Methods and Applications of Analysis.
Vol. 15, No. 3, pp. 277–284, September 2008

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p-MEMS EQUATION ON A BALL*

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Abstract. We investigate qualitative properties of the MEMS equation involving the $p-$Laplace operator, $1 < p \leq 2$, on a ball B in $\mathbb{R}^N$, $N \geq 2$. We establish uniqueness results for semi-stable solutions and stability (in a strict sense) of minimal solutions. In particular, along the minimal branch we show monotonicity of the first eigenvalue for the corresponding linearized operator and radial symmetry of the first eigenfunction.

Key words.

AMS subject classifications. 35B05, 35B65, 35J70

1. Introduction and statement of the main results. Let us consider the problem

$$
\begin{cases} -\Delta_p u = \frac{\lambda}{(1-u)^2} & \text{in } \Omega \\ u < 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}
$$

where $\Delta_p(\cdot) = \text{div}\,(|\nabla(\cdot)|^{p-2}\nabla(\cdot))$, $p > 1$, denotes the p-Laplace operator, $\lambda > 0$ and $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a smooth domain.

For $p = 2$ equation (1) arises in the study of Micro-Electromechanical Systems (MEMS), where electronics combines with micro-size mechanical devices to design various types of microscopic components of modern sensors in various areas. Mathematical modeling of MEMS devices has been studied rigourously just recently, see [7, 8, 9, 14, 15, 16, 19] and [10, 11, 12, 13] for the corresponding parabolic version.

We are interested here to establish some qualitative properties of semi-stable solutions of the quasilinear version (1) of the MEMS equation. In the semilinear context, this follows by comparison arguments which become highly non trivial when $p-$Laplace operator, $p \neq 2$, is involved.

Due to the singular/degenerate character of the elliptic operator Δ_p, by [6, 17, 20] the best regularity for a weak-solution u of (1) is $u \in C^{1,\alpha}(\Omega)$, for some $\alpha \in (0,1)$. A classical solution u of (1) then will be a $C^{1,\alpha}(\Omega)-$function, $\alpha \in (0,1)$, which satisfies the equation in a weak sense

$$
\int_\Omega |\nabla u|^{p-2}(\nabla u, \nabla \phi)\, dx = \lambda \int_\Omega \frac{\phi}{(1-u)^2}\, dx \qquad \forall\, \phi \in W_0^{1,p}(\Omega). \tag{2}
$$

Throughout the paper, a solution u of (1) is always assumed to be in a classical sense as specified here. Let us remark that for $1 < p < 2$ solutions might be of class C^2

*Received May 14, 2008; accepted for publication October 21, 2008. Authors' research is supported by *MIUR Metodi variazionali ed equazioni differenziali nonlineari*.

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but the term $|\nabla u|^{p-2}$ is singular where ∇u vanishes. Therefore, also in this case, a classical solution is meant to satisfy the equation just in a weak sense.

We continue here the investigation of (1) we started in [2]. Setting

$$\lambda^* = \sup\{\lambda > 0 : (1) \text{ has a solution}\},$$

in [2] we showed that $\lambda^* < +\infty$ and for every $\lambda \in (0, \lambda^*)$ there is a minimal (and semi-stable) solution u_λ (i.e. u_λ is the smallest positive solution of (1) in a pointwise sense). Further, the family $\{u_\lambda\}$ is non-decreasing in λ and the function

$$u^* = \lim_{\lambda \uparrow \lambda^*} u_\lambda$$

is a weak solution (in a suitable sense) of (1) at $\lambda = \lambda^*$. In low dimensions the function u^* satisfies $\|u^*\|_\infty < 1$ and is then a classical solution.

To make things more precise, let us recall a few definitions. For $1 < p \leq 2$ (the case we will be later concerned with) let $\rho = |\nabla u|^{p-2}$ and introduce a weighted L^2-norm of the gradient: $|\phi| = \left(\int_\Omega \rho |\nabla \phi|^2\right)^{\frac{1}{2}}$. According to [4, 5], define $\mathcal{A}_u$ as the following subspace of $H_0^1(\Omega)$:

$$\mathcal{A}_u = \{\phi \in H_0^1(\Omega) : \ |\phi| < +\infty\}.$$

Since $\int_\Omega |\nabla \phi|^2 \leq \|\nabla u\|_\infty^{2-p} |\phi|^2$, the space $(\mathcal{A}_u, |\cdot|)$ is an Hilbert space. We can then give the following

Definition 1.1. *A solution u of (1) is* ***semi-stable*** *(resp.* ***stable****) if*

$$\int_\Omega |\nabla u|^{p-2} |\nabla \phi|^2 \, dx + (p-2) \int_\Omega |\nabla u|^{p-4} (\nabla u, \nabla \phi)^2 \, dx - 2\lambda \int_\Omega \frac{\phi^2}{(1-u)^3} \, dx$$
$$\geq 0 \ (\textit{resp.} \ > 0)$$

for every $\phi \in \mathcal{A}_u \setminus \{0\}$.

The space $\mathcal{A}_u$ allows to define the pair first eigenvalue/eigenfunction in the p-Laplace context as given by the following

Theorem 1.2. *([2]) Let u be a solution of (1). The infimum*

$$\mu_{1,\lambda}(u) := \inf_{\phi \in \mathcal{A}_u \setminus \{0\}} \frac{\int_\Omega |\nabla u|^{p-2} |\nabla \phi|^2 \, dx + (p-2) \int_\Omega |\nabla u|^{p-4} (\nabla u, \nabla \phi)^2 \, dx - 2\lambda \int_\Omega \frac{\phi^2}{(1-u)^3} \, dx}{\int_\Omega \phi^2}$$

is attained at some function $\phi_1 = \phi_{1,\lambda,u} > 0$ a.e. in Ω, and any other minimizer is proportional to ϕ_1.

By duality a linearized operator L_u can be defined as an operator from $\mathcal{A}_u$ into itself. The first eigenfunction solves $L_u(\phi_1) = \mu_{1,\lambda}(u)\phi_1$ in a weak sense:

$$\begin{aligned} L_u(\phi_1)[\psi] := & \int_\Omega |\nabla u|^{p-2} (\nabla \phi_1, \nabla \psi) \, dx + (p-2) \int_\Omega |\nabla u|^{p-4} (\nabla u, \nabla \phi_1)(\nabla u, \nabla \psi) \, dx \\ & -2\lambda \int_\Omega \frac{\phi_1 \psi}{(1-u)^3} \, dx \\ & = \mu_{1,\lambda}(u) \int_\Omega \phi_1 \psi \, dx. \end{aligned}$$

There are the following issues which were left open in [2]:

- uniqueness of u_λ among the semi-stable solutions of (1);
- stability of the minimal solution u_λ.

On the ball $B := B(0,1)$ there is a positive answer to these questions for $1 < p \leq 2$. In this case, by [3] any solution of (1) is radial and radially decreasing. Since $u' \leq 0$, the key property will be that the function $s \to g(s) := |s|^{p-2}s$ is convex in $(-\infty, 0]$ whenever $1 < p \leq 2$.
Some of our results make use of first eigenfunctions for the linearized operator. This is a first application of theorem 1.2 which in our opinion might have other useful consequences.

Our arguments work as well if we replace $(1-u)^{-2}$ with a general nondecreasing and nonnegative convex nonlinearity $f(u)$:

$$\begin{cases} -\Delta_p u = \lambda f(u) & \text{in } B \\ u = 0 & \text{on } \partial B. \end{cases} \tag{3}$$

The function $f(u)$ can be either smooth on $[0, +\infty)$ or singular at $u = 1$. A classical solution u of (3) is meant to be bounded in the first case and to be < 1 in the second one. Moreover, in the definition 1.1 we have to replace $2(1-u)^{-3}$ with $f'(u)$.

We have the following uniqueness result

Theorem 1.3. *Let us assume $1 < p \leq 2$ and let u be a semi-stable solution of problem (3) on B. Then $u \equiv u_\lambda$ where u_λ is the minimal solution.*

We now investigate the properties of the first eigenvalue $\mu_{1,\lambda}(u)$ and the corresponding eigenfunction $\phi_{1,\lambda,u}$, which is the content of the following

Theorem 1.4. *On B $\phi_{1,\lambda,u}$ is radial and radially decreasing with $\phi'_{1,\lambda,u}(r) < 0$ for $r \in (0,1]$. The first eigenvalue is strictly decreasing along the minimal branch: $\mu_\lambda := \mu_{1,\lambda}(u_\lambda) \downarrow$ as $\lambda \uparrow \lambda^*$. In particular, $\mu_\lambda > 0$ for every $0 < \lambda < \lambda^*$ and u_λ is a stable solution of (3) on B.*

We are able to prove a stronger uniqueness property for problem (3) when the first egenvalue is zero, as highlighted by this

Theorem 1.5. *Let $1 < p \leq 2$. Let u be a solution of problem (3) so that $\mu_{1,\lambda}(u) = 0$. Then, $\lambda = \lambda^*$, $u = u^*$ and any other solution v of (3) coincides with u.*

Let us stress that theorem 1.5 might be established in a more general way by the arguments in [1, 18] based directly on the definition of λ^*. We do not pursue this approach since we prefer a more classical one based on comparison arguments.

In the next sections we will give the proofs of theorems 1.3 through 1.5.

2. Proof of theorem 1.3. Let u be a semi-stable solution of (3). By [3] we know that u is radial, radially decreasing and have an unique critical point at the origin with $u'(r) \approx r^{\frac{1}{p-1}}$ as $r \to 0$. In particular, $u' < 0$ in $(0,1)$.
Since u'_λ and u' behave as $r^{\frac{1}{p-1}}$ as $r \to 0$, it is easily seen that $u, u_\lambda \in \mathcal{A}_u \cap W_0^{1,p}(B)$. Therefore, $u_\lambda - u$ can be used as a test function both in the equation and in the linearized operator at u.

By taking $u_\lambda - u$ as test function in (2) we get

$$\int_B |\nabla u|^{p-2}(\nabla u, \nabla(u_\lambda - u))\, dx = \lambda \int_B f(u)(u_\lambda - u)\, dx$$

and

$$\int_B |\nabla u_\lambda|^{p-2}(\nabla u_\lambda, \nabla(u_\lambda - u))\,dx = \lambda \int_B f(u_\lambda)(u_\lambda - u)\,dx.$$

Taking into account radial symmetry, the difference leads to

$$0 = \int_B (|u_\lambda'|^{p-2}u_\lambda' - |u'|^{p-2}u')(u_\lambda' - u')\,dx - \lambda \int_B (f(u_\lambda) - f(u))(u_\lambda - u)\,dx.$$

Since $f(u_\lambda) \geq f(u) + f'(u)(u_\lambda - u)$ by convexity, we have that

$$0 \geq \int_B (|u_\lambda'|^{p-2}u_\lambda' - |u'|^{p-2}u')(u_\lambda' - u')\,dx - \lambda \int_B f'(u)(u_\lambda - u)^2\,dx$$

in view of $u_\lambda \leq u$ by minimality of u_λ. Since in $(0,1)$

$$-(r^{N-1}|u_\lambda'|^{p-2}u_\lambda')' = \lambda r^{N-1} f(u_\lambda) \leqslant \lambda r^{N-1} f(u) = -(r^{N-1}|u'|^{p-2}u')',$$

for $0 < \varepsilon < r < 1$ we get

$$\begin{aligned} &r^{N-1}|u'(r)|^{p-2}u'(r) - \varepsilon^{N-1}|u'(\varepsilon)|^{p-2}u'(\varepsilon) \\ &\leqslant r^{N-1}|u_\lambda'(r)|^{p-2}u_\lambda'(r) - \varepsilon^{N-1}|u_\lambda'(\varepsilon)|^{p-2}u_\lambda'(\varepsilon) \end{aligned}$$

and by letting $\varepsilon \to 0$ it follows

$$|u'(r)|^{p-2}u'(r) \leqslant |u_\lambda'(r)|^{p-2}u_\lambda'(r) \qquad \text{in } (0,1). \tag{4}$$

Since $u', u_\lambda' < 0$ in $(0,1)$, it gives $|u'(r)| \geqslant |u_\lambda'(r)|$ or equivalently $u'(r) \leqslant u_\lambda'(r)$ for every $r \in (0,1)$.

We now take into account that the function $g(s) = |s|^{p-2}s$ is strictly convex in $(-\infty, 0)$ for $1 < p < 2$. Therefore, in $(0,1)$ we have

$$(|u_\lambda'|^{p-2}u_\lambda' - |u'|^{p-2}u')(u' - u_\lambda') > (p-1)|u'|^{p-2}(u_\lambda' - u')$$

whenever $u' < u_\lambda'$. Since $u' \leqslant u_\lambda'$ in $(0,1)$, if $u \neq u_\lambda$ in turn we get

$$0 > \int_B (p-1)|u'|^{p-2}(u_\lambda' - u')^2 - \lambda f'(u)(u_\lambda - u)^2\,dx. \tag{5}$$

At the same time, by the semi-stability of u we have

$$\int_B (p-1)|u'|^{p-2}(u_\lambda' - u')^2 - \lambda f'(u)(u_\lambda - u)^2\,dx \geqslant 0 \tag{6}$$

and a contradiction arises unless $u = u_\lambda$.

Consider now the case $p = 2$. Since now $g(s)$ is linear, we have only $\geq$ in (5). However, if $\mu_{1,\lambda}(u) > 0$ we have a strict inequality in (6) and a contradiction still arises unless $u = u_\lambda$.

We have therefore to deal with the case $p = 2$, $\mu_{1,\lambda}(u) = 0$ and $u \neq u_\lambda$: by the variational characterization of the first eigenvalue it follows that $u - u_\lambda = \beta\phi_1$, $\beta > 0$, where ϕ_1 is the (positive) first eigenfunction of the linearized operator L_u. We define in this case

$$G(t) = -\Delta(tu+(1-t)u_\lambda)-\lambda f(tu+(1-t)u_\lambda) = \lambda\big[tf(u)+(1-t)f(u_\lambda)-f(tu+(1-t)u_\lambda)\big].$$

Since f is convex, then $G(t) \geqslant 0$. Since

$$G'(t) = -\Delta(u - u_\lambda) - \lambda f'(tu + (1 - t)u_\lambda)(u - u_\lambda)$$

and $u - u_\lambda = \beta\phi_1$, we have that

$$G'(1) = -\Delta(u - u_\lambda) - \lambda f'(u)(u - u_\lambda) = 0.$$

Also, $G''(t) = -\lambda f''(tu + (1 - t)u_\lambda)(u - u_\lambda)^2 < 0$ thanks to the convexity of f. But this is not consistent with $G(1) = 0$, $G'(1) = 0$ and $G(t) \geqslant 0$. The proof is done. □

3. Proof of theorem 1.4. Let us consider a hyperplane P, passing trough the origin. Setting for simplicity $\phi_1 = \phi_{1,\lambda,u}$, define $\phi_1^P(x) = \phi_1(x_P)$ where x_P is symmetric to x with respect to the hyperplane P. Since u is radial, it follows that ϕ_1^P still minimizes the quotient in theorem 1.2 and is then proportional to ϕ_1: $\phi_1^P = \beta\phi_1$. Since ϕ_1^P and ϕ_1 coincide on P, it follows that $\beta = 1$ and $\phi_1^P = \phi_1$, that is ϕ_1 is symmetric with respect to P. Since P is arbitrary chosen, it follows that ϕ_1 is radial.

Let us now show that $\phi_1'(r) < 0$ for $r \in (0, 1]$.
Note that, since ϕ_1 is radial as we showed above, then it fulfills the following equation

$$-(p-1)(r^{N-1}|u'(r)|^{p-2}\phi_1'(r))' = r^{N-1}(\lambda f'(u(r))\phi_1(r) + \mu_\lambda\phi_1(r)) \tag{7}$$

where $\mu_\lambda := \mu_{1,\lambda}(u_\lambda) \geqslant 0$. Since f' is positive, we therefore have that
the term $r^{N-1}|u'(r)|^{p-2}\phi_1'(r)$ is decreasing for $r \in (0, 1]$.
Also by (7), we get

$$\frac{(r^{N-1}|u'(r)|^{p-2}\phi_1'(r))'}{r^{N-1}} \underset{r\to 0}{\longrightarrow} c \tag{8}$$

exploiting de l'Hôpital we get that

$$\frac{r^{N-1}|u'(r)|^{p-2}\phi_1'(r)}{r^N} \underset{r\to 0}{\longrightarrow} c \tag{9}$$

therefore

the term $r^{N-1}|u'(r)|^{p-2}\phi_1'(r) \to 0$ for $r \to 0$.

Since as showed above $r^{N-1}|u'(r)|^{p-2}\phi_1'(r)$ is decreasing for $r \in (0, 1]$, then $r^{N-1}|u'(r)|^{p-2}\phi_1'(r) < \varepsilon^{N-1}|u'(\varepsilon)|^{p-2}\phi_1'(\varepsilon)$ for $0 < \varepsilon < r \leqslant 1$. Letting $\varepsilon \to 0$, we get

$$r^{N-1}|u'(r)|^{p-2}\phi_1'(r) < 0$$

for $r \in (0, 1]$, showing the thesis.

To prove monotonicity of the first eigenvalue , we start noticing that $u_\lambda \leqslant u_\beta$ for $\lambda < \beta$ yields to $u_\beta' \leq u_\lambda' < 0$ in $(0, 1)$ with the same argument as in (4). Let us assume that the first eigenfunctions $\phi_\lambda := \phi_{1,\lambda,u_\lambda}$ and $\phi_\beta := \phi_{1,\beta,u_\beta}$ are normalized to have

$$\int_B \phi_\lambda^2 = \int_B \phi_\beta^2 = 1.$$

Since u_λ, u_β, ϕ_λ and ϕ_β are radial, we now have that

$$\begin{aligned}\mu_\beta &\le (p-1)\int_B |u_\beta'|^{p-2}(\phi_\lambda')^2\,dx - \beta\int_B f'(u_\beta)\phi_\lambda^2\,dx\\ &< (p-1)\int_B |u_\lambda'|^{p-2}|(\phi_\lambda')^2 - \lambda\int_B f'(u_\lambda)\phi_\lambda^2\,dx = \mu_\lambda\end{aligned}$$

in view of $u_\lambda \ne u_\beta$, and the thesis follows.

4. Proof of theorem 1.5. Let u be a solution of (3) so that $\mu_{1,\lambda}(u) = 0$. First, we have that $\lambda \ge \lambda^*$. Indeed, for $\lambda < \lambda^*$ by theorem 1.3 we would have that $u \equiv u_\lambda$ and then $\mu_{1,\lambda}(u) > 0$ by theorem 1.4. Since by the defintion of λ^* $\lambda \le \lambda^*$, we get that $\lambda = \lambda^*$. Since $u^* \le u$ and u is a classical solution, we get that also u^* is a classical solution and by theorem 1.3 $u = u^*$.

Let v be another solution of (3) and let ϕ_1 be the first eigenfunction of L_u. Define

$$\hat G(t) := \int_B |tv' + (1-t)u'|^{p-2}(tv' + (1-t)u')\phi_1'\,dx - \lambda\int_B f(tv + (1-t)u)\phi_1\,dx.$$

By the radial symmetry of u, v, ϕ_1 and the convexity of $g(s) = |s|^{p-2}s$ in $(-\infty, 0)$ for $1 < p \le 2$, we get that

$$\begin{aligned}\hat G(t) &= \int_B g(tv' + (1-t)u')\phi_1'\,dx - \lambda\int_B f(tv + (1-t)u)\phi_1\,dx\\ &\ge t\int_B g(v')\phi_1'\,dx + (1-t)\int_B g(u')\phi_1'\,dx - \lambda\int_B f(tv + (1-t)u)\phi_1\,dx\\ &= \lambda\int_B \left[tf(v) + (1-t)f(u) - f(tv + (1-t)u)\right]\phi_1\,dx \geqslant 0\end{aligned}$$

in view of $\phi_1' \le 0$ by theorem 1.4. Let us now note that $\hat G(0) = 0$ by the equation satisfied by u. Compute now the first derivative

$$\hat G'(t) = (p-1)\int_B |tv' + (1-t)u'|^{p-2}(v' - u')\phi_1'\,dx - \lambda f'(tv + (1-t)u)(v-u)\phi_1\,dx.$$

Since $L_u(\phi_1) = \mu_{1,\lambda}(u)\phi_1 = 0$ and $v - u \in \mathcal{A}_u$, we get that $\hat G'(0) = 0$. By $\hat G(0) = \hat G'(0) = 0$ and $\hat G(t) \geqslant 0$, it follows $\hat G''(0) \geqslant 0$. But

$$\begin{aligned}\hat G''(0) &= (p-1)(p-2)\int_B |u'|^{p-4}u'(v' - u')^2\phi_1' - \lambda f''(u)(v-u)^2\phi_1\,dx\\ &\le -\lambda\int_B f''(u)(v-u)^2\phi_1\,dx\end{aligned}$$

in view of $u', \phi_1' \le 0$ and $1 < p \le 2$. Since $f'' > 0$, $\lambda > 0$ and $\phi_1 > 0$ a.e. in B it follows that $\hat G''(0) < 0$ unless $u = v$. Therefore the thesis follows. ∎

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