Omega)$ be a solutions to (1.1), that is

(2.16)
$$\int_\Omega |\nabla u|^{p-2}(\nabla u\,,\,\nabla\varphi) = \int_\Omega f(x)\varphi \qquad \text{for every} \quad \varphi \in W_0^{1,p}(\Omega)\,.$$

Note that actually $u \in C^{1,\alpha}(\Omega)$ under our assumptions. We will get our regularity results exploiting the linearized operator at u. To define it let us start recalling some known facts about weighted Sobolev spaces. It is natural to distinguish the case $1 < p < 2$ from the case $p \geq 2$.

The case $p \geq 2$.

The weighted Sobolev space (with weight ρ) $W^{1,2}(\Omega,\rho)$ can be defined as the set of those functions having distributional derivative for which the norm:

$$\|v\|_\rho = \|v\|_{L^2(\Omega)} + \|\nabla v\|_{L^2(\Omega,\rho)} \tag{2.17}$$

is bounded. We are interested in particular to the case

$$\rho = |\nabla u|^{p-2}$$

which is the weight that is naturally associated to our problem. Furthermore we can also define the weighted Sobolev space $H^{1,2}(\Omega,\rho)$ as the closure of $C^\infty(\Omega)$ in $W^{1,2}(\Omega,\rho)$ w.r.t. the norm in (2.17). Note that, since $p \geq 2$, then $\rho \in L^\infty(\Omega)$ so that $C^\infty(\Omega) \subset W^{1,2}(\Omega,\rho)$ and furthermore it follows that $H^{1,2}(\Omega,\rho) \subseteq W^{1,2}(\Omega,\rho)$.

A posteriori, according to Theorem 3.1 (see also Theorem 4.1), we have that $\rho^{-1} \in L^1(\Omega)$. This allows to exploit the result of Meyers and Serrin [16], as remarked in [30] and to deduce that in any domain actually $H^{1,2}(\Omega,\rho) = W^{1,2}(\Omega,\rho)$. Note now that $H_0^{1,2}(\Omega,\rho)$ may be defined as the closure of $C_c^\infty(\Omega)$ in $W^{1,2}(\Omega,\rho)$ w.r.t. the norm in (2.17). $H_0^{1,2}(\Omega,\rho)$ also coincides with the completion of $C_c^\infty(\Omega)$ w.r.t. the norm in (2.17). This will be a consequence of the regularity results (namely Theorem 3.1) that we are going to prove. To do this, at the beginning, we have to choose a space where to define the linearized operator. We define (in the case $p \geq 2$) the linearized operator in the space $H^{1,2}(\Omega,\rho)$ and, for any $v \in H^{1,2}(\Omega,\rho)$, we consider the linearized equation $L_u(v,\cdot) = 0$ associated to (1.1) at a given solution u defined by

(2.18)

$$L_u(v,\phi) = \int_\Omega |\nabla u|^{p-2}(\nabla v, \nabla\phi) + (p-2)\int_\Omega |\nabla u|^{p-4}(\nabla u, \nabla v)(\nabla u, \nabla\phi) - \int_\Omega f_i\phi$$

for every $\phi \in H^{1,2}_0(\Omega,\rho)$, where we have posed: $f_i(x) = \frac{\partial f}{\partial x_i}(x)$.
It follows by [5] that, setting $u_i(x) = \frac{\partial u}{\partial x_i}(x)$, then $u_i \in H^{1,2}(\Omega,\rho)$ and

$$L_u(u_i,\phi) = 0 \qquad \forall\phi \in H^{1,2}_0(\Omega,\rho)\,. \tag{2.19}$$

The case $1 < p < 2$.
In this case, a posteriori, according to Theorem 3.1 (see also Theorem 4.1), the above construction can be carried out as well, since we will prove that $\rho \in L^1(\Omega)$. At the beginning, to achieve Theorem 3.1, we start defining the linearized operator in the space

$$\mathcal{A}_u := \left\{ v \in H^1_0(\Omega) \,:\, \|\nabla v\|_{L^2(\Omega,\rho^2)} < \infty \right\}.$$

It follows by [5] that $u_i \in \mathcal{A}_u$ and

$$L_u(u_i,\phi) = 0 \qquad \forall\phi \in H^{1,2}_0(\Omega)\,. \tag{2.20}$$

We have the following:

Theorem 2.1. *Let $u \in W^{1,p}(\Omega)$ be a solution to (1.1), $1 < p < \infty$. Assume that (1.3) holds and let $x_0 \in \mathcal{Z}_u$ with $B_{2\rho}(x_0) \subset \Omega$. Fix $0 \le \beta < 1$, $0 \le \mu < \mu_*$ and $\gamma < n-2$ for $n \ge 3$ while $\gamma = 0$ if $n = 2$. Then, if $f \in W^{1,m}(\Omega)$ for some $m > \frac{n}{2-\mu}$, it follows that:*

$$\int_{B_\rho(x_0)} \frac{|\nabla u|^{p-2-\beta}|u_{i,j}|^2}{|x-x_0|^\mu |x-y|^\gamma} \leqslant \mathcal{C} \qquad \forall\, i,j \in \{1,\ldots,n\}\,, \tag{2.21}$$

with $\mathcal{C} = \mathcal{C}(x_0,\rho,f,n,p,\beta,\gamma,\mu)$ that does not depend on $y \in \Omega$.

Proof. We start recalling that, since we assumed that (1.3) is satisfied, then the results in [27] apply and (1.4) holds.
Let us fix some notations. We consider $G_\varepsilon(t) = (2t-2\varepsilon)\chi_{[\varepsilon\,,\,2\varepsilon]}(t) + t\chi_{[2\varepsilon\,,\,\infty)}(t)$ for $t \geqslant 0$, while $G_\varepsilon(t) = -G_\varepsilon(-t)$ for $t \leqslant 0$ ($\chi_{[a,b]}(\cdot)$ denoting the characteristic function of a set).
Consider a cut-off function φ_ρ, in such a way that $\varphi_\rho \in C^\infty_c(B_{2\rho}(x_0))$, $\varphi_\rho = 1$ in $B_\rho(x_0)$ and $|\nabla\varphi_\rho| \leqslant \frac{2}{\rho}$. For shortness we will write φ instead of φ_ρ.
Also, for $0 \leqslant \beta < 1$, $\gamma < (n-2)$ if $n \ge 3$, $\gamma = 0$ if $n = 2$ and $\mu < \mu_*$ fixed, we set

$$T_\varepsilon(t) = \frac{G_\varepsilon(t)}{|t|^\beta} \qquad \text{and} \qquad H_\delta(t) = \frac{G_\delta(t)}{|t|^{\gamma+1}}\,, \tag{2.22}$$

and consider the test function

$$\phi = T_\varepsilon(u_i) \cdot \frac{H_\delta(|x-y|)}{|x-x_0|^\mu} \cdot \varphi^2 .$$

Note that ϕ can be plugged into (2.19) (see also (2.20)) since it is regularized near the critical set $\mathcal{Z}_u$ by the definition of T_ε (note also that $x_0 \in \mathcal{Z}_u$). Moreover ϕ is regularized near any y because of the definition of H_δ. Finally ϕ is smooth elsewhere by standard regularity theory. Therefore we have

(2.23)

$$\begin{aligned}
&\int_\Omega |\nabla u|^{p-2}|\nabla u_i|^2 \cdot T'_\varepsilon(u_i) \cdot \frac{H_\delta(|x-y|)}{|x-x_0|^\mu} \cdot \varphi^2 \\
&+ (p-2)|\nabla u|^{p-4}(\nabla u\,,\, \nabla u_i)^2 \cdot T'_\varepsilon(u_i) \cdot \frac{H_\delta(|x-y|)}{|x-x_0|^\mu} \cdot \varphi^2 \\
&+ \int_\Omega |\nabla u|^{p-2}(\nabla u_i\,,\, \nabla_x H_\delta(|x-y|)) \cdot T_\varepsilon(u_i) \cdot \frac{\varphi^2}{|x-x_0|^\mu} \\
&+ (p-2)|\nabla u|^{p-4}(\nabla u\,,\, \nabla u_i)(\nabla u\,,\, \nabla_x H_\delta(|x-y|)) \cdot T_\varepsilon(u_i) \cdot \frac{\varphi^2}{|x-x_0|^\mu} \\
&- \mu \int_\Omega |\nabla u|^{p-2}(\nabla u_i\,,\, \nabla|x-x_0|) \cdot T_\varepsilon(u_i) \cdot \frac{H_\delta(|x-y|)}{|x-x_0|^{\mu+1}}\varphi^2 \\
&- \mu(p-2) \int_\Omega |\nabla u|^{p-4}(\nabla u\,,\, \nabla u_i)(\nabla u\,,\, \nabla|x-x_0|) \cdot T_\varepsilon(u_i) \cdot \frac{H_\delta(|x-y|)}{|x-x_0|^{\mu+1}}\varphi^2 \\
&+ 2\int_\Omega |\nabla u|^{p-2}(\nabla u_i\,,\, \nabla\varphi) \cdot T_\varepsilon(u_i) \cdot \frac{H_\delta(|x-y|)}{|x-x_0|^\mu} \cdot \varphi \\
&+ 2(p-2)|\nabla u|^{p-4}(\nabla u\,,\, \nabla u_i)(\nabla u\,,\, \nabla\varphi) \cdot T_\varepsilon(u_i) \cdot \frac{H_\delta(|x-y|)}{|x-x_0|^\mu} \cdot \varphi \\
&= \int_\Omega f_i(x) \cdot T_\varepsilon(u_i) \cdot \frac{H_\delta(|x-y|)}{|x-x_0|^\mu} \cdot \varphi^2 .
\end{aligned}$$

The main idea in the following computations is to estimate all the terms in the previous inequality and then close the estimates via Hölder's inequality. We start observing that

$$\begin{aligned}
&\min\{(p-1)\,,\,1\} \int_\Omega |\nabla u|^{p-2}|\nabla u_i|^2 \cdot T'_\varepsilon(u_i) \cdot \frac{H_\delta(|x-y|)}{|x-x_0|^\mu} \cdot \varphi^2 \leqslant \\
&\leqslant \int_\Omega |\nabla u|^{p-2}|\nabla u_i|^2 \cdot T'_\varepsilon(u_i) \cdot \frac{H_\delta(|x-y|)}{|x-x_0|^\mu} \cdot \varphi^2 \\
&+ (p-2)|\nabla u|^{p-4}(\nabla u\,,\, \nabla u_i)^2 \cdot T'_\varepsilon(u_i) \cdot \frac{H_\delta(|x-y|)}{|x-x_0|^\mu} \cdot \varphi^2
\end{aligned}$$

and get by (2.23) that

$$
\begin{aligned}
&C^{-1}\int_\Omega |\nabla u|^{p-2}|\nabla u_i|^2\cdot T'_\varepsilon(u_i)\cdot\frac{H_\delta(|x-y|)}{|x-x_0|^\mu}\cdot\varphi^2\\
&\leq \int_\Omega |\nabla u|^{p-2}|\nabla u_i|\,|\nabla_x H_\delta(|x-y|)|\cdot|T_\varepsilon(u_i)|\cdot\frac{\varphi^2}{|x-x_0|^\mu}\\
&+\int_\Omega |\nabla u|^{p-2}|\nabla u_i|\;\cdot|T_\varepsilon(u_i)|\cdot\frac{H_\delta(|x-y|)}{|x-x_0|^{\mu+1}}\varphi^2\\
&\int_\Omega |\nabla u|^{p-2}|\nabla u_i|\,|\nabla\varphi|\cdot|T_\varepsilon(u_i)|\cdot\frac{H_\delta(|x-y|)}{|x-x_0|^\mu}\cdot\varphi\\
&+\int_\Omega |f_i(x)|\cdot|T_\varepsilon(u_i)|\cdot\frac{H_\delta(|x-y|)}{|x-x_0|^\mu}\cdot\varphi^2\,.
\end{aligned}
\tag{2.24}
$$

Here and in the following we denote with $C = C(x_0, \rho, f, n, p, \beta, \gamma, \mu)$ a generic constant that we allow to vary each line. For $\varepsilon > 0$ fixed, we exploit the dominated convergence theorem to let $\delta \to 0$ and get

$$
\begin{aligned}
&C^{-1}\int_\Omega \frac{|\nabla u|^{p-2}|\nabla u_i|^2 T'_\varepsilon(u_i)}{|x-x_0|^\mu|x-y|^\gamma}\cdot\varphi^2\\
&\leq \int_\Omega \frac{|\nabla u|^{p-2}|\nabla u_i||T_\varepsilon(u_i)|}{|x-x_0|^\mu|x-y|^{\gamma+1}}\cdot\varphi^2\\
&+\int_\Omega \frac{|\nabla u|^{p-2}|\nabla u_i|\,|T_\varepsilon(u_i)|}{|x-x_0|^{\mu+1}|x-y|^\gamma}\cdot\varphi^2\\
&+\int_\Omega \frac{|\nabla u|^{p-2}|\nabla u_i|\,|T_\varepsilon(u_i)|}{|x-x_0|^\mu|x-y|^\gamma}\cdot|\nabla\varphi|\varphi\\
&+\int_\Omega \frac{|f_i(x)||T_\varepsilon(u_i)|}{|x-x_0|^\mu|x-y|^\gamma}\cdot\varphi^2\,.
\end{aligned}
\tag{2.25}
$$

We use Young's inequality $ab \leqslant \vartheta a^2 + \frac{b^2}{4\vartheta}$ and the fact that $|T_\varepsilon(t)| \leqslant t^{1-\beta}$ and get:

$$
\begin{aligned}
&\int_\Omega \frac{|\nabla u|^{p-2}|\nabla u_i||T_\varepsilon(u_i)|}{|x-x_0|^\mu|x-y|^{\gamma+1}}\cdot\varphi^2\\
&\leqslant \int_\Omega \frac{|\nabla u|^{\frac{p-2}{2}}|\nabla u_i|\chi_{\{|u_i|\geq\varepsilon\}}}{|x-y|^{\frac{\gamma}{2}}|x-x_0|^{\frac{\mu}{2}}|u_i|^{\frac{\beta}{2}}}\varphi\cdot\frac{|\nabla u|^{\frac{p-2}{2}}|u_i|^{\frac{2-\beta}{2}}}{|x-y|^{\frac{\gamma+2}{2}}|x-x_0|^{\frac{\mu}{2}}}\varphi\\
&\leqslant \vartheta\int_\Omega \frac{|\nabla u|^{p-2}|\nabla u_i|^2\chi_{\{|u_i|\geq\varepsilon\}}}{|x-y|^\gamma|x-x_0|^\mu|u_i|^\beta}\varphi^2\\
&+\frac{C}{4\vartheta}\int_\Omega \frac{1}{|x-y|^{\gamma+2}}\left(\frac{|\nabla u-\nabla u(x_0)|}{|x-x_0|^{\frac{\mu}{p-\beta}}}\right)^{p-\beta}\varphi^2\\
&\leqslant \vartheta\int_\Omega \frac{|\nabla u|^{p-2}|\nabla u_i|^2\chi_{\{|u_i|\geq\varepsilon\}}}{|x-y|^\gamma|x-x_0|^\mu|u_i|^\beta}\varphi^2+\frac{C}{4\vartheta}\,,
\end{aligned}
$$

since we assumed that $\gamma < n-2$ and $\mu < \min\{\alpha_M^-, \frac{s-n}{(p-1)s}\}(p-1)$. Exploiting again Young's inequality we get:

$$\begin{aligned}
&\int_\Omega \frac{|\nabla u|^{p-2}|\nabla u_i||T_\varepsilon(u_i)|}{|x-x_0|^{\mu+1}|x-y|^\gamma}\cdot\varphi^2\\
&\leqslant \int_\Omega \frac{|\nabla u|^{\frac{p-2}{2}}|\nabla u_i|\chi_{\{|u_i|\geq\varepsilon\}}}{|x-y|^{\frac{\gamma}{2}}|x-x_0|^{\frac{\mu}{2}}|u_i|^{\frac{\beta}{2}}}\varphi\cdot\frac{|\nabla u|^{\frac{p-2}{2}}|u_i|^{\frac{2-\beta}{2}}}{|x-y|^{\frac{\gamma}{2}}|x-x_0|^{\frac{\mu+2}{2}}}\varphi\\
&\leqslant \vartheta\int_\Omega \frac{|\nabla u|^{p-2}|\nabla u_i|^2\chi_{\{|u_i|\geq\varepsilon\}}}{|x-y|^\gamma|x-x_0|^\mu|u_i|^\beta}\varphi^2\\
&+\frac{C}{4\vartheta}\int_\Omega\frac{1}{|x-y|^\gamma}\frac{|\nabla u-\nabla u(x_0)|^{p-\beta}}{|x-x_0|^{\mu+2}}\varphi^2\\
&\leqslant \vartheta\int_\Omega \frac{|\nabla u|^{p-2}|\nabla u_i|^2\chi_{\{|u_i|\geq\varepsilon\}}}{|x-y|^\gamma|x-x_0|^\mu|u_i|^\beta}\varphi^2\\
&+\frac{C}{\vartheta}\int_\Omega\frac{1}{|x-y|^\gamma}\frac{1}{|x-x_0|^2}\varphi^2\\
&\leqslant \vartheta\int_\Omega \frac{|\nabla u|^{p-2}|\nabla u_i|^2\chi_{\{|u_i|\geq\varepsilon\}}}{|x-y|^\gamma|x-x_0|^\mu|u_i|^\beta}\varphi^2+\frac{C}{4\vartheta}\,.
\end{aligned}$$

since we assumed that $\gamma < n-2$ and $\mu < \min\{\alpha_M^-, \frac{s-n}{(p-1)s}\}(p-1)$. We now use the fact that $|\nabla\varphi| \le \frac{1}{\rho}$ and get:

$$\begin{aligned}
&\int_\Omega \frac{|\nabla u|^{p-2}|\nabla u_i|\,|T_\varepsilon(u_i)|}{|x-x_0|^{\mu}|x-y|^\gamma}\cdot|\nabla\varphi|\varphi\\
&=\frac{2}{\rho}\int_\Omega \frac{|\nabla u|^{\frac{p-2}{2}}|\nabla u_i|\chi_{\{|u_i|\geq\varepsilon\}}}{|x-x_0|^{\frac{\mu}{2}}|x-y|^{\frac{\gamma}{2}}|u_i|^{\frac{\beta}{2}}}\varphi\cdot\frac{|\nabla u|^{\frac{p-2}{2}}|u_i|^{\frac{2-\beta}{2}}}{|x-x_0|^{\frac{\mu}{2}}|x-y|^{\frac{\gamma}{2}}}\\
&\leqslant \vartheta\int_\Omega \frac{|\nabla u|^{p-2}|\nabla u_i|^2\chi_{\{|u_i|\geq\varepsilon\}}}{|x-x_0|^\mu|x-y|^\gamma|u_i|^\beta}\varphi^2\\
&+\frac{C}{\vartheta}\int_{B_{2\rho}(x_0)}\frac{1}{|x-y|^\gamma}\Big(\frac{|\nabla u-\nabla u(x_0)|}{|x-x_0|^{\frac{\mu}{p-\beta}}}\Big)^{p-\beta}\\
&\leqslant \vartheta\int_\Omega \frac{|\nabla u|^{p-2}|\nabla u_i|^2\chi_{\{|u_i|\geq\varepsilon\}}}{|x-y|^\gamma|u_i|^\beta}\varphi^2+\frac{C}{\vartheta}\,,
\end{aligned}$$

since we assumed that $\gamma < n-2$ and $\mu < \min\{\alpha_M^-, \frac{s-n}{(p-1)s}\}(p-1)$. Finally we have:

$$\begin{aligned}
&\int_\Omega \frac{|f_i(x)||T_\varepsilon(u_i)|}{|x-x_0|^\mu|x-y|^\gamma}\varphi^2\\
&\leqslant C\left(\int_\Omega|\nabla f|^m\right)^{\frac{1}{m}}\left(\int_\Omega\Big(\frac{1}{|x-x_0|^\mu|x-y|^\gamma}\Big)^{m'}\right)^{\frac{1}{m'}}\le C
\end{aligned}$$

since $\mu < 1$, $\gamma < n-2$ and $f \in W^{1,m}(\Omega)$ for some $m > \frac{n}{2-\mu}$ by assumption.

Taking into account (2.25), exploiting the above estimates and evaluating T'_ε, we get

$$\int_\Omega \frac{|\nabla u|^{p-2}|\nabla u_i|^2}{|u_i|^\beta |x-x_0|^\mu |x-y|^\gamma}\big(G'_\varepsilon(u_i) - \beta\frac{G_\varepsilon(u_i)}{|u_i|} - C\vartheta\chi_{\{|u_i|\geq\varepsilon\}}\big)\cdot\varphi^2 \leqslant C\,.$$

Since for for ϑ small $(G'_\varepsilon(u_i) - \beta\frac{G_\varepsilon(u_i)}{|u_i|} - C\vartheta\chi_{\{|u_i|\geq\varepsilon\}})$ is strictly positive we conclude that

$$\int_\Omega \frac{|\nabla u|^{p-2-\beta}|\nabla u_i|^2}{|x-x_0|^\mu |x-y|^\gamma}\cdot\varphi^2 \leq \int_\Omega \frac{|\nabla u|^{p-2}|\nabla u_i|^2}{|u_i|^\beta |x-x_0|^\mu |x-y|^\gamma}\cdot\varphi^2 \leqslant C\,,$$

and the thesis follows by the definition of φ. □

Corollary 2.2. *Let $u \in W^{1,p}(\Omega)$ be a solution to* (1.1). *Assume that* (1.3) *holds and let*

$$1 < p < 3\,.$$

Assume that $f \in W^{1,m}(\Omega)$ for some $m \geq \frac{n}{2-\mu_*}$ and consider a critical point $x_0 \in \mathcal{Z}_u$ with $B_{2\rho}(x_0) \subset \Omega$. Then we have

$$\int_{B_\rho(x_0)} \frac{\|D^2 u\|^2}{|x-x_0|^\eta} \leqslant \mathcal{C}\,, \tag{2.26}$$

with $\mathcal{C} = \mathcal{C}(p\,,\,n\,,\,f\,,\,u\,,\,x_0)$ and for any η such that

$$\eta < \eta_* := n + \big(2\,min\{\alpha_M^-, \frac{s-n}{(p-1)s}\} - 2\big)\,.$$

Since $\mu_* \leq 1$, the result holds in particular if $f \in W^{1,n}(\Omega)$. As a consequence we also get that $u \in W^{2,2}_{loc}(\Omega)$.

Proof. The proof follows by directly by Theorem 2.1 and (1.4) since we have that $p-3 < 0$ by assumption. □

3. Local summability of the weight

We prove in this section a summability property of $|\nabla u|^{-1}$, extending the results of [5] to the case of vanishing source terms. We will assume that f satisfies the condition (I_{μ_*}) and prove the following:

Theorem 3.1. *Let $1 < p < \infty$ and let $u \in W^{1,p}(\Omega)$ be a solution of* (1.1). *Assume that f satisfies* (1.3) *and assume that (I_{μ_*}) holds. Then, for any $x_0 \in \mathcal{Z}_u$ and for some $\rho = \rho(x_0) > 0$, we have*

$$\int_{B_\rho(x_0)} \frac{1}{|\nabla u|^{(p-1)^-}}\frac{1}{|x-y|^{(n-2)^-}} \leqslant \mathcal{C}. \tag{3.27}$$

Namely

$$\int_{B_\rho(x_0)} \frac{1}{|\nabla u|^t} \frac{1}{|x-y|^\gamma} \leqslant \mathcal{C}\,,$$

with $0 \leqslant t < p-1$, $\gamma < n-2$ *if* $n \geq 3$ *and* $\gamma = 0$ *if* $n = 2$ *and* $\mathcal{C} = \mathcal{C}(t, \gamma, p, f, \rho, x_0)$ *not depending on* $y \in \Omega$.

Proof. Consider

$$\phi = \frac{1}{|\nabla u|^t + \varepsilon} \cdot \frac{1}{|x-x_0|^\mu + \delta} \cdot \frac{G_\delta(f)}{|f|} \cdot H_\delta(|x-y|) \cdot \varphi^2$$

with $\varepsilon, \delta > 0$, H_δ and G_δ defined according to (2.22), $t < p-1$ and $0 < \mu < \mu_*$ with μ_* defined in (1.5) such that (1.12) and (1.13) hold, by (I_{μ_*}). Here, as above, we assume that the ball $B_{2\rho}(x_0)$ is contained in Ω, and we consider a cut-off function $\varphi = \varphi_\rho$, in such a way that $\varphi_\rho \in C_c^\infty(B_{2\rho}(x_0))$, $\varphi_\rho = 1$ in $B_\rho(x_0)$ and $|\nabla \varphi_\rho| \leqslant \frac{2}{\rho}$. In particular we may and do assume that (I_{μ_*}) holds in $B_{2\rho}(x_0)$. We will write φ instead of φ_ρ.
We use ϕ as test function in (2.16) and get

(3.28)

$$\begin{aligned}
&\int_{B_{2\rho}(x_0)} f(x)\phi = \int_{B_{2\rho}(x_0)} |\nabla u|^{p-2}(\nabla u\,,\,\nabla\phi) = \\
&= -\int_{B_{2\rho}(x_0)} \frac{t|\nabla u|^{t-1}}{(|\nabla u|^t+\varepsilon)^2} \frac{|\nabla u|^{p-2}(\nabla u\,,\,\nabla|\nabla u|)}{|x-x_0|^\mu+\delta} \cdot \frac{G_\delta(f)}{|f|} \cdot H_\delta \cdot \varphi^2 \\
&- \int_{B_{2\rho}(x_0)} \frac{\mu|x-x_0|^{\mu-1}}{(|x-x_0|^\mu+\delta)^2} \frac{|\nabla u|^{p-2}(\nabla u\,,\,\nabla|x-x_0|)}{|\nabla u|^t+\varepsilon} \cdot \frac{G_\delta(f)}{|f|} \cdot H_\delta \cdot \varphi^2 \\
&+ \int_{B_{2\rho}(x_0)} \frac{|\nabla u|^{p-2}(\nabla u\,,\,\nabla_x H_\delta(|x-y|))}{|\nabla u|^t+\varepsilon} \cdot \frac{1}{|x-x_0|^\mu+\delta} \cdot \frac{G_\delta(f)}{|f|}\varphi^2 \\
&+ 2\int_{B_{2\rho}(x_0)} \frac{|\nabla u|^{p-2}(\nabla u\,,\,\nabla\varphi)}{|\nabla u|^t+\varepsilon} \cdot \frac{1}{|x-x_0|^\mu+\delta} \cdot \frac{G_\delta(f)}{|f|} \cdot H_\delta\varphi \\
&+ \int_{B_{2\rho}(x_0)} \cdot \Big(\frac{G_\delta(t)}{|t|}\Big)' \,\Big|_{t=f} \, \frac{\chi_{\{\delta \leq |f| \leq 2\delta\}}}{|\nabla u|^t+\varepsilon} \cdot \frac{|\nabla u|^{p-2}(\nabla u\,,\,\nabla f)}{|x-x_0|^\mu+\delta} \cdot H_\delta\varphi^2\,,
\end{aligned}$$

that gives

$$
\begin{aligned}
&C^{-1}\int_{B_{2\rho}(x_0)} f\,\phi\\
&\leqslant \int_{B_{2\rho}(x_0)} |\nabla u|^{p-1}\sum_{i=1}^{N}|\nabla u_i|\frac{|\nabla u|^{t-1}}{(|\nabla u|^t+\varepsilon)^2}\cdot\frac{H_\delta(|x-y|)}{|x-x_0|^\mu+\delta}\cdot\frac{|G_\delta(f)|}{|f|}\cdot\varphi^2\\
&+\int_{B_{2\rho}(x_0)} \frac{|\nabla u|^{p-1}}{|x-x_0|^{\mu+1}}\cdot\frac{1}{|\nabla u|^t+\varepsilon}\cdot\frac{|G_\delta(f)|}{|f|}\cdot H_\delta(|x-y|)\cdot\varphi^2\\
&+\int_{B_{2\rho}(x_0)} \frac{|\nabla u|^{p-1}H'_\delta(|x-y|)}{|\nabla u|^t+\varepsilon}\cdot\frac{1}{|x-x_0|^\mu+\delta}\cdot\frac{|G_\delta(f)|}{|f|}\varphi^2+\\
&+\int_{B_{2\rho}(x_0)} \frac{|\nabla u|^{p-1}\,|\nabla\varphi|}{|\nabla u|^t+\varepsilon}\cdot\frac{1}{|x-x_0|^\mu+\delta}\cdot\frac{|G_\delta(f)|}{|f|}\cdot H_\delta(|x-y|)\varphi\\
&+\int_{B_{2\rho}(x_0)} \frac{|\nabla u|^{p-1}|\nabla f|}{|\nabla u|^t+\varepsilon}\cdot\frac{1}{|x-x_0|^\mu+\delta}\cdot\frac{H_\delta(|x-y|)}{|f|}\varphi^2\,,
\end{aligned}
$$

where we have estimated the last term exploiting the fact that

$$
\Big|(\frac{G_\delta(t)}{|t|})'\,|_{t=f}\,\Big|\leq\frac{4}{|f|}\quad\text{for}\quad\delta\leq|f|\leq 2\delta\,.
$$

Here and in the following we denote with $C=C(x_0,\rho,f,n,p,t,\gamma,\mu)$ a generic constant that we allow to vary each line. We now let $\delta\to 0$ and, by the dominated convergence theorem and (1.12), (1.13) (see (I_{μ_*})), we get:

(3.29)

$$
\begin{aligned}
&C^{-1}\int_{B_{2\rho}(x_0)} \frac{1}{|\nabla u|^t+\varepsilon}\cdot\frac{1}{|x-y|\gamma}\cdot\varphi^2\leqslant\\
&\leqslant\int_{B_{2\rho}(x_0)} |f|\cdot\frac{1}{|\nabla u|^t+\varepsilon}\cdot\frac{1}{|x-x_0|^\mu}\cdot\frac{1}{|x-y|\gamma}\cdot\varphi^2\\
&\leqslant\int_{B_{2\rho}(x_0)} |\nabla u|^{p-1}\sum_{i=1}^{N}|\nabla u_i|\frac{|\nabla u|^{t-1}}{(|\nabla u|^t+\varepsilon)^2}\cdot\frac{1}{|x-x_0|^\mu}\cdot\frac{1}{|x-y|^\gamma}\cdot\varphi^2\\
&+\int_{B_{2\rho}(x_0)} \frac{|\nabla u|^{p-1}}{|x-x_0|^{\mu+1}}\cdot\frac{1}{|\nabla u|^t+\varepsilon}\cdot\frac{1}{|x-y|^\gamma}\cdot\varphi^2\\
&+\int_{B_{2\rho}(x_0)} \frac{|\nabla u|^{p-1}}{|\nabla u|^t+\varepsilon}\cdot\frac{1}{|x-x_0|^\mu}\frac{1}{|x-y|^{\gamma+1}}\cdot\varphi^2\\
&+\int_{B_{2\rho}(x_0)} \frac{|\nabla u|^{p-1}\,|\nabla\varphi|}{|\nabla u|^t+\varepsilon}\cdot\frac{1}{|x-x_0|^\mu}\cdot\frac{1}{|x-y|^\gamma}\varphi\\
&+\int_{B_{2\rho}(x_0)} \frac{|\nabla u|^{p-1}|\nabla f|}{|\nabla u|^t+\varepsilon}\cdot\frac{1}{|x-x_0|^{2\mu}}\cdot\frac{1}{|x-y|^\gamma}\varphi^2\,.
\end{aligned}
$$

The reader can easy check that, since $\mu < 1$ and $\gamma < n-2$ and $\varepsilon > 0$, then the dominated convergence theorem applies. Let us now estimate the terms on the righthand side of (3.29). We have:

$$
\begin{aligned}
&\int_{B_{2\rho}(x_0)} |\nabla u|^{p-1}\sum_{i=1}^{N}|\nabla u_i|\frac{|\nabla u|^{t-1}}{(|\nabla u|^t+\varepsilon)^2}\cdot\frac{1}{|x-x_0|^{\mu}}\cdot\frac{1}{|x-y|^{\gamma}}\cdot\varphi^2\\
&\leqslant C\int_{B_{2\rho}(x_0)}\frac{1}{(|\nabla u|^t+\varepsilon)^{\frac12}}\cdot\frac{1}{|x-y|^{\frac{\gamma}{2}}}\cdot\frac{\varphi^2}{|x-x_0|^{\mu}}\cdot\frac{|\nabla u|^{p+t-2}}{(|\nabla u|^t+\varepsilon)^{\frac32}}\frac{\|D^2u\|}{|x-y|^{\frac{\gamma}{2}}}\\
&\leqslant \vartheta\int_{B_{2\rho}(x_0)}\frac{1}{(|\nabla u|^t+\varepsilon)}\cdot\frac{1}{|x-y|^{\gamma}}\cdot\frac{\varphi^2}{|x-x_0|^{\mu}}\\
&+\frac{C}{\vartheta}\int_{B_{2\rho}(x_0)}\frac{|\nabla u|^{(p-2)-(2+t-p)}}{|x-y|^{\gamma}}\cdot\frac{\|D^2u\|^2\varphi^2}{|x-x_0|^{\mu}}\\
&\leqslant \vartheta\int_{B_{2\rho}(x_0)}\frac{1}{(|\nabla u|^t+\varepsilon)}\frac{1}{|x-y|^{\gamma}}\cdot\frac{\varphi^2}{|x-x_0|^{\mu}}+\frac{C}{\vartheta}\,,
\end{aligned}
$$

exploiting Theorem 2.1 (note that (1.12) is enough) with $\beta = 2+t-p$. Note that the assumption $t < p-1$ implies $\beta < 1$. The assumption $t < p-1$ also yields

$$
\begin{aligned}
&\int_{B_{2\rho}(x_0)}\frac{|\nabla u|^{p-1}}{|x-x_0|^{\mu+1}}\cdot\frac{1}{|\nabla u|^t+\varepsilon}\cdot\frac{1}{|x-y|^{\gamma}}\cdot\varphi^2\\
&\leqslant C\int_{B_{2\rho}(x_0)}\frac{1}{|x-x_0|^{\mu+1}}\cdot\frac{1}{|x-y|^{\gamma}}\cdot\varphi^2\\
&\leqslant C
\end{aligned}
$$

since $\mu < 1$ (see (1.5)) and $\gamma < n-2$. Moreover

$$
\begin{aligned}
&\int_{B_{2\rho}(x_0)}\frac{|\nabla u|^{p-1}}{|\nabla u|^t+\varepsilon}\cdot\frac{1}{|x-x_0|^{\mu}}\frac{1}{|x-y|^{\gamma+1}}\cdot\varphi^2\\
&\leqslant C\int_{B_{2\rho}(x_0)}\frac{1}{|x-x_0|^{\mu}}\cdot\frac{1}{|x-y|^{\gamma+1}}\cdot\varphi^2\\
&\leqslant C\,,
\end{aligned}
$$

since $\mu < 1$ and $\gamma < n-2$. Also we have:

$$
\begin{aligned}
&\int_{B_{2\rho}(x_0)}\frac{|\nabla u|^{p-1}\,|\nabla\varphi|}{|\nabla u|^t+\varepsilon}\cdot\frac{1}{|x-x_0|^{\mu}}\cdot\frac{1}{|x-y|^{\gamma}}\varphi\\
&\leqslant\frac{C}{\rho}\int_{B_{2\rho}(x_0)}\frac{|\nabla\varphi|}{|x-x_0|^{\mu}}\frac{\varphi}{|x-y|^{\gamma}}\\
&\leqslant C\int_{B_{2\rho}(x_0)}\frac{1}{|x-x_0|^{\mu}}\frac{\varphi}{|x-y|^{\gamma}}\leqslant C\,,
\end{aligned}
$$

since $t < p-1$, $\mu < 1$, $\gamma < n-2$. Note that here the constant that we get depends on ρ as stated in the theorem. We also have

$$\begin{aligned}
&\int_{B_{2\rho}(x_0)} \frac{|\nabla u|^{p-1}|\nabla f|}{|\nabla u|^t+\varepsilon}\cdot\frac{1}{|x-x_0|^{2\mu}}\cdot\frac{1}{|x-y|^\gamma}\varphi^2\\
&\quad\leqslant \int_{B_{2\rho}(x_0)} |\nabla f|\cdot\frac{1}{|x-x_0|^{2\mu}}\cdot\frac{1}{|x-y|^\gamma}\varphi^2\\
&\quad\leqslant C\left(\int_\Omega |\nabla f|^m\right)^{\frac{1}{m}}\left(\int_\Omega\Big(\frac{1}{|x-x_0|^{2\mu}|x-y|^\gamma}\Big)^{m'}\right)^{\frac{1}{m'}}\leq C\,,
\end{aligned}$$

exploiting that $f\in W^{1,m}(\Omega)$ for some $m>\frac{n}{2(1-\mu)}$ (see (1.12)) and the fact that $t<p-1$ and $\mu<1$ and $\gamma<n-2$ ($\gamma=0$ is $n=2$).

Collecting the above estimates, by (3.29), we get

$$(3.30)\qquad (1-C\vartheta)\int_{B_{2\rho}(x_0)}\frac{1}{(|\nabla u|^t+\varepsilon)}\cdot\frac{1}{|x-y|^\gamma}\cdot\frac{1}{|x-x_0|^\mu}\cdot\varphi^2\leqslant C\,.$$

We now take ϑ small, say ϑ small such that in (3.30) we have $(1-C\vartheta)\geq\frac{1}{2}$, and get

$$\int_{B_{2\rho}(x_0)}\frac{1}{(|\nabla u|^t+\varepsilon)}\cdot\frac{1}{|x-y|^\gamma}\cdot\frac{1}{|x-x_0|^\mu}\cdot\varphi^2\leqslant C\,,$$

that gives the thesis by letting $\varepsilon\to 0$ and exploiting Fatou's Lemma. □

Corollary 3.2. *Let $p\geq 3$ and let $u\in W^{1,p}(\Omega)$ be a solution of (1.1). Assume that (1.3) and (I_{μ_*}) hold and assume that $f\in W^{1,m}(\Omega)$ for some $m>\frac{n}{2-\mu_*}$. Then, for $x_0\in\mathcal{Z}_u$ (and for some $\rho=\rho(x_0)>0$) and for any*

$$q<\frac{p-1}{p-2}\,,$$

we have:

$$(3.31)\qquad \int_{B_\rho(x_0)}\frac{\|D^2u\|^q}{|x-x_0|^\tau}\leqslant\mathcal{C}\,,$$

for any

$$(3.32)\qquad \tau<\tau_*\,:=\,n-2+\frac{q}{2}\mu_*\,,$$

with μ_* defined in (1.5). The result holds in particular if $f\in W^{1,n}(\Omega)$.

Proof. We can rewrite any $\tau<\tau_*$ as:

$$\tau:=\gamma+\frac{q}{2}(\mu_*-\varepsilon)$$

for some $\gamma < n-2$ if $n \geq 3$, $\gamma = 0$ if $n = 2$ and $\varepsilon > 0$ (small). Then, for any $0 \leq \beta < 1$ and $1 < q < 2$, we have

$$\begin{aligned}
&\int_{B_\rho(x_0)} \frac{\|D^2u\|^q}{|x-x_0|^\tau}\,dx \\
&\leqslant \int_{B_\rho(x_0)} \frac{|\nabla u|^{(p-2-\beta)\frac{q}{2}}\|D^2u\|^q}{|x-x_0|^{\frac{q}{2}(\gamma+\mu_*-\varepsilon)}} \cdot \frac{1}{|\nabla u|^{(p-2-\beta)\frac{q}{2}}|x-x_0|^{\gamma\frac{2-q}{2}}}\,dx \\
&\leq \Big(\int_{B_\rho(x_0)} \frac{|\nabla u|^{(p-2-\beta)}\|D^2u\|^2}{|x-x_0|^{\gamma+\mu_*-\varepsilon}}\,dx\Big)^{\frac{q}{2}} \Big(\int_{B_\rho(x_0)} \frac{1}{|\nabla u|^{(p-2-\beta)\frac{q}{2-q}}|x-x_0|^\gamma}\,dx\Big)^{\frac{2-q}{2}} \\
&\leq C\,,
\end{aligned}$$

where we used Holder inequality with exponents $\frac{2}{q}$ and $\frac{2}{2-q}$, Theorem 2.1 with $y = x_0$ (since we assumed that $f \in W^{1,m}(\Omega)$ for some $m > \frac{n}{2-\mu_*}$) and Theorem 3.1. Note that, to apply Theorem 3.1, we can choose $0 < \beta < 1$ such that $(p-2-\beta)\frac{q}{2-q} < p-1$ because of the assumption $q < \frac{p-1}{p-2}$.

□

4. The strong comparison principle

The summability properties of $|\nabla u|^{-1}$ obtained in Theorem 3.1 allows to prove a weighted Sobolev inequality that we recall here below:

Theorem 4.1 ([5]). *Let $p > 2$ and $u \in C^1(\Omega)$ be a solution of* (1.1). *Assume that f satisfies* (1.3) *and* (I_{μ_*}) *(see* (1.12) *and* (1.13)*). Setting $\rho = |\nabla u|^{p-2}$, for any $\Omega' \subset\subset \Omega$, we have*

$$\int_{\Omega'} \frac{1}{\rho^t \cdot |x-y|^\gamma} \leqslant \mathcal{C}\,, \tag{4.33}$$

with $t = \frac{p-1}{p-2}r$, $\frac{p-2}{p-1} < r < 1$, $\gamma < n-2$ if $n \geq 3$ while $\gamma = 0$ if $n = 2$, and $y \in \Omega$.
Assuming in the case $n \geq 3$ with no loose of generality that $\gamma > n-2t$, then it follows that the space $H_0^{1,2}(\Omega', \rho)$ is continuously embedded in L^q, for $1 \leqslant q < 2^*(p)$, where

$$\frac{1}{2^*(p)} = \frac{1}{2} - \frac{1}{n} + \frac{p-2}{p-1} \cdot \frac{1}{n}\,.$$

More precisely there exists a constant $\mathcal{C}_S$ such that

$$\|w\|_{L^q(\Omega')} \leqslant \mathcal{C}_S \|\nabla w\|_{L^2(\Omega',\rho)} = \mathcal{C}_S \left(\int_{\Omega'} |\nabla w|^2 \rho\right)^{\frac{1}{2}}\,, \tag{4.34}$$

for any $w \in H_0^{1,2}(\Omega', \rho)$ and $1 \leqslant q < 2^*(p)$.

Moreover the embedding is compact.

Proof. Once Theorem 3.1 is proved then (4.33) follows by a simple covering argument. We can therefore exploit (4.33) to repeat verbatim the proof of Theorem 3.1 in [5]. Finally the embedding is compact by [2, Lemma 5.1]. □

As a consequence we have

Theorem 4.2. *Let* $u, v \in C^1(\Omega)$ *such that*

$$-\Delta_p u - f(x) \leq -\Delta_p v - f(x) \qquad \text{in } \Omega\,.$$

Assume that u *or* v *is a solution to* (1.1) *with* $\frac{2n+2}{n+2} < p < \infty$*, and assume that* f *satisfies* (1.3) *and* (I_{μ_*}) *(see* (1.12) *and* (1.13)*). Then, for* $x_0 \in \mathcal{Z}_u$ *fixed, if we assume that* $u \leq v$ *in* $B_{5\rho}(x_0) \subset \Omega$*, it follows that*

$$\sup_{B_\rho(x_0)} (v-u) \leq C_H \inf_{B_{2\rho}(x_0)} (v-u) \tag{4.35}$$

for some constant $C_H = C_H(u, v, f, n, p, \rho, x_0)$.

Proof. The proof follows by the Moser-type iteration scheme, as developed in [30]. The technique in [30] actually is mainly based on Sobolev embedding and works in our case thanks to Theorem 4.1. The details can be found in [6, Theorem 3.3]. □

We are now ready to prove Theorem 1.3 and Theorem 1.4.

Proof of Theorem 1.3:

The proof is a consequence of the Harnack inequality in Theorem 4.2. Let us set

$$\mathcal{C}_{u,v} := \left\{ x \in \Omega' \ : \ u(x) = v(x) \right\}.$$

Since u and v are continuous, it follows that $\mathcal{C}_{u,v}$ is closed. Let us consider now $x_0 \in \mathcal{C}_{u,v}$ and $\rho = \rho(x_0) > 0$ such that $B_{5\rho}(x_0) \subset \Omega'$. Note that actually $\nabla u(x_0) = \nabla v(x_0)$ since u touches v from below at x_0.
If $\nabla u(x_0) \neq 0$ then, eventually reducing ρ, we have that $u = v$ in $B_\rho(x_0)$ by the classical strong comparison principle [3] (see also [10]).
If else $\nabla u(x_0) = 0 = \nabla v(x_0)$, eventually reducing ρ, we exploit Theorem 4.2 to prove that $u = v$ in $B_\rho(x_0)$.

Therefore $\mathcal{C}_{u,v}$ is also open and the thesis follows recalling that Ω' is connected.

Proof of Theorem 1.4: Let us consider the case when u is a nonnegative solution to the equation (1.10) and set

$$f(x) := |x|^\sigma g(u(x))\,.$$

Since $\Delta_p u \leq 0$, by the strong maximum principle [32] it follows that u is strictly positive, since we assumed that it is not trivial. Therefore,

$g(u(0)) > 0$ since $g(s) > 0$ for $s > 0$ and, by the assumption $0 \leq \sigma < \mu_*^\infty$, it follows that f satisfies the condition $(I_{\mu_*^\infty})$ (it is easy to check the validity of (1.12)). Therefore Theorem 3.1 holds and Theorem 4.1 follows as well. This allows to repeat verbatim the proof of [6, Theorem 3.3] and deduce also in this case the validity of the Harnack inequality, namely Theorem 4.2. The proof now can be finished arguing exactly as in the proof of Theorem 1.3.

5. Examples

Example 5.1. *Let $1 < p < \infty$ and consider $u \in C^{1,\alpha}(B)$ weak solution to*

$$\begin{cases} -\Delta_p u = u^q & \text{in } B \\ u > 0 & \text{in } B \\ u = 0 & \text{on } \partial B\,, \end{cases} \tag{5.36}$$

where B is the unit ball in $\mathbb{R}^n$ centered at zero. Then, by [4, 5]*, it follows that u is radial and radially decreasing with*

$$\mathcal{Z}_u := \{\nabla u = 0\} \equiv \{0\}\,.$$

Therefore, arguing as in [26]*, by l'Hopital's rule we have*

$$|\nabla u| \approx |x|^{\frac{1}{p-1}} \qquad \text{and} \qquad \|D^2 u\| \approx |x|^{\frac{2-p}{p-1}}.$$

Example 5.2. *Let $p > 1$. Then the function*

$$u(x_1, \ldots, x_N) = \frac{|x_1|^{p'}}{p'}$$

solves

$$\Delta_p\, u = 1\,,$$

and

$$\mathcal{Z}_u = \{x_1 = 0\}\,.$$

Remark 5.3. *It is easy to see that, for $p \geq 3$, the solution given in Example 5.2 is no more regular than what we obtained in Corollary (3.2). On the contrary, the solutions in Example 5.1 are more regular (just use polar coordinates to see this).*
We might guess that the geometry of the critical set $\mathcal{Z}_u$ plays a role.

Proposition 5.4. *Let $u \in C^{1,\alpha}(\Omega)$ be a weak solution of* (1.1) *in $\Omega \subset \mathbb{R}^n$. Suppose that $x_0 \in \mathcal{Z}_u$, namely $\nabla u(x_0) = 0$, and suppose that: for some $\bar\rho, \bar\vartheta > 0$ we have that $f \geq \bar\vartheta$ in $B_{\bar\rho}(x_0)$. Then (in a neighborhood of x_0) we have*

$$\alpha \leq \frac{1}{p-1}.$$

In particular, for $p > 2$, we have that $u \notin C^2(\Omega)$.

Proof. Let us consider the balls $B_\rho(x_0)$ and $B_{2\rho}(x_0)$ centered in x_0 with $2\rho \leq \bar{\rho}$. Consider a cut-off function φ_ρ such that, $\varphi_\rho \geq 0$ in $B_{2\rho}(x_0)$, $\varphi_\rho \in C_c^\infty(B_{2\rho}(x_0))$ and

$$\begin{cases} \varphi_\rho \equiv \frac{1}{\rho^n} & \text{in} \quad B_\rho(x_0) \\ |\nabla \varphi_\rho| \leq \frac{c}{\rho^{n+1}} & \text{in} \quad B_{2\rho}(x_0) \setminus B_\rho(x_0)\,. \end{cases} \tag{5.37}$$

Using φ_ρ as test-function in (1.1) we get

$$\int_\Omega |\nabla u|^{p-2}(\nabla u, \nabla \varphi_\rho)\, dx = \int_\Omega f\, \varphi_\rho\, dx\,. \tag{5.38}$$

We have that:

$$\int_\Omega |\nabla u|^{p-2}(\nabla u, \nabla \varphi_\rho)\, dx \leq \int_\Omega |\nabla u - \nabla u(x_0)|^{p-1} |\nabla \varphi_\rho|\, dx \leq$$
$$\leq c\, \frac{\rho^{\alpha(p-1)} \rho^n}{\rho^{n+1}} \leq c\, \rho^{\alpha(p-1)-1}$$

so that

$$\int_\Omega |\nabla u|^{p-2}(\nabla u, \nabla \varphi_\rho)\, dx \underset{\rho \to 0}{\longrightarrow} 0 \qquad \text{if} \quad (\alpha(p-1)-1) > 0\,. \tag{5.39}$$

On the other hand

$$\int_\Omega f\, \varphi_\rho\, dx \geq \bar{\vartheta} \int_\Omega \varphi_\rho\, dx \geq \bar{\vartheta}\, |B_1| > 0 \tag{5.40}$$

for ρ sufficiently small. Therefore, for $(\alpha(p-1)-1) > 0$, we get a contradiction by (5.38), (5.39) and (5.40). As a consequence we get that near x_0 necessarily $(\alpha(p-1)-1) \leq 0$ and the thesis. Note now that, if $p > 2$, then $\frac{1}{p-1} < 1$ so that $u \notin C^2(\Omega)$. □

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Dipartimento di Matematica, UNICAL,, Ponte Pietro Bucci 31B,, 87036 Arcavacata di Rende, Cosenza, Italy

E-mail address: sciunzi@mat.unical.it