ON A POINCARÉ TYPE FORMULA FOR SOLUTIONS OF SINGULAR AND DEGENERATE ELLIPTIC EQUATIONS

ALBERTO FARINA, BERARDINO SCIUNZI, AND ENRICO VALDINOCI

Abstract. We provide a geometric Poincaré type formula for stable solutions of $-\Delta_p(u) = f(u)$. From this, we derive a symmetry result in the plane. This work is a refinement of previous results obtained by the authors under further integrability and regularity assumptions.

Let $p \in [2, +\infty)$, $f \in C^1(\mathbb{R})$ and Ω be a non-empty, open subset of $\mathbb{R}^N$. Assume that $u \in W^{1,p}_{\rm loc}(\Omega)$ is a weak solution of

$$-\Delta_p(u) = f(u) \qquad \text{in } \Omega. \tag{1}$$

We consider the weighted Sobolev space with weight $\rho = |\nabla u|^{p-2}$ (see[1] [4]). Such space, denoted by $H^{1,2}_\rho(\Omega)$ may be defined as the closure of $C^1(\Omega)$ with respect to the $\|\cdot\|_{H^{1,2}_\rho(\Omega)}$-norm defined as

$$\begin{aligned} \|v\|_{H^{1,2}_\rho(\Omega)} &:= \|v\|_{L^2(\Omega)} + \|\nabla v\|_{L^2_\rho(\Omega)} \\ &= \sqrt{\int_\Omega |v(x)|^2\,dx} + \sqrt{\int_\Omega |\nabla v(x)|^2\,\rho(x)\,dx}. \end{aligned}$$

We also define $H^{1,2}_{\rho,0}(\Omega)$ to be the closure of $C^1_0(\Omega)$ with respect to the $H^{1,2}_\rho(\Omega)$-norm. We suppose that u is stable, that is

$$\int_\Omega |\nabla u|^{p-2}|\nabla\varphi|^2 + (p-2)|\nabla u|^{p-4}\,(\nabla u, \nabla\varphi)^2 - f'(u)\varphi^2\,dx \;\geqslant\; 0 \tag{2}$$

for any $\varphi \in C^1_0(\Omega)$ – or, equivalently, by density, for any $\varphi \in H^{1,2}_{\rho,0}(\Omega)$. This stability condition is classical in the calculus of variation framework (for instance, local minima are

---

1991 *Mathematics Subject Classification.* 35B05, 35B65, 35J70.

Addresses: *AF* – LAMFA – CNRS UMR 6140 – Université de Picardie Jules Verne – Faculté de Mathématiques et d'Informatique – 33, rue Saint-Leu – 80039 Amiens CEDEX 1, France. Email: alberto.farina@u-picardie.fr.
BS – Università della Calabria – Dipartimento di Matematica – Ponte Pietro Bucci, 31 B – I-87036 Arcavacata di Rende, Cosenza Italy. Email: sciunzi@mat.unical.it.
EV – Università di Roma Tor Vergata – Dipartimento di Matematica – Via della ricerca scientifica, 1 – I-00133 Rome, Italy. Email: valdinoci@mat.uniroma2.it.
EV is supported by FIRB "Analysis and Beyond" and GNAMPA "Equazioni nonlineari su varietà: proprietà qualitative e classificazione delle soluzioni".

⋆ ⋆ ⋆

.
[1]We remark that the weight ρ is locally integrable, since we suppose here $p \geqslant 2$. The results of this paper, for $1 < p \leqslant 2$, are already contained in [7], since, in this case, $\nabla u \in W^{1,2}_{\rm loc}(\Omega)$, thanks to [13].

stable solutions; see, e.g., [1, 7]). At any $x \in \Omega \cap \{\nabla u \neq 0\}$, we consider the level set of u, namely

$$\mathcal{L}_{u,x} = \big\{ y \in \Omega \text{ s.t. } u(y) = u(x) \big\}.$$

By well-known regularity theory (see, e.g., [6, 13]), one has that

$$u \in C^{1,\alpha}_{\rm loc}(\Omega) \cap C^{2,\alpha}_{\rm loc}(\Omega \cap \{\nabla u \neq 0\}) \tag{3}$$

and so $\mathcal{L}_{u,x}$ is a C^2-hypersurface. In particular, we can consider the principal curvatures $k_{l,u}$ of $\mathcal{L}_{u,x}$, for $l = 1, \dots, N-1$. Also, given $g \in C^1(\Omega)$, one can define its tangential gradient with respect to $\mathcal{L}_{u,x}$, that is

$$\nabla_{\mathcal{L}_{u,x}} g = \nabla g - \big(\nabla g, \frac{\nabla u}{|\nabla u|}\big) \frac{\nabla u}{|\nabla u|}. \tag{4}$$

With this notation, the following inequality holds:

Theorem 1.

$$\begin{aligned} (p-1)\int_\Omega |\nabla u|^p |\nabla \varphi|^2 \, dx \geqslant (p-1) \int_{\Omega \cap \{\nabla u \neq 0\}} |\nabla u|^{p-2} \varphi^2 |\nabla_{\mathcal{L}_{u,x}} |\nabla u||^2 \, dx \\ + \int_{\Omega \cap \{\nabla u \neq 0\}} |\nabla u|^p \varphi^2 \sum_{1 \leqslant l \leqslant N-1} k^2_{l,u} \, dx \end{aligned} \tag{5}$$

for any $\varphi \in C^1_0(\Omega)$.

Notice that the left hand side of (5) is finite, since $u \in W^{1,p}_{\rm loc}(\Omega)$ and $\varphi \in C^1_0(\Omega)$. Formula (5) was proved in [7], under the additional assumption that $\nabla u \in W^{1,2}_{\rm loc}(\Omega)$, see (2.10) in [7], and several delicate approximation estimates will be needed here to remove such unnecessary hypothesis. When $p = 2$, formula (5) reduces to the important Poincaré type formula of [11, 12]. In fact, [11, 12] first introduced these types of weigthed inequalities, in which the weighted norms of any test function φ is bounded by a weighted norm of its gradient, and the weights involve the geometry of the level sets of a stable solution. By an appropriate choice of the test function, as shown in [7], it is possible to deduce several interesting results on the geometry of the solution itself. In particular, following some ideas of [7], the symmetry result in the plane which is stated here below is a consequence of Theorem 1:

Corollary 2. *If u is a stable solution of $-\Delta_p(u) = f(u)$ in the whole $\mathbb{R}^2$, with $|\nabla u| \in L^\infty(\mathbb{R}^2)$, then it possesses onedimensional symmetry, that is there exists $u_o : \mathbb{R} \to \mathbb{R}$ and $\varpi \in \mathrm{S}^1$ such that*

$$u(x) = u_o(\varpi \cdot x) \qquad \textit{for any } x \in \mathbb{R}^2.$$

Under the additional assumption that $\nabla u \in W^{1,2}_{\rm loc}(\mathbb{R}^2)$, Corollary 2 was proven in [7] (see, in particular, Theorem 1.1 there). Therefore, Corollary 2 is a refinement of a previous result of [7] which drops an unnecessary assumption. When $p = 2$, Corollary 2 is related to a celebrated problem posed by De Giorgi (see [5, 9, 3] and also [2, 1, 10, 8]). The proofs of Theorem 1 and Corollary 2 are contained in the forthcoming Sections 1 and 2, respectively.

1. Proof of Theorem 1

From Proposition 2.2 of [4], we know that[2]

$$\frac{\partial u}{\partial x_j} = u_j \in H^{1,2}_{\rho,\mathrm{loc}}(\Omega) \qquad \text{for any } j = 1, \dots, N. \tag{6}$$

Moreover, as proved in [4],

$$|\nabla u|^{p-2}\nabla u \in H^1_{\mathrm{loc}}(\Omega, \mathbb{R}^N) \tag{7}$$

Now, we consider (4) and we see that

$$|\nabla_{\mathcal{L}_{u,x}} g|^2 = |\nabla g|^2 - (\nabla g, \frac{\nabla u}{|\nabla u|})^2. \tag{8}$$

Also, by a direct computation,

$$\frac{\partial |\nabla u|}{\partial x_j} = \frac{(\nabla u, \nabla u_j)}{|\nabla u|} \qquad \text{in } \{\nabla u \neq 0\}. \tag{9}$$

From (8) and (9), we obtain that, in $\{\nabla u \neq 0\}$,

$$\begin{aligned} &|\nabla u|^{p-2}\Big[|\nabla|\nabla u||^2 - \sum_{j=1}^N |\nabla u_j|^2\Big] - (p-2)|\nabla u|^{p-2}|\nabla_{\mathcal{L}_{u,x}}|\nabla u||^2 = \\ &|\nabla u|^{p-2}|\nabla|\nabla u||^2 + (p-2)|\nabla u|^{p-4}(\nabla u, \nabla|\nabla u|)^2 - \\ &\sum_{j=1}^N |\nabla u|^{p-2}|\nabla u_j|^2 - (p-2)\sum_{j=1}^N |\nabla u|^{p-4}(\nabla u, \nabla u_j)^2. \end{aligned} \tag{10}$$

Now, we observe that, by Cauchy-Schwarz inequality,

$$|\nabla u|^{p-2}\|D^2 u\|\,|\nabla\psi| \leqslant |\nabla u|^{p-2}\|D^2 u\|^2\chi_\psi + |\nabla u|^{p-2}|\nabla\psi|^2 \in L^1(\Omega), \tag{11}$$

for any $\psi \in C^1_0(\Omega)$, where χ_ψ is the characteristic function of the support of $|\nabla\psi|$, thanks to (3), (6) and the fact that $p \geqslant 2$. From (1),

$$0 = -\int_\Omega |\nabla u|^{p-2}(\nabla u, \nabla\psi_j) - f(u)\psi_j \, dx, \tag{12}$$

for any $\psi \in C^\infty_0(\Omega)$. We remark that we can integrate by parts in (12), by means of (7). What is more, the distributional derivatives of $|\nabla u|^{p-2}u_i$ may be computed via a standard calculus, due to Remark 2.3 of [4] and Stampacchia's Theorem (for the latter, see, e.g., Theorem 1.56 on page 79 of [14]), giving

$$\partial_j(|\nabla u|^{p-2}u_i) = |\nabla u|^{p-2}u_{i\,j} + (p-2)|\nabla u|^{p-4}(\nabla u, \nabla u_j)u_i. \tag{13}$$

with $\partial_j(|\nabla u|^{p-2}u_i) = 0$ in the critical set $\{\nabla u = 0\}$, according to Stampacchia's Theorem. Therefore, (12), an integration by parts, (11) and (13) give that

$$0 = \int_\Omega |\nabla u|^{p-2}(\nabla u_j, \nabla\psi) + (p-2)|\nabla u|^{p-4}(\nabla u, \nabla u_j)(\nabla u, \nabla\psi) - f'(u)u_j\psi \, dx, \tag{14}$$

[2] We remark that in some parts of [4] some further assumptions are taken, such as boundary conditions or sign hypotheses on the nonlinearity. This is not the case for the part of [4] that we use here, since it only deals with the local integrability properties of the solutions.

for any $\psi \in C_0^\infty(\Omega)$, and, in fact, by density and (11), for any $\psi \in C_0^1(\Omega)$. Now, we take $\varphi \in C_0^1(\Omega)$ as in the statement of Theorem 1 and we define, for a fixed j,

$$\psi := u_j \cdot \varphi^2.$$

Let $\Omega' \subset \Omega$ be a bounded open set containing the support of φ and consider a sequence $w_\epsilon \in C^1(\Omega')$ which approaches u_j in the $\|\cdot\|_{H^{1,2}_\rho(\Omega')}$-norm as $\epsilon \to 0^+$ (the existence of this sequence follows from (6) and our definition of $H^{1,2}_\rho$). Let $\psi_\epsilon := w_\epsilon \cdot \varphi^2$. Notice that $\psi_\epsilon \in C_0^1(\Omega)$ and so we can apply to it formula (14), obtaining

$$\begin{aligned} 0 &= \int_\Omega |\nabla u|^{p-2}(\nabla u_j, \nabla\psi_\epsilon) + (p-2)|\nabla u|^{p-4}(\nabla u, \nabla u_j)(\nabla u, \nabla\psi_\epsilon) - f'(u)u_j\psi_\epsilon \\ &= \int_\Omega |\nabla u|^{p-2} w_\epsilon(\nabla u_j, \nabla\varphi^2) + |\nabla u|^{p-2}\varphi^2(\nabla u_j, \nabla w_\epsilon) \\ &\quad + (p-2)|\nabla u|^{p-4}(\nabla u, \nabla u_j) w_\epsilon(\nabla u, \nabla\varphi^2) \\ &\quad + (p-2)|\nabla u|^{p-4}(\nabla u, \nabla u_j)\varphi^2(\nabla u, \nabla w_\epsilon) \\ &\quad - f'(u)u_j\varphi^2 w_\epsilon\, dx. \end{aligned} \tag{15}$$

Furthermore, for any bounded open set $\tilde\Omega \subset \Omega$, by Cauchy-Schwarz inequality and (6),

$$\begin{aligned} &\int_{\tilde\Omega} |\nabla u|^{p-2}\|D^2u\|\,|\nabla w_\epsilon - \nabla u_j|\,dx \\ &\leqslant \sqrt{\int_{\tilde\Omega} |\nabla u|^{p-2}\|D^2u\|^2\,dx}\sqrt{\int_{\tilde\Omega} |\nabla u|^{p-2}|\nabla w_\epsilon - \nabla u_j|^2\,dx}, \end{aligned}$$

which goes to zero as $\epsilon \to 0^+$, thanks to (3), (6) and the fact that $p \geqslant 2$. Consequently, by passing $\epsilon \to 0^+$ in (15), then summing over j, and recalling (13), we obtain that

$$\begin{aligned} &\sum_{j=1}^N \int_\Omega \big(|\nabla u|^{p-2}(\nabla u_j, \nabla(u_j\varphi^2)) + (p-2)|\nabla u|^{p-4}(\nabla u, \nabla u_j)(\nabla u, \nabla(u_j\varphi^2))\big)\,dx \\ &= \int_\Omega f'(u)|\nabla u|^2\varphi^2\,dx \end{aligned} \tag{16}$$

Now, fix $\varepsilon > 0$ and look at the stability condition (2) for the test function

$$\varphi \cdot G_\varepsilon(|\nabla u|)$$

with $G_\varepsilon \in C^\infty(\mathbb{R})$ such that $G_\varepsilon(t) = t$ if $|t| \geqslant 2\varepsilon$, $G_\varepsilon(t) = 0$ if $|t| \leqslant \varepsilon$ and $|G'(t)| \leqslant 3$ for any $t \in \mathbb{R}$.

Note that $\varphi \cdot G_\varepsilon(|\nabla u|)$ is a admissible as test function in the linearized equation (2) (that is $\varphi \cdot G_\varepsilon(|\nabla u|) \in H^{1,2}_{\rho,0}(\Omega)$) since G_ε is locally Lipschitz continuous and vanishes in a neighborhood of 0. Consequently $G_\varepsilon(|\nabla u|)$ is identically zero in a neighborhood of the critical set $\nabla u = 0$, while elsewhere we can exploit (3): more precisely, $\varphi \cdot G_\varepsilon(|\nabla u|) \in H_0^{1,2}(\Omega)$, and $H_0^{1,2}(\Omega) \subset H^{1,2}_{\rho,0}(\Omega)$ since $p \geqslant 2$ (see [4]). Consequently the distributional derivatives may be calculated in a standard way and (2) gives

$$
\begin{aligned}
&\int_\Omega |\nabla u|^{p-2}(G_\varepsilon(|\nabla u|))^2\cdot|\nabla\varphi|^2+(p-2)|\nabla u|^{p-4}(\nabla u,\nabla\varphi)^2\cdot(G_\varepsilon(|\nabla u|))^2\,dx+\\
&\int_\Omega |\nabla u|^{p-2}(G'_\varepsilon(|\nabla u|))^2\cdot|\nabla|\nabla u||^2\varphi^2+\\
&\qquad (p-2)|\nabla u|^{p-4}\varphi^2(G'_\varepsilon(|\nabla u|))^2(\nabla u,\nabla|\nabla u|)^2\,dx+\\
&2\int_\Omega |\nabla u|^{p-2}(G_\varepsilon(|\nabla u|))(G'_\varepsilon(|\nabla u|))\varphi(\nabla|\nabla u|,\nabla\varphi)+\\
&\qquad (p-2)|\nabla u|^{p-4}\varphi(G'_\varepsilon(|\nabla u|))(G_\varepsilon(|\nabla u|))(\nabla u,\nabla|\nabla u|)(\nabla u,\nabla\varphi)\,dx-\\
&\int_\Omega f'(u)(G_\varepsilon(|\nabla u|))^2\cdot\varphi^2\,dx \;\geqslant\; 0.
\end{aligned}
\tag{17}
$$

Now, we set $\Omega_o := \Omega\cap\{\nabla u\neq 0\}$. Notice that, for any $m>0$ and any $\Psi\in L^1(\mathbb{R}^N)$, with $\Psi=0$ outside Ω, we have

$$
\begin{aligned}
&\lim_{\varepsilon\to 0^+}\int_\Omega (G'_\varepsilon(|\nabla u|))^m\Psi-\int_{\Omega_o}\Psi\\
&\qquad\leqslant \lim_{\varepsilon\to 0^+}\int_{\Omega_o} 3^m|\Psi|\chi_{\varepsilon\leqslant|\nabla u|\leqslant 2\varepsilon}=0.
\end{aligned}
\tag{18}
$$

We now let $\varepsilon\to 0^+$ in (17): using (6), (18) and the dominated convergence theorem, we conclude that

$$
\begin{aligned}
&\int_{\Omega_o} |\nabla u|^p|\nabla\varphi|^2+(p-2)|\nabla u|^{p-2}(\nabla u,\nabla\varphi)^2\,dx+\\
&\int_{\Omega_o} |\nabla u|^{p-2}|\nabla|\nabla u||^2\varphi^2+(p-2)|\nabla u|^{p-4}\varphi^2(\nabla u,\nabla|\nabla u|)^2\,dx+\\
&2\int_{\Omega_o} |\nabla u|^{p-2}|\nabla u|\varphi(\nabla|\nabla u|,\nabla\varphi)+\\
&+(p-2)|\nabla u|^{p-4}\varphi|\nabla u|(\nabla u,\nabla|\nabla u|)(\nabla u,\nabla\varphi)\,dx-\\
&\int_\Omega f'(u)|\nabla u|^2\varphi^2\,dx \;\geqslant\; 0.
\end{aligned}
\tag{19}
$$

We now use (16) and (19) to get

$$\begin{aligned}
&\int_{\Omega_o} |\nabla u|^p|\nabla\varphi|^2+(p-2)|\nabla u|^{p-2}(\nabla u,\nabla\varphi)^2\,dx+\\
&\int_{\Omega_o} |\nabla u|^{p-2}|\nabla|\nabla u||^2\varphi^2+(p-2)|\nabla u|^{p-4}\varphi^2(\nabla u,\nabla|\nabla u|)^2\,dx+\\
&2\int_{\Omega_o} |\nabla u|^{p-2}|\nabla u|\varphi(\nabla|\nabla u|,\nabla\varphi)+\\
&(p-2)|\nabla u|^{p-4}\varphi|\nabla u|(\nabla u,\nabla|\nabla u|)(\nabla u,\nabla\varphi)\,dx\geqslant\\
&\geqslant\sum_{j=1}^N\int_\Omega |\nabla u|^{p-2}(\nabla u_j,\nabla(u_j\varphi^2))+(p-2)|\nabla u|^{p-4}(\nabla u,\nabla u_j)(\nabla u,\nabla(u_j\varphi^2))\,dx\\
&=2\sum_{j=1}^N\int_\Omega |\nabla u|^{p-2}u_j\varphi(\nabla u_j,\nabla\varphi)+(p-2)|\nabla u|^{p-4}u_j\varphi(\nabla u,\nabla u_j)(\nabla u,\nabla\varphi)\,dx+\\
&\sum_{j=1}^N\int_\Omega |\nabla u|^{p-2}\varphi^2|\nabla u_j|^2+(p-2)|\nabla u|^{p-4}\varphi^2(\nabla u,\nabla u_j)^2\,dx.
\end{aligned}$$

After an elementary, but remarkable, simplification, the above inequality implies that

$$\begin{aligned}
&\int_{\Omega_o} |\nabla u|^p|\nabla\varphi|^2+(p-2)|\nabla u|^{p-2}(\nabla u,\nabla\varphi)^2\,dx+\\
&\int_{\Omega_o} |\nabla u|^{p-2}|\nabla|\nabla u||^2\varphi^2+(p-2)|\nabla u|^{p-4}\varphi^2(\nabla u,\nabla|\nabla u|)^2\,dx\geqslant\\
&\geqslant\sum_{j=1}^N\int_\Omega |\nabla u|^{p-2}\varphi^2|\nabla u_j|^2+(p-2)|\nabla u|^{p-4}\varphi^2(\nabla u,\nabla u_j)^2\,dx.
\end{aligned}\tag{20}$$

From this and (10), we conclude that

$$\begin{aligned}
&(p-1)\int_{\Omega_o} |\nabla u|^p|\nabla\varphi|^2\,dx\geqslant\\
&\geqslant\int_{\Omega_o} |\nabla u|^{p-2}\varphi^2\Big[\sum_{j=1}^N|\nabla u_j|^2-|\nabla|\nabla u||^2\Big]+(p-2)|\nabla u|^{p-2}\varphi^2|\nabla_{\mathcal{L}_{u,x}}|\nabla u||^2\,dx.
\end{aligned}\tag{21}$$

On the other hand, by formula (2.1) of [12],

$$\sum_{j=1}^N|\nabla u_j|^2-|\nabla|\nabla u||^2=|\nabla u|^2\sum_{1\leqslant l\leqslant N-1}k_{l,u}^2+|\nabla_{\mathcal{L}_{u,x}}|\nabla u||^2.$$

Plugging this into (21) and rearranging the terms, we obtain (5). This completes the proof of Theorem 1. ■

2. Proof of Corollary 2

We observe that (5) holds true and therefore the hypotheses of Corollary 2.6 of [7] are satisfied. Consequently, Corollary 2.6 of [7] implies that both $\nabla_{\mathcal{L}_{u,x}}|\nabla u|$ and $k_{1,u}$ vanish on $\{\nabla u \neq 0\}$. This and Lemma 2.11 of [7] entail that u possesses one-dimensional symmetry. ■

References

[1] G. Alberti, L. Ambrosio, X. Cabré. On a long-standing conjecture of E. De Giorgi: symmetry in 3D for general nonlinearities and a local minimality property. Special issue dedicated to Antonio Avantaggiati on the occasion of his 70th birthday. *Acta Appl. Math.*, 65(1-3):9–33, 2001.

[2] L. Ambrosio, X. Cabré. Entire solutions of semilinear elliptic equations in $\mathbb{R}^3$ and a conjecture of De Giorgi *J. Am. Math. Soc.*, 13(4):725–73, 2000.

[3] H. Berestycki, L. A. Caffarelli, L. Nirenberg, Further qualitative properties for elliptic equations in unbounded domains. *Ann. Sc. Norm. Super. Pisa, Cl. Sci. (4)*, 25(1-2):69–94, 1997.

[4] L. Damascelli, B. Sciunzi. Regularity, monotonicity and symmetry of positive solutions of m-Laplace equations. *J. Differential Equations*, 206(2):483–515, 2004.

[5] E. De Giorgi, Convergence problems for functionals and operators. *Proceedings of the International Meeting on Recent Methods in Nonlinear Analysis (Rome, 1978)*, pp. 131–188, Pitagora, Bologna, 1979.

[6] E. DiBenedetto. $C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations. *Nonlinear Anal.*, 7(8):827–850, 1983.

[7] A. Farina, B. Sciunzi, E. Valdinoci. Bernstein and De Giorgi type problems: new results via a geometric approach. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 7(4):741–791, 2008.

[8] A. Farina, E. Valdinoci. The state of the art for a conjecture of De Giorgi and related problems. Du, Y., Ishii, H., Lin, W.-Y. (eds) *Recent Progress on Reaction Diffusion Systems and Viscosity Solutions.* Series on Advances in Mathematics for Applied Sciences, pp. 372. World Scientific, Singapore, 2008.

[9] N. A. Ghoussoub, C. Gui, On a conjecture of De Giorgi and some related problems. *Math. Ann.*, 311(3):481–491, 1998.

[10] O. Savin, Regularity of flat level sets in phase transitions. *Ann. Math. (2)*, 169(1):41–78, 2009.

[11] P. Sternberg, K. Zumbrun. A Poincaré inequality with applications to volume-constrained area-minimizing surfaces. *J. Reine Angew. Math.*, 503:63–85, 1998.

[12] P. Sternberg, K. Zumbrun. Connectivity of phase boundaries in strictly convex domains. *Arch. Rational Mech. Anal.*, 141(4):375–400, 1998.

[13] P. Tolksdorf. Regularity for a more general class of quasilinear elliptic equations. *J. Differential Equations*, 51(1):126–150, 1984.

[14] G.M. Troianiello. Elliptic differential equations and obstacle problems. The University Series in Mathematics. Plenum Press, New York, xiv+353 pp., 1987.